ALAN ROSS ANDERSON
NUEL D. BELNAP JR.
J. MICHAEL DUNN

Entailment

The Logic of Relevance and Necessity, Volume II

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ENTAILMENT

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THE LOGIC OF RELEVANCE AND NECESSITY

by

ALAN ROSS ANDERSON

and

NUEL D. BELNAP, JR.

and

J. MICHAEL DUNN

with contributions by

KIT FINE
ALASDAIR URQUHART

and further contributions by

DANIEL COHEN
STEVE GIAMBRONE
DOROTHY L. GROVER
ANIL GUPTA
GLEN HELMAN
ERROL P. MARTIN
MICHAEL A. MCROBBIE
STUART SHAPIRO

and including a Bibliography of Entailment by

ROBERT G. WOLF

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Dedicated to six generous parents

SELMA WETTELAND ANDERSON · ROSS ELMER ANDERSON
ELIZABETH DAFTER BELNAP · NUEL DINSMORE BELNAP
PHILOMENA LAUER DUNN · JON HARDIN DUNN

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PREFACE

THIS VOLUME continues the line of investigation into the logic of relevance and necessity—Entailment, we say—commenced by Ackermann in 1956 and reported on by ARA and NDB in Volume I of this book in 1975. At that time what we had planned for Volume II was well in hand, and, rather against the suggestion of our gracious editor, we explicitly and publicly projected the second volume for "about a year" after Volume I, that is, for about 1976 or 1977. In the meantime ARA died, NDB and R. K. Meyer entered an industrious collaboration that in the end did not succeed, and finally NDB and JMD, whose joint work on these topics goes back about a quarter of a century, have completed this volume within at least one year, as Russell would say, of the publication of Volume I.

In that volume we passed on our belief that the earliest versions of relevance logic were those of Moh 1950 and Church 1951. Though hardly guilty of a howler in the sense of §20.2, we certainly missed the truth by over two decades: relevance logic was already treated with insight and rigor by Orlov 1928! This we first learned from the engaging report of Došen 1990. We subsequently learned to our increased chagrin that the work of Orlov had already been brought to light by V. M. Popov in 1978. This fact is recorded for instance in a recommendable 1988 book, previously unknown to us, by E. K. Vojšhvillo of Moscow State University: *Philosophico-methodological aspects of relevance logic* (Russian) (Moscow: Izdatél'stvo Moskovskogo Universiteta). We learned of these matters and much else in the course of a memorable and instructive visit to Moscow in the winter of 1991, hospitably arranged for one of us (NDB) by V. A. Smirnov of the Institute for Philosophy of the Academy of Sciences. Under the leadership of Smirnov and Vojšhvillo, the city supports a large and interesting group of relevance logicians. Life is full of delights and hazards. Among the former lies the making of new and valued friends in distant places. Among the latter one must number errors in print. Still, although our history was not accurate, the measure of our debt to Moh and Church is undiminished.

Intertwining of the philosophical and mathematical voices continues from Volume I, as does the interspersal of the odd joke, though we regret that there are fewer of the latter than we used to enjoy with ARA. Much of the philosophical argle-bargle depends on that in Volume I, or on good sense, so some readers may be disappointed that we do not answer more recent critics in any concentrated fashion. Our reasons are two: first, we do not enjoy finding fault with individual arguments or lodging criticisms of individual writers (in contrast with Movements in the Large, such as the Officers) and we do not take ourselves to be good at it; and, second, tired readers would inevitably fasten on the emotion of such disputes, thus drifting away from what we think constitute our positive contributions. In any event, the mathematics here is almost entirely independent, and it is made clear whenever consultation with Volume I is required. As before, readers are encouraged to listen to the voice of polemic, proof, or pun according to their preferences.

We have laid out the chapters and sections in a good order for reading, but within this volume most chapters and even many sections are independent. It is not possible to be much more precise; the text has come from many hands over an extended period and is substantially more like a conversation than like a monologue. For further guidance we have provided an analytical table of contents with helpful titles, and we have caused each section to explain how it fits into the whole. For the reader, however, who has not seen Volume I or any of the related literature, a reading of §35 should give a useful introduction to the scheme of intuitions that lie behind relevance logic and entailment.

Policies on cross-references, citations, and notation are the same as for Volume I, except that in a few indicated cases we have let stand without tinkering the choice of fonts and symbols of a contributed manuscript, so as to reduce the chance of introduced errors. We think everything is plain except for discontinuities in our numbering of sections. The cause is this: later chapters were not only executed but even dreamt up over many years, so we early decided to start numbering the first section of these chapters "on the decade" so as to leave plenty of room for intellectual expansion. We are glad we did, and hope that readers will not be disconcerted.

We think that there is a large quantity of good stuff in this volume; we are pleased to have carried out some of the research ourselves, and we are proud to present a series of excellent studies by others. Among the latter, we may single out as the most recent and so the most exciting contributions (1) the proof by Urquhart in §65 that the concept of relevance essentially outruns any attempt fully to capture it by mechanical means, and (2) the new and original account by Fine in §53 of what quantifiers "for all" and "there exists" can and must mean in the context of relevance logic. We end this volume with a door-opening survey by Shapiro of the actual and potential impact of relevance logic on present-day computer science.

ACKNOWLEDGMENTS

THIS BOOK has been in preparation since 1959, and indeed some of the sections of this volume were composed long before Volume I was sent off to the printers. Three decades make it inevitable that sources of crucial suggestions or ideas for Volume II shall have been forgotten, so that a wretchedly impersonal wholesale expression of apologetic gratitude is required to cover all. Here we must make partial amends with a scattering of notes.

Our deepest debt is to the distinguished logicians, listed on the title page, who have given us permission to include their work as part of this volume, some of it taken from other publications and some written especially for this work. An extra measure of our gratitude is due those whose specially commissioned contributions have threatened to molder unpublished as the weary years declined. And although readers will of course award merit to contributions as they see fit, we as authors must single out Robert G. Wolf, who not only compiled the distinguished bibliography that he permitted us to publish as part of this volume, but has read over the entire manuscript, using his vast knowledge in order to provide us with hundreds of corrections and suggestions.

We renew and indeed redouble our thanks to Sanford Thatcher of Princeton University Press for much, but especially (1) for his intervolume patience, (2) for giving us access to Leigh Cauman, whom we join one—and probably many—of our friends in labeling "the best editor that ever was," and (3) for turning over to Syntax International Pte, Ltd, and its remarkable crew of craftsmen the task of turning our appalling henscratches into graceful print. Beth Gianfagna carefully supervised the final stages, and Gretchen Oberfranc used her skill and sensitivity in making Wolf's bibliography camera-ready.

To Kate Maloy we are obliged for the work exhibited in the frontispiece, which deftly recreates an old and typical fragment of ARA's smile-provoking wit.

Publishers and journals who have kindly given their permission for republication of works by our contributors or by us also deserve our thanks. Detailed information appears in the bibliography at the end of this volume under the heading given below; section numbers in parentheses indicate where in this volume (a portion of) the cited material appears.

D. Reidel Publishing Company: Dunn 1986 (§§42 and 48); Belnap 1977 (§81); 1979 (§82). **The journal of philosophical logic:** Fine 1974 (§51); 1988 (§53); Belnap and Gupta and Dunn 1980 (§61); Belnap 1982 (§62); Dunn 1987a

(§74). **The journal of philosophy:** Anderson and Belnap 1961 (§35). **The journal of symbolic logic:** Belnap 1967 (§40); Urquhart 1974 (§62.2); Urquhart 1984 (§65.2). Kluwer Academic Publications: Belnap and Dunn 1981 (§80). **Logique et analyse:** Urquhart 1983 (§65.1). **Mathematische Annalen:** Dunn and Belnap 1968a (§40). **Philosophical studies:** Dunn 1976 (§50). **Studia logica:** Dunn 1976b (§49); Dunn 1976d (§49); McRobbie and Belnap 1979 (§60); Dunn 1980 (§73).

We have a special debt to R. K. Meyer. As indicated in the tentative contents incautiously displayed by ARA and NDB in Volume I, Meyer was scheduled to be an important contributor to Volume II. Later NDB and Meyer collaborated for a period, during which Meyer constructively read over much of what ARA and NDB had produced, and wrote or partially wrote a number of pieces beyond those projected in the tentative contents. Since the close of the NDB-Meyer collaboration these pieces have generally been issued elsewhere by Meyer, but their influence certainly remains, often in the form of sections that we ourselves have written about Meyer's research; and if in the odd phrase or two we have quite unknowingly retained Meyer's actual words, we can only ask the reader's understanding and indulgence.

Some of the early portions of the manuscript came from the hands of the ten distinguished secretaries listed with admiration in Volume I; the remainder of Volume II, however, including Wolf's bibliography, is entirely from Collie Henderson, who has been awesome indeed. We don't know anyone wiser or more intelligent, and we feel greatly fortunate that she is on our side. We also wish to thank Susan Quinn and Luzia Martins for their excellent editorial assistance.

NDB and JMD spent three critical terms together at Indiana University in the late seventies. NDB was sponsored as Oscar R. Ewing Visiting Professor by the Ewing Fund, which also provided a superb research assistant in Daniel Cohen.

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Our final recorded debt is an enduring one to Carolyn W. Anderson, constant and cooperative friend of the enterprise.

SUMMARY REVIEW OF VOLUME I

ALTHOUGH THIS VOLUME is squarely a continuation of Volume I, still, many of the technical results do not depend thereon. And although we hope that most readers will not be dipping deeply into this volume without covering at least the introductory portions of the first, still, we do not wish to penalize too severely the reader who is patiently working through this volume while floating on a raft in a swimming pool, having left Volume I up at the house. Hence, we have provided the following highly compressed summaries of the grammar, axiomatics, and natural deduction formulations of the most important systems developed in Volume I, and of some of the key concepts used in investigating them.

§R1. Grammatical review. We make this resurvey chiefly to emphasize one or two points, and to hedge on one or two others. For further reference, we note that there is a list of special symbols at the back of this volume, and some helpful entries under "notation for" in the Index.

All the systems of Volume I have a denumerable stock of propositional (we also say "sentential") variables, and at least → as a binary connective. For truth functions we use $A\&B$ (sometimes AB) for conjunction, $A\vee B$ for disjunction, and for negation we use whichever of $\bar{A}$ or $\sim A$ seems convenient.

In §27.1.4 we pretty much settled on "co-tenability" for the connective $\circ$, since in **R** one has $A\circ B \rightleftarrows \sim(A\to\sim B)$. But since $\circ$ has uses in connection with other systems where this equivalence fails, we follow the growing tendency in the literature to call it "fusion," a term which is happily free of unhappy associations.

In connection with **E**, we generally think of $\Box A$ as defined by $A\to A\to A$ (§4.3), but we also want to think of it as added to **R** as a new primitive, yielding $\mathbf{R}^{\Box}$.

We take $\boldsymbol{t}$, and sometimes $\boldsymbol{f}$, as propositional constants. And also sometimes $\boldsymbol{T}$ and $\boldsymbol{F}$.

We waffle a bit as to which connectives and constants are present in which systems, partly as a result of the magnitude of the number of years over which this volume has been in the writing. Mostly it does not matter, except when what is at issue is some delicate question of conservative extension or the like—and on these occasions we make a point of trotting out such connectives as we have in our pocket. As a general rule, however, we think of things as being like this:

E	$\to, \&, \vee, \sim$
R	$\to, \&, \vee, \sim, \circ, \boldsymbol{t}$
T	$\to, \&, \vee, \sim$
RM	$\to, \&, \vee, \sim$
EM	$\to, \&, \vee, \sim$

All the connectives except negation are thought of as *positive*, and are present in the positive fragment $\mathbf{S}_+$ of each system **S**. Whenever necessity is added, we shall record this explicitly—though $\mathbf{R}^\Box$ is the only case we much discuss.

Sometimes we may want a name for the set of formulas. In those circumstances we shall use "SL," thinking of it as having $\to$, &, $\sim$, and $\vee$ always, and $\circ$ and $\boldsymbol{t}$ depending on context. Then we shall use superscripts and subscripts to indicate additions to and deletions from the vocabulary of a system, as described in the Preface to Volume I.

§R2. Axiomatic review. For purposes of reference we here lay out the chief systems of propositional logic of Volume I. This resurvey draws on §21.1 and (principally) §27.1.

First the axioms (some with multiple names) on which we draw:

R1		A1	$A\to A$
	E1	A2	$A\to A\to B \to B$
R2	E2		$A\to B \to. B\to C\to .A\to C$
R3			$A \to. A\to B\to B$
		A3	$A\to B \to. C\to A\to .C\to B$
R4	E3	A4	$(A\to .A\to B) \to. A\to B$
R5	E4	A5	$A\&B \to A$
R6	E5	A6	$A\&B \to B$
R7	E6	A7	$(A\to B)\&(A\to C) \to. A\to .B\&C$
R8	E8	A8	$A \to A\vee B$
R9	E9	A9	$B \to A\vee B$
R10	E10	A10	$(A\to C)\&(B\to C) \to. (A\vee B)\to C$
R11	E11	A11	$A\&(B\vee C) \to (A\&B)\vee C$
	E12	A12	$A\to \bar{A} \to \bar{A}$
R12	E13	A13	$A\to \bar{B} \to. B\to \bar{A}$
R13	E14	A14	$\bar{\bar{A}}\to A$
		A15	$A\to B \to. A\to B\to C\to C$
	E7	A16	$\Box A\&\Box B \to \Box(A\&B)$, where $\Box A =_{\mathrm{df}} A\to A\to A$
		A17	$A \to. A\to A\to A$
	EM0	A18	$A\to B\to. A\to B\to .A\to B$

RM0 $A \to . A \to A$

□1 $\Box A \to A$
□2 $\Box(A \to B) \to . \Box A \to \Box B$
□3 $\Box A \& \Box B \to \Box(A \& B)$
□4 $\Box A \to \Box\Box A$

$\boldsymbol{t}0$ $\boldsymbol{t} \to A \to A$

$f1$ $f \rightleftarrows {\sim}\boldsymbol{t}$
$f2$ $A \to f \to f \to A$

$\boldsymbol{T}1$ $A \to \boldsymbol{T}$
$\boldsymbol{F}1$ $\boldsymbol{F} \to A$

Then there are seven rules:

→E. From $A \to B$ and A to infer B.
&I. From A and B to infer $A \& B$.
□I. If A is a theorem, so is $\Box A$.
∘I. If $A \to . B \to C$ is a theorem, so is $A \circ B \to C$.
∘E. If $A \circ B \to C$ is a theorem, so is $A \to . B \to C$.
$\boldsymbol{t}$I. If A is a theorem, so is $\boldsymbol{t} \to A$.
$\boldsymbol{t}$E. If $\boldsymbol{t} \to A$ is a theorem, so is A.

The systems may now be defined as follows. (Numerous other formulations are to be found in Volume I.) All have the first two rules →E and &I.

T A1–A14
E E1–E14
R R1–R13, ∘I, ∘E, $\boldsymbol{t}$I, $\boldsymbol{t}$E
EM E1–E14, EM0
RM R1–R13, RM0

Any of the calculuses may also be conservatively extended to a formulation with ∘ governed by ∘I and ∘E, and $\boldsymbol{t}$ governed by $\boldsymbol{t}$I, $\boldsymbol{t}$E (in the case of the modal systems **E** and **EM**, $\boldsymbol{t}$0 is an additional explicit postulate). We also define

$\mathbf{R}^{\Box}$ **R** + □1–□4, □I

Each of these calculuses **S** has a positive fragment $\mathbf{S}_{+}$ obtained by dropping negation from the vocabulary and also dropping all its negation axioms.

§R3. Natural deduction review. This section does for the natural deduction systems of Volume I, as summarized in §27.2, what §R2 did for the axiomatic. We define F-formulations of each **E**, **R**, **T**, **RM**, and **EM**, as well as $\mathbf{R}^{\Box}$.

The rules as stated below hold for all systems, unless otherwise indicated.

Structural rules

Hyp. A step may be introduced as the hypothesis of a new subproof, and each new hypothesis receives a unit class $\{k\}$ of numerical subscripts, where k is the rank (§8.1) of the new subproof.

Rep. A_{a} may be repeated, retaining the relevance indices a.

Reit. A_{a} may be reiterated (retaining subscripts) into hypothetical subproofs in **FR** and **FRM** with no proviso, and in the others provided A has the form $\boldsymbol{t}$ or $B\to C$.

Intensional rules

$\to$I. From a proof of B_{a} on the hypothesis $A_{\{k\}}$ to infer $A\to B_{\mathrm{a}-\{k\}}$, provided k is in a.

$\to$E. From A_{a} and $A\to B_{\mathrm{b}}$ to infer $B_{\mathrm{a}\cup\mathrm{b}}$, where for FT, $\max(\mathrm{b})\leq\max(\mathrm{a})$.

Mixed rules

$\sim$I. From $A\to{\sim}A_{\mathrm{a}}$ to infer ${\sim}A_{\mathrm{a}}$.

$\sim$E. From ${\sim}B_{\mathrm{a}}$ and $A\to B_{\mathrm{b}}$ to infer ${\sim}A_{\mathrm{a}\cup\mathrm{b}}$, where for FT, $\max(\mathrm{b})\leq\max(\mathrm{a})$.

$\vee$E. From $A\vee B_{\mathrm{a}}$, $A\to C_{\mathrm{b}}$, and $B\to C_{\mathrm{b}}$, to infer $C_{\mathrm{a}\cup\mathrm{b}}$, where for FT $\max(\mathrm{b})\leq\max(\mathrm{a})$.

Extensional rules

$\sim\sim$I. From A_{a} to infer ${\sim}{\sim}A_{\mathrm{a}}$.
$\sim\sim$E. From ${\sim}{\sim}A_{\mathrm{a}}$ to infer A_{a}.
&I. From A_{a} and B_{a} to infer $A\&B_{\mathrm{a}}$.
&E. From $A\&B_{\mathrm{a}}$ to infer A_{a}. From $A\&B_{\mathrm{a}}$ to infer B_{a}.
$\vee$I. From A_{a} to infer $A\vee B_{\mathrm{a}}$. From B_{a} to infer $A\vee B_{\mathrm{a}}$.
&$\vee$. From $A\&(B\vee C)_{\mathrm{a}}$ to infer $(A\&B)\vee C_{\mathrm{a}}$.

Mingle rules

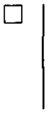
From A_{a} and A_{b} to infer $A_{\mathrm{a}\cup\mathrm{b}}$ (for **FRM** and **FEM** only; for **FEM** it is required that A have the form $B\to C$).

We add natural deduction rules, not to be found in Volume I, for $\boldsymbol{t}$ and $\circ$.

$\boldsymbol{t}$. From $\boldsymbol{t}\to A_{\varnothing}$ to infer $A_{\varnothing}$, and conversely.
$\circ$I. From A_{a} to infer $B\to.A\circ B_{\mathrm{a}}$.
$\circ$E. From $A\to.B\to C_{\varnothing}$ and $A\circ B_{\mathrm{a}}$ to infer C_{a}.

These fusion and $\boldsymbol{t}$ rules are ugly because (a) we wanted to give them in forms that would work for any system, and (b) the particular form of natural deduction we have been using is not well suited to fusion or to ***t***. We note that the reader will be able to find more satisfying rules in the context of **R**.

In order to be able to add rules for □, so as to be able to get $\mathbf{R}^{\Box}$, we follow Fitch 1952 in defining a new kind of subproof: a *strict categorical* subproof has no hypotheses and is marked with a □:



Then reiteration into strict categorical subproofs is limited to formulas $\Box A$ and ***t***. The rules for □:

□I. From a strict categorical subproof with last item A_a to infer $\Box A_a$.
□E. From $\Box A_a$ to infer A_a.

For each system FS, its positive fragment FS_+ results from dropping negation from the vocabulary and dropping all negation rules.

ENTAILMENT

CHAPTER VI

THE THEORY OF ENTAILMENT

§30. **Propositional quantifiers.** Except for a brief forward reference in §21.2.2, we have made no formal or explicit use or mention of quantifiers, propositional, individual, or other. We have followed standard, somewhat inexact mathematical practice in using free variables in displayed formulas to indicate generality, or universal quantification (using mainly metalinguistic variables ranging over formulas, with occasional other *ad hoc* devices when it was, or was deemed, necessary to talk about propositional variables). In the future we will be using quantifiers explicitly, and we start by airing a few prejudices about the topic, and straightening out notation.

§30.1. **Motivation.** This section may be skipped by any readers who have no worries about propositional quantifiers: it says nothing special about entailment, and it is not needed for reading the rest of the book. We include it for those readers who think that employment of propositional quantifiers requires a special defense.

Propositional quantifiers have been neglected in the classical literature in the interest of exploiting individual quantifiers, which come second for us, both in this book and *in rerum natura.* We don't have terribly fierce feelings about logical or metaphysical priority when it comes to quantifiers, but to the extent that one takes the "logic of unanalyzed propositions" as somehow more fundamental, or something, than (say) the subject-predicate analysis of English sentences or the Fregean analysis provided by first-order quantification theory—to that extent we think propositional quantifiers precede others. Having started out with propositional or sentential connectives and with free variables which *must* have the generality interpretation, it seems to us reasonable to begin next to make this interpretation explicit, before going on to finer analysis of the propositions themselves.

Although Church 1962 may not agree with us in finding some vague sense of "priority" concerning propositional quantification, *we* certainly agree with *his* characterization of the natural and obvious character of propositional quantification, and we quote his (as it seems to us) compelling considerations at some length (though of course we disagree with the account at the end of his second sentence of how analytic truths arise—which disagreement is irrelevant to the point under consideration):

> That logic does not therefore consist merely in a metatheory of some object language arises in the following way. It is found that ordinary theories, and perhaps any satisfactory theory, of deductive reasoning in the form of a metatheory will lead to analytic sentences in the object language, i.e., to sentences which, on the theory in question, are consequences of any arbitrary set of hypotheses, or it may be, of any arbitrary nonempty set of hypotheses. These analytic sentences lead in turn to certain generalizations; e.g., the infinitely many analytic sentences $A \vee \sim A$, where A ranges over all sentences of the object language, lead to the generalization $p \vee \sim p$, or more explicitly $(p)(p \vee \sim p)$; and in similar fashion $(F)(y)[(x)F(x) \supset F(y)]$ may arise by generalization from infinitely many analytic sentences of the appropriate form. These generalizations are common to many object languages on the basis of what is seen to be in some sense the same theory of deductive reasoning for the different languages. Hence they are considered to belong to logic, as not only is natural but has long been the standard terminology.
>
> Against the suggestion, which is sometimes made from a nominalistic motivation, to avoid or omit these generalizations, it must be said that to have, e.g., all of the special cases $A \vee \sim A$ and yet not allow the general law $(p).p \vee \sim p$ seems to be contrary to the spirit of generality in mathematics, which I would extend to logic as the most fundamental branch of mathematics. Indeed such a situation would be much as if one had in arithmetic $2+3 = 3+2$, $4+5 = 5+4$, and all other particular cases of the commutative law of addition, yet refused to accept or formulate a general law, $(x)(y).x+y = y+x$ (pp. 181–182).

The parallel seems to us clear and obvious; yet it is strikingly noticeable that very little attention has been paid to propositional quantification in the literature. (Happily, this neglect has not been so marked in recent years as previously, and so we note with gratification that the discussion below, drafted some time back, might well take a less stern tone today. That right and justice will triumph in the end, as they always do, may be anticipated, for example, from the good work of Bull 1969, Fine 1970, Grover 1972, 1972a, 1973, and Gabbay 1972a. We note moreover, to anticipate remarks immediately to be made, that it is exactly as the theory of propositional quantifiers has become nontrivial—for instance, in modal and intuitionist contexts—that it has begun to assume its proper role in the literature. Nevertheless, lest wrong and injustice make a comeback, we shall retain our stern tone.)

There are probably several reasons, historically, for the neglect of the topic that Russell 1906a calls "the theory of propositions" and that Church 1956 calls "the extended propositional calculus" (see Church 1956, §28, who gives a history of the matter, including references to the intimations of Russell 1903 and to the work of Łukasiewicz and Tarski 1930); no doubt the principal reason is that the theory is not very interesting. We explain briefly why this is so.

If, to some standard formulation of the two-valued calculus with "material *modus ponens*" as sole rule, we add propositional quantifiers, together with a complete set of axioms and rules for propositional quantification, we can always find a formula $\boldsymbol{F}$ that has "falsehood" as its *Bedeutung*. This could be done in a variety of ways, e.g., by taking $\boldsymbol{F}$ as primitive, with the axiom $\boldsymbol{F} \supset A$, or else by treating $\boldsymbol{F}$ as short for $p \& {\sim} p$, for some propositional variable p, or else by taking $\boldsymbol{F}$ as short for $\forall pp$, or the like. No matter how this is done, we get the result that $\boldsymbol{F}$ is provably materially "equivalent" to, and hence intersubstitutable with, *any* contradictory truth function; and letting $\boldsymbol{T}$ be (say) ${\sim}\boldsymbol{F}$, we have dually that $\boldsymbol{T}$ is intersubstitutable with any tautological truth function.

Standard treatments then produce the following theorems:

$$\forall pA(p) \equiv .\ A(\boldsymbol{T}) \& A(\boldsymbol{F})$$

and

$$\exists pA(p) \equiv .\ A(\boldsymbol{T}) \vee A(\boldsymbol{F})$$

(where $A(p)$ is some context about p and $A(\boldsymbol{T})$ is the result of putting $\boldsymbol{T}$ for p therein, etc.; we use this notation informally for a short space before introducing it formally). These have the force of guaranteeing that all the work done by propositional quantifiers can also be done by finite conjunctions and disjunctions—indeed, by *short* ones—so nothing much is gained except some new notation of doubtful value (for the purely truth-functional case).

This lack of novelty is accompanied by some embarrassment in trying to render propositional quantifiers in English or other natural languages. Relative (and other) pronouns provide handy locutions for reading individual quantifiers in English. Thus $\exists xFx$ can come out "There is something that has the property ef." Or, for short, "Something is ef." We used to think our mother tongue deficient in providing similar locutions for reading propositional quantifiers. This, however, was being needlessly uncharitable. We were set straight by Grover 1972, which lays the groundwork for a more unified treatment of propositional and individual quantifiers. In this treatment, further developed in Grover, Camp, and Belnap 1975, it turns out that English does have *prosentences* analogous to pronouns, though these prosentences are not always so readily available or so easy to find. For further details, see the cited papers.

We shall not worry too much here, however, about the precise degree to which English gracefully adapts itself to propositional quantifiers. Truth-functionally, $\exists qq$ comes to $\boldsymbol{T} \vee \boldsymbol{F}$, and if we can bring ourselves to utter "tee or ef," we ought to be able to say "there's a queue such that queue" without feeling too red-faced. And if, as in the interesting non-truth-functional cases we treat in this chapter, there are perhaps infinitely many propositions in our domain, then we can think of $\exists qq$ as the disjunction of all of them.

So, to the degree that English is thought recalcitrant on this point, we shall treat it as no serious obstacle. For, as Richard Montague put it, near enough, in addressing the American Philosophical Association, why should we be bound to a language that is becoming obsolescent? So we render propositional quantifiers into our mother tongue as best we can, treating the reader with equal charity when he says, "There's an ekks such that ef of ekks, and, moreover, if it gees it aitches."

Somewhat more serious than either the notational redundancy or the translational inconvenience, is a third feature of the situation which is forced on us by the intended interpretation of the whole calculus. Natural and obvious axioms and rules for quantifiers lead, as mentioned above, to

$$A(\boldsymbol{T})\&A(\boldsymbol{F})\supset\forall pA(p),$$

which stale custom will probably cause us to read "if both tee and eff satisfy a condition, then all propositions do." Even discounting the blunder in reading the horseshoe, this formula has the effect of saying that $\boldsymbol{T}$ and $\boldsymbol{F}$ are the only propositions recognized by the classical theory. And this fact reinforces what we all knew, but tried to conceal, all along, namely, that the "propositional calculus" is not a calculus of propositions *at all.* As a calculus it is faultless, but havering about its interpretation—a maundering induced in part by the mind-muddling misnomer "propositional calculus"—has created a host of bogus philosophical problems, some of which we hope to dispel in this chapter. We hereby make a firm resolve to remember in the future to call the calculus by its right name, "the (two-valued) truth value calculus" **TV**, and refer to the result $\mathbf{TV}^{\forall\exists p}$ of adding truth value quantifiers as "the extended truth value calculus," or "the theory [as opposed to the (free-variable) calculus] of truth values." How well we can adhere to our resolution remains for us and the reader to see. Old habits die hard, and the difficulty of unlearning a well-entrenched and habitual error may be insurmountable, or virtually so, especially since we want promptly to resuscitate the terms "propositional calculus," "extended propositional calculus," and "theory of propositions" for systems like **R**, **E**, **T**, $\mathbf{R}^{\forall\exists p}$, $\mathbf{E}^{\forall\exists p}$, and $\mathbf{T}^{\forall\exists p}$, where the free variables and variables of quantification may range over propositions.

In this connection we mention one final motivation for the usual neglect of truth value quantifiers. Under the misapprehension that truth value quantifiers range over propositions, these bits of notation have been viewed with apprehension by those who have been touted off propositions by neo-nominalists. (We apologize if it appears that our preoccupation with this issue amounts to a disease, but it must be remembered that in 1947 Goodman and Quine announced, eloquently and influentially, that the subject we are discussing had been abolished. It is disquieting to be told that righteousness demands allegiance to truth values alone and that propositions are only for the unregenerate who hanker after spooks.) But, in view of the equivalences mentioned earlier, it would seem that truth value quantifiers should be acceptable even to the most abstemious. We shall consider truth value quantifiers later, but we treat full-blooded propositional quantifiers first.

§30.2. Notation. Here we lay out our notation for propositional quantification.

1. *Propositional variables.* We assume that we have a collection of propositional (sometimes *sentential*) variables, at most denumerable, and that an alphabetical order is imposed on them. We use p, q, r, s (perhaps subscripted) as ranging over them.

In some contexts it is convenient to assume that the variables are divided into two *disjoint* series. The principal reason for this is narrowly technical: it is easy to become confused when faced with the necessity of instantiating with a variable in a context in which, in spite of oneself, it can get grabbed by a quantifier (see below). In such cases, we call *parameters* those variables which never get tied to any quantifier. (They are also called "variables of instantiation," or, following Russell, "real variables.") Parameters, we note, are convenient for calculational purposes, but rarely play any semantic role. The other variables, those which can get tied to a quantifier, we call (when we are making the distinction) simply *variables.* (They are also called "variables of quantification," or, following Russell, "apparent variables.") Sometimes in proof theory it is convenient to insist that such variables are always bound to a quantifier, but such an insistence tends to get in the way of semantic discussions.

2. *Quantifiers.* We use $\forall$ and $\exists$, respectively, for universal and existential quantification. In contexts where the parameter-variable distinction is live, they can bind only variables, never parameters.

3. *Formulas* are built from propositional variables by connectives, as usual; and, if A is a formula, so are $\forall pA$ and $\exists pA$. As before, A, B, etc. range over formulas, but we now add to this stock the metalinguistic expressions $A(p)$, $A(q)$, $B(r)$, etc., and, more rarely, $B(p, q)$, $C(q, r, s)$, etc., which are also to be construed, like A, B, etc., as metavariables ranging over formulas (in which the indicated propositional variables may or may not occur).

Our next topic is the rather tedious one of characterizing the *proper substitution* of a formula B for p in $A(p)$, the result of this substitution being (notationally) the formula $A(B)$. The logically mature reader (and the reader who just doesn't care) may take it that he knows well enough what the notation is supposed to mean: put B in for *all* free occurrences of p, and don't confuse any bound variables. But we have a definite policy as to how we are going to build in the "don't confuse any bound variables" clause in the characterization of proper substitution, and we are determined to state that policy (and to use it officially below, whenever we are forced to return to this depressing topic—which will be as rarely as is consistent with avoiding actual technical mistakes). At any rate, the confident (or indifferent) reader may move on immediately to §31, referring back to our exact explanations below when necessary to supplement what he knowns about proper substitution of a term t for a variable x in $A(x)$ in the analogous classical first-order case, the result of this substitution being $A(t)$. "Proper substitution" as used above is in the sense analogous to that of Kalish and Montague 1964; but the same notion, down to matters of minor technical import and terminological practice, is that which the reader will have learned from any good text (e.g., Kleene 1952, Curry 1963) applied here—as in Church 1956—to propositional quantification.

One case is simple; if B contains no variables, but only parameters, then $A(B)$ is just the result of replacing all *free* occurrences of p with B in $A(p)$. Otherwise, we need a policy such that no undesirable confusion of free and bound variables results, as it might, for example, if we were to put $\exists r(r\rightarrow q)$ for p in $\forall q(p\vee q)$, where the resulting formula $\forall q(\exists r(r\rightarrow q)\vee q)$ has all occurrences of q bound, though q was free in $\exists r(r\rightarrow q)$. Difficulty arises only when the substituend B contains a free occurrence of a variable q, and $A(p)$ contains a part of the form $\forall qC$ in which p occurs free; in this case any free occurrence of q in B will become bound on substitution, a procedure which may lead us from a formula that is valid (let $A(p)$ be $\exists q(q\rightleftarrows p)$) to one that is not (since putting $\sim q$ for p gives $\exists q(q\rightleftarrows\sim q)$).

We therefore say that $A(p)$ *is ready for substitution of B for p* iff $A(p)$ has no part $\forall qC$ such that p is free in $\forall qC$ and q is free in B. (Kleene 1952, §18, would say "B is *free for p in $A(p)$*.") We then fix our notation so that, if $A(p)$ is not ready, we *first get it* ready, and *then* substitute. We accomplish this as follows (adapting a device of Curry and Feys 1958 and Kripke 1959a).

Suppose $A(p)$ is not ready for substitution of B for p; let $\forall qC$ be the leftmost troublesome part of $A(p)$. Then replace every occurrence of q in $\forall qC$ by the first variable in alphabetic order which does not occur in either $A(p)$ or B (let this variable be r), getting $\forall rC'$; then replace $\forall qC$ in $A(p)$ by $\forall rC'$, getting $A(p)_1$. If $A(p)_1$ is still not ready, we repeat the procedure, until we arrive at a formula $A(p)_n$ such that $A(p)_n$ is ready for substitution of B for p. Then conventions for $A(B)$ are as follows:

(i) If p does not occur free in $A(p)$, $A(B)$ is $A(p)$;

(ii) if p occurs free in $A(p)$, and $A(p)$ is ready for substitution of B for p, then $A(B)$ is the result of replacing all free occurrences of p in $A(p)$ by B; and

(iii) If p occurs free in $A(p)$, and $A(p)$ is not ready for substitution of B for p, then $A(B)$ is the result of replacing all free occurences of p in $A(p)_n$ by B, where $A(p)_n$ is as above.

It may appear that we are smuggling a rule for alphabetic change of bound variables into our notation. With respect to what we *intend*, we have to plead guilty. Where C' results from C by *rewriting* the bound variable p as the bound variable q (in the sense of mere alphabetic change), we take C and C' to be mere *notational* variants, differing not at all in logical content. So we always *anticipate* having admissible rules that permit alphabetic change (in the usual sense) without restriction, and we care not a whit if our notational conventions facilitate tricky proofs of the admissibility of a rule which is always wanted in the systems considered in this book. However, it is to be noted that we have not *yet* smuggled in alphabetic change, even in part, and cannot do so until we have stated some axioms and rules. Thus far, it is just a *notational convention* that, if $A(p)$ is $\forall q(p \vee q)$ and B is $\exists r(r \rightarrow q)$, then, where s is the next variable in alphabetic order, $A(B)$ is $\forall s(\exists r(r \rightarrow q) \vee s)$. That is, we are merely, at this point, finding notation to avoid repeated statement of the condition that $A(p)$ either *be* ready or *be gotten* ready before we make substitutions. So, although perhaps we have made it easier, the rule for alphabetic change of bound variables will still have to be derived if it is to be available.

§31. Natural deduction: $\mathbf{FE}^{\forall\exists p}$.

The *theory of entailment* is that *theory* $\mathbf{E}^{\forall\exists p}$ obtained from the (free variable) propositional *calculus* **E** of entailment upon addition of propositional quantifiers. We will also discuss the analogous quantificational extensions of **R**, **T**, **RM**, and **EM**. All the differences in these systems come from their implication fragments, so we will be able to treat their truth-functional and propositional quantificational aspects wholesale. We will also consider various formulations of these systems, and we begin by describing the formulation that is most useful for carrying out proofs: the Fitch-style natural deduction formulation $\mathbf{FE}^{\forall\exists p}$.

Before embarking on this project, however, we interject a remark about the intended interpretation of the quantifiers. In order to motivate discussion thereof, we shall from time to time rely on the familiar analogy with individual quantifiers, according to which $\forall xF(x)$ is thought of as a (possibly infinite) conjunction, and $\exists xFx$ as a (possibly infinite) disjunction. That is, we think of $\forall pA(p)$ as $A(p_1)\&A(p_2)\& \ldots \&A(p_i)\& \ldots$, where the p_i run through all the (possibly infinitely many) propositions recognized by the theory in question; and similarly for $\exists pA(p)$. We are going to use this interpretation in these sections only to provide heuristic guidelines in explaining why entailments hold or fail, particularly when infinite cases would, according to reasonable expectations, be simply generalizations of finite cases. Nevertheless, it is perhaps worth pointing out that we do in fact take the interpretation seriously. What makes this easy is having available propositions as well as sentences. For although talk of an infinite conjunction of sentences, in a sense in which the conjuncts are recoverable parts of the conjunction, is with respect to any usual language sheer nonsense (although logicians have in fact devised mathematical creations which can be described in such terms), still there is nothing very peculiar in speaking of infinite conjunctions of propositions. Reason: although the concept of sentential conjunction suggests that the conjuncts be literal "parts," so that infinity becomes unbearable, with propositions we need have no such suggestion. Propositions have no parts; so we are *naturally* led to another understanding of conjunction—and one which is *naturally* indifferent as to how many propositions are being conjoined. We think of propositions as being ordered by an implication or entailment relation, and, given a set of propositions, we take their conjunction to be the greatest lower bound of the set. Which is to say, the conjunction of a set of propositions is the logically weakest propositions that is strong enough to entail each member of the set. And this makes sense regardless of whether or not the set is finite.

With this understanding, and without being fussy, we would then say that $A\&B$ expresses the propositional conjunction of the set of propositions expressed by A and by B, while $\forall xF(x)$ expresses the propositional conjunction of the set of propositions expressed by $F(x)$ as the value of x runs through the domain of individuals, and $\forall pA(p)$ expresses the propositional conjunction of the set of propositions expressed by $A(p)$ as p runs through the appropriate domain of propositions.

But, to repeat, we shall be using the view that universal quantification is a generalization of conjunction only heuristically. Indeed, we learn in §§52–53 below that there is another and more sophisticated view which appears even more coherent with the spirit of relevance logic.

We assume all the apparatus of §23 for FE (summarized in §R3), and, as before, we wish to have an introduction and an elimination rule for the universal and for the existential quantifier (both of which we take as primitive, so as to be able more readily to consider the positive part of $\mathrm{FE}^{\forall\exists p}$ subsequently).

§31.1. Universal quantification. The elimination rule for the universal quantifier, $\forall$E, is obvious: from $\forall pA(p)_{\mathrm{a}}$ to infer $A(B)_{\mathrm{a}}$, for arbitrary B.

For $\forall$I it is most convenient to introduce a new style of subproof which we call, following Fitch, "general categorical." *General categorical* subproofs are distinguished from the hypothetical subproofs we have used earlier in three ways:

(a) They have no hypotheses. Under one restriction (to be stated below), any step may be reiterated into such proofs. Such proofs may, by way of reiteration, use hypotheses of hypothetical proofs to which they are subordinate, but they have no hypotheses of their own.

(b) They are flagged with a variable, thus:

$$\begin{array}{l|l}
p & \\
 & \vdots \\
 &
\end{array}$$

The purpose of the flagging variable is, intuitively, to indicate that the proof is *general with respect to* the flagged variable, in the sense that nothing inside the proof depends on the choice of p. This condition would be violated unless we restricted reiteration in the following way:

(c) A_a may be reiterated into a categorical proof that is general with respect to p only if p fails to occur free in A.

This condition guarantees that, as it is common to say, p is an arbitrary specimen; nothing previously established about p can be used in the subproof anyhow, so it might as well be anybody. (Note: we *may* reiterate into such proofs formulas containing *bound* occurrences of the flagging variable.)

We can now state the rule ∀I for introduction of the universal quantifier: from a categorical subproof which is general with respect to p, having $A(p)_a$ as its final step, to infer $\forall pA(p)_a$.

We give some examples.

$$\begin{array}{lll}
1 & \vert\underline{\ \ }\ \forall p\forall qA(p,q)_{\{1\}} & \text{hyp} \\
2 & \vert\quad q\ \vert\ \forall p\forall qA(p,q)_{\{1\}} & \text{1 reit} \\
3 & \vert\quad \ \vert\ p\ \vert\ \forall p\forall qA(p,q)_{\{1\}} & \text{2 reit} \\
4 & \vert\quad \ \vert\ \ \vert\ \forall qA(p,q)_{\{1\}} & 3\ \forall\text{E} \\
5 & \vert\quad \ \vert\ \ \vert\ A(p,q)_{\{1\}} & 4\ \forall\text{E} \\
6 & \vert\quad \ \vert\ \forall pA(p,q)_{\{1\}} & 3\text{–}5\ \forall\text{I} \\
7 & \vert\ \forall q\forall pA(p,q)_{\{1\}} & 2\text{–}6\ \forall\text{I} \\
8 & \forall p\forall qA(p,q)\to\forall q\forall pA(p,q) & 1\text{–}7\ {\to}\text{I}
\end{array}$$

That is, universal quantifiers permute. They also distribute over conjunction.

$$\begin{array}{lll}
1 & \vert\underline{\ \ }\ \forall pA(p)\ \&\ \forall qB(q)_{\{1\}} & \text{hyp} \\
2 & \vert\quad r\ \vert\ \forall pA(p)\ \&\ \forall qB(q)_{\{1\}} & \text{1 reit} \\
3 & \vert\quad \ \vert\ \forall pA(p)_{\{1\}} & 2\ \&\text{E} \\
4 & \vert\quad \ \vert\ A(r)_{\{1\}} & 3\ \forall\text{E} \\
5 & \vert\quad \ \vert\ \forall qB(q)_{\{1\}} & 2\ \&\text{E} \\
6 & \vert\quad \ \vert\ B(r)_{\{1\}} & 5\ \forall\text{E} \\
7 & \vert\quad \ \vert\ A(r)\ \&\ B(r)_{\{1\}} & 4\ 6\ \&\text{I} \\
8 & \vert\ \forall r(A(r)\ \&\ B(r))_{\{1\}} & 2\text{–}7\ \forall\text{I} \\
9 & (\forall pA(p)\&\forall qB(q))\to\forall r(A(r)\&B(r)) & 1\text{–}8\ {\to}\text{I}
\end{array}$$

Proof of the converse will be left to the reader.

Although our restrictions on reiteration are exactly what one would expect from Fitch 1952, for the case when both modality and quantifiers are present, as they are (e.g.) in $\mathbf{FE}^{\forall p}$, some slight care is needed to assure that quantificational and modal restrictions are honored together. (This problem does not arise for $\mathbf{FR}^{\forall p}$, since in this (and stronger) systems, modal restrictions are inoperative and only the quantifier restriction counts.) Where modality is present, however, it is important to recall that, legally, reiterations are permitted only from a subproof into the *next innermost* subproof. We relax this policy, *as a shortcut*, at the end of §31.3 below, allowing in practice (for the modal systems) the reiteration of $A\rightarrow B$, properly subscripted, from a given subproof into any subproof nested within it. Our new reiteration rule, however, allows us to reiterate *anything* into a (flagged) general categorical subproof, whether it has the form of an entailment or not. While the reader has by now, we trust, developed the correct intuitions, note that the following is an *invalid* derivation in $\mathbf{FE}^{\forall p}$:

$$\begin{array}{lllllll}
1 & \vert\underline{\ p_{\{1\}}\ } & & & & & \text{hyp} \\
2 & \vert\ q & \vert & \vert\underline{\ p\rightarrow q_{\{2\}}\ } & & & \text{hyp} \\
3 & \vert & \vert & \vert\ p_{\{1\}} & & & \text{1 reit ???} \\
4 & \vert & \vert & \vert\ q_{\{1,2\}} & & & 3\text{–}4 \rightarrow\text{E} \\
5 & \vert & \vert & p\rightarrow q\rightarrow q_{\{1\}} & & & 2\text{–}4 \rightarrow\text{I} \\
6 & \vert & \forall q(p\rightarrow q\rightarrow q)_{\{1\}} & & & & 2\text{–}5\ \forall\text{I} \\
7 & \vert & p\rightarrow q\rightarrow q_{\{1\}} & & & & 6\ \forall\text{E} \\
8 & p\rightarrow .p\rightarrow q\rightarrow q & & & & & 7 \rightarrow\text{I}
\end{array}$$

Since 8 is an instance of the law of assertion R3 (§27.1.1), this particular formula being crucially invalid in **E** as a fallacy of modality, the reader is (we trust) already prepared to believe that there is something wrong with it, as a derivation in $\mathbf{FE}^{\forall p}$ (there is, of course, nothing wrong with the above as a derivation in $\mathbf{FR}^{\forall p}$, save that it is a little long-winded). The difficulty, as indicated, is in step 3. For note that step 2 simultaneously introduces *two* new subproofs, an outer general categorical one flagged by q, and an inner hypothetical one for which $p\rightarrow q$ is assumed on hypothesis. For our derivation to be strictly legal, the reiterated p of step 1 must *first* be reiterated into the subproof flagged by q (giving us a new step 1a, which is legal because q is not free in p) and *then* reiterated into the hypothetical subproof begun with $p\rightarrow q$. This step is illegal, since p, not being necessitive in the sense of §5.2, is no more a candidate for reiteration into *hypothetical* subproofs than ever, in systems based on **E** (or **T** or **EM**).

That our old subproofs, as opposed to our new ones, are hypothetical has little to do with the point just made. Rather, we should think of *each* of our subproofs as equipped with a *barrier*, signified by the vertical line, in the fine terminology of Thomason 1970. We may imagine each barrier equipped with a guard, who examines the credentials of a formula A for admission to that particular subproof. Before, in FE, all guards got the same instructions; let A through, by reiteration, only if it is of the form $B \rightarrow C$, and otherwise kick A back into the subproof that it came from. (In **FR**, being a guard was an absolute sinecure; any old formula that showed up for admission through the barrier was permitted to pass.) But now there are infinitely many kinds of guards; there are p-guards and q-guards and r-guards and so forth, each instructed to knock back any formula in which its particular variable occurs free, and to admit all others. Hypothetical subproofs are then guarded as before, with a qualification immediately to be made; i.e., in the weaker systems there is a necessitive guard on hypothetical subproofs, who is sacked in $\mathbf{FR}^{\forall p}$, though he would remain on duty in an $\mathbf{FR}^{\Box\forall p}$, guarding in this case not hypothetical subproofs but strict categorical ones in the sense of the introductory summary review section §R3. We might, incidentally, assimilate the FE guard more strictly to the $\mathbf{FR}^{\Box}$ guard, by thinking of all hypothetical FE subproofs as immediately embedded within strict categorical subproofs in the $\mathbf{R}^{\Box}$-motivated way. Indeed, it was exactly this kind of thought, applied to Fitch 1952, which produced the reiteration restrictions on FE from §4 on.

Now we may restore our old shortcut method. Reiterate a formula A directly through as many subproofs as you please, nested inwardly, so long as A is not stopped by any guard on its inward trip. Again, this is just a shortcut, and not a narrowly legal statement of the reiteration rule. But our practice will be to take this shortcut. Note that it would still eliminate the fallacious step 3 above in $\mathrm{FE}^{\forall p}$, since the shortcut requires that a formula get past *all* guards.

Addition of universal quantifiers requires one more adjustment in the rule of reiteration in order to carry out proofs of formulas we would reasonably expect to have as theorems. It will be recalled from the discussion of E7 in §23.4 that FE permits reiteration not only of entailments (as a condition in the statement of the primitive rule of reiteration), but also of conjunctions of entailments (derivatively, by first using &E, then reiterating the results, then using &I). For obvious reasons, this effect cannot be got in a similar way in $\mathrm{FE}^{\forall\exists p}$ for universally quantified entailments $\forall p_1 \forall p_2 \ldots \forall p_n(A \rightarrow B)$ (which we understand as possibly infinite conjunctions of entailments). We therefore relax the rule of reiteration to allow reiteration of such formulas as well as those of the form $A \rightarrow B$, in $\mathrm{FE}^{\forall\exists p}$.

Let us note, however, that the adjustment just made is substantive and not merely symbol-shifting in character; it will produce, in our Fitch-style system $\mathrm{FE}^{\forall\exists x}$ what has come in the literature to be known as the *Barcan formula*, $\forall x \Box Fx \rightarrow \Box \forall x Fx$ (which was set out in Barcan Marcus 1946; around it passions still rage). We note here merely that, in other modal logics, the

Barcan formula is wanted as a theorem only from certain semantical perspectives, as Prior 1955 and (especially) Kripke 1963a may be taken to have pointed out. Although we favor those perspectives which support the Barcan formula here (for one thing, they are easier to think through, in an area of relevance logic where much semantical work remains to be done), the topic is not yet closed.

The rule of reiteration then assumes two forms: (i) A_{a} may be reiterated into a hypothetical subproof if it has the form of an entailment preceded by zero or more universal quantifiers; and (ii) A_{a} may be reiterated into a general categorical subproof if it contains no free occurrence of the flagging variable. (We use "reit" to refer to both (i) and (ii), since the form of an application will always make it clear which is being used.)

This then allows us to prove, e.g.,

$$
\begin{array}{rllll}
1 & \mid\underline{\ \forall p(A(p)\to B(p))_{\{1\}}\ } & & & \text{hyp}\\
2 & \mid\ \ \mid\underline{\ \forall pA(p)_{\{2\}}\ } & & & \text{hyp}\\
3 & \mid\ \ \mid\ \ p & \mid\ \forall pA(p)_{\{2\}} & & 2\ \text{reit}\\
4 & \mid\ \ \mid & \mid\ A(p)_{\{2\}} & & 3\ \forall\text{E}\\
5 & \mid\ \ \mid & \mid\ \forall p(A(p)\to B(p))_{\{1\}} & & 1\ \text{reit (twice)}\\
6 & \mid\ \ \mid & \mid\ A(p)\to B(p)_{\{1\}} & & 5\ \forall\text{E}\\
7 & \mid\ \ \mid & \mid\ B(p)_{\{1,2\}} & & 4\ 6\ \to\text{E}\\
8 & \mid\ \ \mid & \forall pB(p)_{\{1,2\}} & & 3\text{–}7\ \forall\text{I}\\
9 & \mid\ \ \forall pA(p)\to\forall pB(p)_{\{1\}} & & & 2\text{–}8\ \to\text{I}\\
10 & \forall p(A(p)\to B(p))\ \to.\ \forall pA(p)\to\forall pB(p) & & & 1\text{–}9\ \to\text{I}
\end{array}
$$

§31.2. Existential quantification. Again one rule is obvious, the rule ∃I for the introduction of existential quantifiers: from $A(B)_{\mathrm{a}}$ to infer $\exists pA(p)_{\mathrm{a}}$.

We note here that, as for ∀E, the "context" $A(p)$ or $A(B)$, in which p or B occurs, might be void. That is, as a case of ∀E we have: from $\forall pp$ to infer B, and, for ∃I: from B to infer $\exists pp$.

The rule ∃E is modeled on the analogy of ∨E, which requires the three premisses $A\vee B_{\mathrm{a}}$, $A\to C_{\mathrm{b}}$, and $B\to C_{\mathrm{b}}$. The difference lies in the fact that $\exists pA(p)_{\mathrm{a}}$ might represent an infinite disjunction, corresponding to the finite disjunction $A\vee B_{\mathrm{a}}$. What then will correspond to what (in effect) is the finite conjunction $(A\to C)\&(B\to C)_{\mathrm{b}}$? Clearly, universal propositional quantification is needed to provide the appropriate (possibly infinite) analogue, which we take as $\forall p(A(p)\to C)$. Moreover, since C is fixed as a single formula in the finite conjunction, we won't have an analogue unless C is also fixed as a single formula in the infinite case; this means, formally, that p cannot occur free in C.

The rule ∃E then reads: from $\exists pA(p)_{\mathrm{a}}$ and $\forall p(A(p)\to B)_{\mathrm{b}}$ to infer $B_{\mathrm{a}\cup\mathrm{b}}$, provided p is not free in B.

Some examples:

1	$\vert\ \forall p(A(p) \to B(p))_{\{1\}}$	hyp
2	$\vert\ \vert\ \exists pA(p)_{\{2\}}$	hyp
3	$\vert\ \vert\ p\ \vert\ \vert\ A(p)_{\{3\}}$	hyp
4	$\vert\ \vert\ \ \ \vert\ \vert\ \forall p(A(p) \to B(p))_{\{1\}}$	1 reit (thrice)
5	$\vert\ \vert\ \ \ \vert\ \vert\ A(p) \to B(p)_{\{1\}}$	4 ∀E
6	$\vert\ \vert\ \ \ \vert\ \vert\ B(p)_{\{1,3\}}$	3 5 →E
7	$\vert\ \vert\ \ \ \vert\ \vert\ \exists pB(p)_{\{1,3\}}$	6 ∃I
8	$\vert\ \vert\ \ \ \vert\ A(p) \to \exists pB(p)_{\{1\}}$	3–7 →I
9	$\vert\ \vert\ \forall p(A(p) \to \exists pB(p))_{\{1\}}$	3–8 ∀I
10	$\vert\ \vert\ \exists pB(p)_{\{1,2\}}$	2 9 ∃E
11	$\vert\ \exists pA(p) \to \exists pB(p)_{\{1\}}$	2–10 →I
12	$\forall p(A(p)\to B(p)) \to .\ \exists pA(p)\to\exists pB(p)$	1–11 →I

1	$\vert\ \exists pA(p)_{\{1\}}$	hyp
2	$\vert\ p\ \vert\ \vert\ A(p)_{\{2\}}$	hyp
3	$\vert\ \ \ \vert\ \vert\ \vert\ \forall p\sim A(p)_{\{3\}}$	hyp
4	$\vert\ \ \ \vert\ \vert\ \vert\ \sim A(p)_{\{3\}}$	3 ∀E
5	$\vert\ \ \ \vert\ \vert\ \sim\forall p\sim A(p)_{\{2\}}$	2 3–4 contrap
6	$\vert\ \ \ \vert\ A(p) \to \sim\forall p\sim A(p)$	2–5 →I
7	$\vert\ \forall p(A(p) \to \sim\forall p\sim A(p))$	2–6 ∀I
8	$\vert\ \sim\forall p\sim A(p)_{\{1\}}$	1 7 ∃E
9	$\exists pA(p) \to \sim\forall p\sim A(p)$	1–8 →I

In the future, we will allow ourselves to omit steps like 6 and 7 immediately above (which may always be supplied in applications of ∃E), going directly from 5 to 8, with the citation "1 2–5 ∃E."

1	$\vert\ \sim\forall p\sim A(p)_{\{1\}}$	hyp
2	$\vert\ \vert\ \sim\exists pA(p)_{\{2\}}$	hyp
3	$\vert\ \vert\ p\ \vert\ \sim\exists pA(p)_{\{2\}}$	2 reit
4	$\vert\ \vert\ \ \ \vert\ \vert\ A(p)_{\{3\}}$	hyp
5	$\vert\ \vert\ \ \ \vert\ \vert\ \exists pA(p)_{\{3\}}$	4 ∃I
6	$\vert\ \vert\ \ \ \vert\ \sim A(p)_{\{2\}}$	3 4–5 contrap
7	$\vert\ \vert\ \forall p\sim A(p)_{\{2\}}$	3–6 ∀I
8	$\vert\ \exists pA(p)_{\{1\}}$	1 2–7 contrap
9	$\sim\forall p\sim A(p) \to \exists pA(p)$	1–8 →I

The last two theorems add up to

$$\exists pA(p) \rightleftarrows \sim\forall p\sim A(p),$$

which means that existential quantifiers may be taken as defined, provided we can prove the required rules ∃I and ∃E from the rules ∀I and ∀E. Proof of ∃I is trivial. We show that ∃E is also provable, in the sense that we may always get $B_{\mathrm{a}\cup\mathrm{b}}$ from $\sim\forall p\sim A(p)_{\mathrm{a}}$ and $\forall p(A(p)\to B)_{\mathrm{b}}$ (with p not free in B), using rules for the arrow, truth functions, and universal quantification alone.

$$\begin{array}{r|l l}
i & \sim\forall p\sim A(p)_{\mathrm{a}} & ? \\
i+1 & \forall p(A(p)\to B)_{\mathrm{b}} & ? \\
i+2 & \big|\underline{\;\sim B_{\{k\}}} & \text{hyp} \\
i+3 & \big|\;\; p \;\big|\; \forall p(A(p)\to B)_{\mathrm{b}} & i+1 \text{ reit (twice)} \\
i+4 & \big|\qquad\big|\; A(p)\to B_{\mathrm{b}} & i+3\ \forall\mathrm{E} \\
i+5 & \big|\qquad\big|\; \sim B_{\{k\}} & i+2 \text{ reit} \\
i+6 & \big|\qquad\big|\; \sim A(p)_{\mathrm{b}\cup\{k\}} & (i+4)\,(i+5) \text{ contrap} \\
i+7 & \big|\; \forall p\sim A(p)_{\mathrm{b}\cup\{k\}} & (i+3)\text{–}(i+6)\ \forall\mathrm{I} \\
i+8 & B_{\mathrm{a}\cup\mathrm{b}} & i(i+2)\text{–}(i+7) \text{ contrap}
\end{array}$$

(Notice that the reiteration at step $i+5$ is justified by the assumption that B does not contain p free.)

The foregoing shows that existential quantifiers may be defined in terms of universal quantifiers, just as disjunction may be defined in terms of conjunction. Or we may do things the other way about: the reader is invited to justify the usual definition of the universal quantifier in terms of the existential by proving

$$\forall pA(p) \rightleftarrows\ \sim\exists p\sim A(p)$$

and by deriving $\forall$I and $\forall$E.

Analogies with §§23.1–23.2 also extend to §23.3: we need, in addition to the four rules stated, two rules for distribution.

§31.3. Distribution of universality over disjunction. This rule is motivated by considerations which are obvious generalizations of those of §23.3. We state the rule $\forall\vee$ as follows: from $\forall p(A\vee B(p))_{\mathrm{a}}$ to infer $A\vee\forall pB(p)_{\mathrm{a}}$, provided p is not free in A. This rule enables us to exploit analogies with finite conjunctions, and also to prove the dual: where p is not free in A,

$$\begin{array}{r l l}
1 & \big|\underline{\;\forall p(\sim A\vee\sim B(p))_{\{1\}}} & \text{hyp} \\
2 & \big|\;\sim A\vee\forall p\sim B(p)_{\{1\}} & 1\ \forall\vee \\
3 & \forall p(\sim A\vee\sim B(p))\to.\ \sim A\vee\forall p\sim B(p) & 1\text{–}2 \to\mathrm{I}
\end{array}$$

Contraposition and De Morgan then yield (using the definition of existential quantification):

$$A\&\exists pB(p)\to\exists p(A\&B(p)) \qquad (p \text{ not free in A}).$$

We will also take the corresponding rule ($\&\exists$) as primitive (for isolation of the positive part of $\mathrm{FE}^{\forall\exists p}$). Here we note only that if existential quantifiers are defined then the rule is derivable from $\forall\vee$.

§31.4. Necessity. We promised at the end of §11 to "fix" the odd definition of necessity $\Box A$ as $A\to A\to A$, and now is the time to do it. By the argument of §45.1 it can be shown that, where $p_1, \ldots, p_n$ are all the variables in A, A is provable in $\mathbf{\Pi'}$ just in case $(\&_{i=1}^{n}(p_i\to p_i))\to A$ is also provable. In view of the theorem of §25, the same is true of **E**, and accordingly **FE**. We notice that $(\&_{i=1}^{n}(p_i\to p_i))\to A$ acts in a way like $\Box A$, since (where now $p_1, \ldots, p_n$ are all the variables in both A and B, and where A^* is $(\&_{i=1}^{n}(p_i\to p_i))\to A$ we have in **E**:

$\vdash A^*\to A$,
$\vdash (A\&B)^* \rightleftarrows. A^*\&B^*$,
$\vdash (A\to B)^* \to. A^*\to B^*$, and
$\vdash A$ iff $\vdash A^*$,

as is easily shown. For **E**, the size of $\&_{i=1}^{n}(p_i\to p_i)$ will vary from formula to formula, but in $\mathbf{FE}^{\forall\exists p}$ we can find one formula that will do the trick for *all* A, namely, $\forall p(p\to p)$, which has each of the finite conjunctions $\&_{i=1}^{n}(p_i\to p_i)$ as a consequence. We therefore now *redefine* $\Box$ in $\mathbf{FE}^{\forall\exists p}$,

$$\Box A =_{\mathrm{df}} \forall p(p\to p)\to A,$$

and prove it equivalent to the definition we offered, with a certain amount of dissatisfaction, ages ago (in §4.3).

$$\begin{array}{lll}
1 & \vert\ \underline{A\to A\to A_{\{1\}}} & \text{hyp}\\
2 & \vert\ \ \vert\ \underline{\forall p(p\to p)_{\{2\}}} & \text{hyp}\\
3 & \vert\ \ \vert\ \ A\to A_{\{2\}} & 2\ \forall\text{E}\\
4 & \vert\ \ \vert\ \ A_{\{1,2\}} & 1\ \text{reit}, 3\ \to\text{E}\\
5 & \vert\ \ \forall p(p\to p)\to A_{\{1\}} & 2\text{–}4\ \to\text{I}\\
6 & A\to A\to A\ \to.\ \forall p(p\to p)\to A & 1\text{–}5\ \to\text{I}
\end{array}$$

$$\begin{array}{lll}
1 & \vert\ \underline{\forall p(p\to p)\to A_{\{1\}}} & \text{hyp}\\
2 & \vert\ \ \vert\ \underline{A\to A_{\{2\}}} & \text{hyp}\\
3 & \vert\ \ \vert\ \ p\ \ \vert\ \ \vert\ \underline{p_{\{3\}}} & \text{hyp}\\
4 & \vert\ \ \vert\ \ \ \ \ \ \vert\ \ \vert\ \ p_{\{3\}} & 3\ \text{rep}\\
5 & \vert\ \ \vert\ \ \ \ \ \ \vert\ \ p\to p & 3\text{–}4\ \to\text{I}\\
6 & \vert\ \ \vert\ \ \forall p(p\to p) & 3\text{–}5\ \forall\text{I}\\
7 & \vert\ \ \vert\ \ A_{\{1\}} & 1\ \text{reit}, 6\ \to\text{E}\\
8 & \vert\ \ \vert\ \ A_{\{1,2\}} & 2\text{–}7\ \to\text{E}\\
9 & \vert\ \ A\to A\to A_{\{1\}} & 2\text{–}8\ \to\text{I}\\
10 & \forall p(p\to p)\to A\ \to.\ A\to A\to A & 1\text{–}9\ \to\text{I}
\end{array}$$

We moralize briefly.

In the first place, it is with surprise and delight that we find here resuscitated one of the hoariest dogmas of logic, that the rock-bottom cornerstone and foundation of all logical inference is the law of identity $\forall p(p\to p)$, and that all the truths of logic flow from this single fount. $\mathbf{FE}^{\forall\exists p}$ therefore lends unexpected support to a respectable tradition, and should alleviate otherwise painful monistic yearnings in logic, as well as strengthen our earlier plea for the recognition of the existence of pure nonnecessitive propositions.

Secondly, it is worth noting that the result requires explicit use of propositional quantifiers, which alone make legitimate our common use of free variables.

Finally, we recall that the new definition renders axiom E7 of **E** (see §R2, and discussion in §21.2.2) otiose: we now have $\Box A\&\Box B\rightarrow\Box(A\&B)$ as a special case of E6, and so the rule allowing reiteration of universally quantified entailments into hypothetical subproofs fits with necessity in a satisfyingly coherent way.

We note in passing that, as among purely truth-functional formulas A, the weaker condition $\forall p({\sim}p\vee p)$ also works like necessity, since (as can be shown without difficulty by considering Boolean conjunctive normal forms (§16.2)) defining

$$\Box' A =_{\mathrm{df}} \forall p({\sim}p\vee p)\rightarrow A$$

leads (for purely truth-functional A and B) to

$\vdash \Box' A\rightarrow A$,
$\vdash \Box' A\&\Box' B\rightarrow\Box'(A\&B)$,
$\vdash \Box'(A\rightarrow B)\rightarrow.\Box' A\rightarrow\Box' B$, and
$\vdash A$ iff $\vdash \Box' A$.

The notion $\Box'$ may have some interest if we are disinclined to send a man to do a boy's (extensional) work, but, in the presence of $\forall p(p\rightarrow p)$, the boy doesn't look very interesting.

We remark that $\forall p(p\rightarrow p)$ will serve as a definition of the constant $\boldsymbol{t}$ of §27.1.2; and of course its negate, or $\exists p{\sim}(p\rightarrow p)$, will do for $\boldsymbol{f}$. These are to be contrasted with $\boldsymbol{T}\rightleftarrows\exists pp$ and $\boldsymbol{F}\rightleftarrows\forall pp$, discussed in §33.

Conversely, if $\boldsymbol{t}$ is already present, subject to the natural deduction rules of the review section §R3 (including the $\mathbf{FE}^{\boldsymbol{t}}$ rule allowing the reiteration of $\boldsymbol{t}$ without restriction), note that the redefinition of $\Box A$ may take the form $\boldsymbol{t}\rightarrow A$, even before propositional quantifiers are introduced. (This was essentially Ackermann's idea, to be discussed in Chapter VIII, and it was a good one.)

§31.5. $\mathrm{FE}^{\forall\exists p}$ and its neighbors: Summary. The quantificational rules for Fitch-style systems of relevance logic with propositional quantifiers are gathered together here for reference. We consider adding to the rules of §R3 the following:

Structural rules

Reit (hypothetical). $\forall p_1\ldots\forall p_n A_{\mathrm{a}}(n\geq 0)$ may be reiterated into a hypothetical subproof (unrestricted for $\mathbf{FR}^{\forall\exists p}$ and $\mathbf{FRM}^{\forall\exists p}$; for $\mathbf{FE}^{\forall\exists p}$, $\mathbf{FEM}^{\forall\exists p}$, and $\mathbf{FT}^{\forall\exists p}$, A must have the form $B\rightarrow C$).

Reit (categorical). A_a may be reiterated into categorical subproofs general with respect to p, provided A_a does not contain p free.

Mixed rule

$\exists$E. From $\exists pA(p)_a$ and $\forall p(A(p)\rightarrow B)_b$ to infer $B_{a\cup b}$, provided p is not free in B (and where for $\mathbf{FT}^{\forall\exists p}$, $\max(b)\leq\max(a)$).

Extensional rules

$\forall$I. From a categorical subproof general with respect to p, having $A(p)_a$ as final step, to infer $\forall pA(p)_a$.

$\forall$E. From $\forall pA(p)_a$ to infer $A(B)_a$.

$\exists$I. From $A(B)_a$ to infer $\exists pA(p)_a$.

$\forall\vee$. From $\forall p(A\vee B(p))_a$ to infer $A\vee\forall pB(p)_a$, provided p is not free in A.

$\&\exists$. From $A\&\exists pB(p)_a$ to infer $\exists p(A\&B(p))_a$, provided p is not free in A.

Adding these rules to **FE**, **FR**, **FT**, **FEM**, and **FRM** as in §27.2 produces $\mathbf{FE}^{\forall\exists p}$, $\mathbf{FR}^{\forall\exists p}$, $\mathbf{FT}^{\forall\exists p}$, $\mathbf{FEM}^{\forall\exists p}$, and $\mathbf{FRM}^{\forall\exists p}$.

And adding them to a system $\mathbf{FS}_+$ produces the corresponding system $\mathbf{FS}_+^{\forall\exists p}$. Further, adding just the rules not involving $\exists$ to **FS** (or $\mathbf{FS}_+$) yields $\mathbf{FS}^{\forall p}$ (or $\mathbf{FS}_+^{\forall p}$). Lastly, adding all the rules to a system **FS** with negation, *but* taking $\forall$ as defined, gives $\mathbf{FS}^{\exists p}$. We do not consider and do not know anything about positive systems with only $\exists$. The difficulty is that the reiteration rule appears to depend upon the universal quantifier. (Of course this makes no difference for **FR**.)

§32. $\mathbf{E}^{\forall\exists p}$ and its neighbors: Summary and equivalence. We consider adding the following axioms to those for the systems **T**, **E**, **R**, **EM**, and **RM** of §R2:

PQ1	$\forall p(A\rightarrow B)\rightarrow.\ \forall pA\rightarrow\forall pB$	
PQ2	$\forall pA\&\forall pB\rightarrow\forall p(A\&B)$	
PQ3	$\forall pA(p)\rightarrow A(B)$	
PQ4	$\forall p(A\rightarrow B)\rightarrow.\ A\rightarrow\forall pB$	(p not free in A)
PQ5	$\forall p(A\vee B)\rightarrow.\ A\vee\forall pB$	(p not free in A)
PQ6	$\forall p(p\rightarrow p)\rightarrow A\rightarrow A$	
PQ7	$A(B)\rightarrow\exists pA(p)$	
PQ8	$\forall p(A\rightarrow B)\rightarrow.\ \exists pA\rightarrow B$	(p not free in B)
PQ9	$\exists pA\&B\rightarrow\exists p(A\&B)$	(p not free in B)

Axiom clause: If A is an axiom, so is $\forall pA$.
Rules: $\rightarrow$E and &I.

In terms of all this, we define the following cornucopia of systems. Let **S** be one of **T**, **E**, **R**, **EM**, or **RM**, as in §27.1.1 or §R2, and let $\mathbf{S}_+$ be the corresponding negation-free system. Then

$$\mathbf{S}^{\forall p} = \mathbf{S} + \mathrm{PQ1}\text{–}\mathrm{PQ5}$$

except for the modal systems $\mathbf{E}^{\forall p}$ and $\mathbf{EM}^{\forall p}$, for which PQ6 is an explicit axiom scheme as well. Similarly,

$$\mathbf{S}_{+}^{\forall p} = \mathbf{S}_{+} + \mathrm{PQ1}\text{–}\mathrm{PQ5}$$

with PQ6 again being added on in the $\mathbf{E}_{+}^{\forall p}$ and $\mathbf{EM}_{+}^{\forall p}$ cases. Further, let

$$\mathbf{S}^{\forall\exists p} = \mathbf{S}^{\forall p} + \mathrm{PQ7}\text{–}\mathrm{PQ9}$$
$$\mathbf{S}_{+}^{\forall\exists p} = \mathbf{S}_{+}^{\forall p} + \mathrm{PQ7}\text{–}\mathrm{PQ9}.$$

The last definition is of particular importance for the positive systems where $\exists$ is not definable. Finally, let

$$\mathbf{S}^{\exists p} = \mathbf{S}^{\forall\exists p}$$

except that $\forall$ is now taken as defined.

COMMENTS. PQ1–PQ2 serve to get us over the inductive step which, with the axiom clause, enables us to prove the admissibility of the rule of *generalization* ("*gen*"):

If $\vdash A$ then $\vdash \forall pA$ (gen).

PQ3 and PQ4 answer rather directly to the corresponding natural deduction rules $\forall$E and $\forall$I. PQ5, which we sometimes call "confinement," corresponds to the distribution principle $\forall\vee$. PQ7–PQ9 are direct representatives of the principles $\exists$I, $\exists$E, and &$\exists$. PQ6, wanted only in those systems in which entailments play a special role, answers to the special reiteration rule; of this more, after a digression.

Some of the feelings expressed in §30.1, implicitly at least, are ruffled slightly by these axioms and rules, which follow the standard practice of allowing theorems containing free variables, and also vacuous quantification. It would be possible, but not worth the trouble, to fix things so that such formulas were not well-formed, but it is easier and equally effective simply to announce that the Theorems of these systems are those theorems which have no free variables and no vacuous quantifiers, and simply let the capital letter reflect the respect we feel toward them. We turn to some observations which we rely on later (nothing deep).

Vacuous quantification, though distasteful, helps appreciably in showing that the Fitch-style systems are contained in their Hilbert-style counterparts. First note that where p is not free in A, we have in all these systems

$$\vdash A \rightleftarrows \forall pA$$

(with $\rightleftarrows$ defined in terms of $\rightarrow$ and &) which follows from $\forall p(A \rightarrow A)$ (axiom A1 of §27.1.1, and axiom clause) and PQ4, with the help of PQ3.

Secondly, notice that, in $\mathbf{E}^{\forall\exists p}$ and $\mathbf{EM}^{\forall\exists p}$, PQ3 and PQ6 make A15 (§27.1.1) otiose. A1 and A15 jointly yield $A\to A\to B\to B$, which, as we saw in §8.3.3, is sufficient (with A2 and A4) for the implicational fragment of **E**. And $A\to A\to B\to B$ follows from PQ3 and PQ6:

1	$\forall p(p\to p) \to .\, A\to A$	PQ3
2	$A\to A\to B \to .\, \forall p(p\to p)\to B$	1 suf
3	$\forall p(p\to p)\to B\to B \to .\, A\to A\to B\to B$	2 suf
4	$A\to A\to B\to B$	3 PQ6 $\to$E

All of which is, on reflection, exactly as it should be.

Thirdly, notice an anomaly in adding quantifiers to these systems: $\mathbf{E}^{\forall\exists p}$ and $\mathbf{EM}^{\forall\exists p}$ apparently require PQ6. The reason lies in the way the systems treat permutation: in **R** and **RM** anything goes; in **E** and **EM** only entailments go; in **T** nothing goes. The point of adding PQ6 to $\mathbf{E}^{\forall\exists p}$, which alone preserves *both* necessity and relevance (and to $\mathbf{EM}^{\forall\exists p}$), is to get the effect axiomatically of the rule for $\mathbf{FE}^{\forall\exists p}$ and $\mathbf{FEM}^{\forall\exists p}$ which lets us reiterate universally quantified formulas $\forall p_1 \ldots \forall p_n A$ into hypothetical proofs *provided* A is an entailment. (For the finite case, compare §21.2.2). The trick here is to replace restricted permutation by something that gives us the effect of restricted universally quantified permutation; and PQ6 does the trick for $\mathbf{E}^{\forall\exists p}$, as we now see. For brevity, we use "suf(cons)" to mean "suffixing in the consequent," i.e., the rule that takes us in all systems from $A\to .\, B\to C$ to $A\to .\, C\to D\to .\, B\to D$.

1	$\forall p(A\to B) \to .\, A\to B$	PQ3
2	$\forall p(A\to B) \to .\, B\to B\to .\mathrm{A}\to B$	1 suf(cons)
3	$\forall p(\forall p(A\to B)\to .B\to B\to .A\to B)$	2 gen
4	$\forall p(A\to B) \to .\, \forall p(B\to B\to .A\to B)$	3 PQ4 $\to$E
5	$\forall p(A\to B) \to .\, \forall p(B\to B)\to\forall p(A\to B)$	4 PQ1(cons)
6	$\forall p(A\to B) \to .\, \forall p(A\to B)\to D\to .\forall p(B\to B)\to D$	5 suf(cons)
7	$\forall p(A\to B) \to .\, \forall p(B\to B)\to D\to D\to .\forall p(A\to B)\to D\to D$	6 suf(cons)
8	$\forall p(p\to p) \to .\, B\to B$	PQ3
9	$\forall p(\forall p(p\to p)\to .B\to B)$	8 gen
10	$\forall p(p\to p) \to \forall p(B\to B)$	9 PQ4 $\to$E
11	$\forall p(B\to B)\to D \to .\, \forall p(p\to p)\to D$	10 suf
12	$\forall p(p\to p)\to D\to D \to .\, \forall p(B\to B)\to D\to D$	11 suf
13	$\forall p(B\to B)\to D\to D$	PQ6 12 $\to$E
14	$\forall p(A\to B) \to .\, \forall p(A\to B)\to D\to D$	7 13 (δ) (see §8.2)
15	$(\forall p(A\to B)\to D\to D\to .C\to D) \to .\, \forall p(A\to B)\to .C\to D$	14 suf
16	$(C\to .\forall p(A\to B)\to D) \to .\, \forall p(A\to B)\to D\to D\to .C\to D$	A2 (§27.1.1)
17	$(C\to .\forall p(A\to B)\to D) \to .\, \forall p(A\to B)\to .C\to D$	16 15 trans

(We apologize for going on so, but it seemed to us best to include one extended proof using propositional quantifiers.)

It will be apparent (to the attentive reader, anyway) that step 17 is simply the (possibly infinite) analogue of an axiom obliquely discussed in §26.1 and §23.4:

$$(A\rightarrow.(B\rightarrow C)\&(E\rightarrow F)\rightarrow D)\rightarrow.\ (B\rightarrow C)\&(E\rightarrow F)\rightarrow.A\rightarrow D.$$

Enough said, except that step 17 is what enables us appropriately to extend the Lemma of §4.1 to $\mathbf{E}^{\forall\exists p}$ and $\mathbf{FEM}^{\forall\exists p}$.

Note, incidentally, the similarity of the roles played by PQ6 in the modal systems with propositional quantifiers and (in §R2) by ***t***0 in the corresponding systems with ***t***. In fact, given ***t***0, PQ6 becomes redundant for the four propositional-quantifier cases involving **E** and **EM**. This is not surprising, given our redefinition of $\square$ in the last section; for we want $\square A\rightarrow A$ as a theorem, in the modal systems, and using either ***t***0 or PQ6 is the way to get it.

In showing that any proof in $\mathbf{FT}^{\forall\exists p}$, or in any of the other Fitch-style systems, can be converted into a proof in the corresponding Hilbert system, we run into only one problem not already familiar from previous discussions (e.g., §23.5). This involves the elimination of general categorical subproofs in proofs or (appropriately defined) quasi-proofs in the starred sytems. As expected, we replace an innermost categorical subproof which is general with respect to p:

$$\text{S:}\qquad p\ \begin{array}{|l} A_1\\ A_2\\ \vdots\\ A_n\end{array}$$

by a sequence of n steps

$$\text{S}'\text{:}\qquad \begin{array}{l} \forall pA_1\\ \forall pA_2\\ \vdots\\ \forall pA_n\end{array}$$

and then fill in steps in the sequence with theorems of the Hilbert-style systems in such a way that the whole resulting Fitch-system quasi-proof remains such, with $\rightarrow$E and &I as sole rules applied among the replacing steps $\forall pA_1,\ldots,\forall pA_n$. Then the step $\forall pA_n$, originally constructed as a consequence of the subproof, can now be taken to be a consequence of the final replacing step by repetition.

The pattern of argument is familiar: we suppose as inductive hypothesis that enough theorems have been inserted to justify all steps prior to $\forall pA_i$, and need to show that we can fix things up for this step as well. For this purpose we need the following theorems, each for the case corresponding to the rule cited on the left.

Reit	$A \to \forall pA$	(p not free in A)
$\to$E	$\forall p(A\to B) \to.\ \forall pA\to\forall pB$	
$\sim$I	$\forall p(A\to\sim A) \to \forall p\sim A$	
$\sim$E	$\forall p(A\to B) \to.\ \forall p\sim B\to\forall p\sim A$	
$\vee$E	$\forall p(A\to C)\&\forall p(B\to C) \to.\ \forall p(A\vee B)\to\forall pC$	
$\exists$E	$\forall p\forall q(A(q)\to B) \to.\ \forall p\exists qA(q)\to\forall pB$	(q not free in B)
$\sim\sim$I	$\forall pA \to \forall p\sim\sim A$	
$\sim\sim$E	$\forall p\sim\sim A \to \forall pA$	
&I	$\forall pA\&\forall pB \to \forall p(A\&B)$	
&E	$\forall p(A\&B) \to \forall pA$	
	$\forall p(A\&B) \to \forall pB$	
$\vee$I	$\forall pA \to \forall p(A\vee B)$	
	$\forall pB \to \forall p(A\vee B)$	
&$\vee$	$\forall p(A\&(B\vee C)) \to \forall p((A\&B)\vee C)$	
$\forall$E	$\forall p\forall qA(q) \to \forall pA(B)$	
$\exists$I	$\forall pA(B) \to \forall p\exists qA(q)$	
$\forall\vee$	$\forall p\forall q(A\vee B(q)) \to \forall p(A\vee\forall qB(q))$	(q not free in A)
&$\exists$	$\forall p(A\&\exists qB) \to \forall p\exists q(A\&B)$	(q not free in A)

All these are theorems of $\mathbf{T}^{\forall\exists p}$, hence also of the other four systems. In all the equivalence proofs it is a trivial matter to show that the axioms of the Hilbert-sytle formulations are provable in the subproof formulations, so we leave all that as an exercise.

For the case of turning $\mathbf{FT}^{\forall\exists p}$ in for $\mathbf{T}^{\forall\exists p}$, we have already explained the strategy. What remains is to keep track of those wretched subscripts, with a view to being sure that the "reduced" quasi-proof is still such, within the meaning of the act, so that the induction can proceed backwards until it comes to a standstill. This involves verifying more cases than any sane person cares to contemplate, so we pick *one* to examine in detail, expecting the reader to agree with us in hoping that the remaining cases for the rules of each system involving connectives will bear close scrutiny.

Let us pretend that we have somehow managed to carry out the procedure for the reduction of the innermost subproof S:

$$\begin{array}{l|l} p & A_1 \\ & A_2 \\ & \vdots \\ & A_i \\ & \vdots \\ & A_n \end{array}$$

described above, as far as i steps, so that we are confident about replacing the first i steps of S by steps of S′, to wit:

$$\begin{array}{l}\forall pA_1\\ \forall pA_2\\ \vdots\\ \forall pA_i\end{array}$$

Suppose furthermore that we are anxious to get on with the work and that A_{i+1} happened to result from an application of ∃E. We would like then to see that $\forall pA_{i+1}$ can be added to our list.

We know, from the fact that A_{i+1} came with the help of ∃E, that somewhere among the predecessors of A_{i+1} in S can be found formulas $\exists qA(q)_{\mathrm{a}}$ and $\forall q(A(q)\to B)_{\mathrm{b}}$, where q is not free in B and where $\max(\mathrm{b}) \leq \max(\mathrm{a})$. We get this information by inspecting ∃E in §31.5, which tells us also that A_{i+1} has the form $B_{\mathrm{a}\cup\mathrm{b}}$.

The inductive hypothesis, on the other hand, tells us that we have already prefixed $\forall p$ to the relevant predecessors of A_{i+1}, and so that among the formulas in S′ we will find

$$\forall p\exists qA(q)_{\mathrm{a}}$$

and

$$\forall p\forall q(A(q)\to B)_{\mathrm{b}}.$$

Consultation of the case-by-case list just given of formulas alleged to be provable reveals the formula for ∃E. If we insert it now into the quasi-proof, and apply →E, we first get

$$\forall p\exists qA(q)\to\forall pB_{\mathrm{b}}$$

where, for the application of →E, subscripts on the minor premiss exceed those on the major (of which there are none) as required by the rule for $\mathbf{T}^{\forall\exists p}$. But now the same condition obtains, since $\max(\mathrm{b}) \leq \max(\mathrm{a})$, so we may again use →E to get

$$\forall pB_{\mathrm{a}\cup\mathrm{b}}$$

as required.

Other cases are similar, and we claim to have proved that $\mathbf{T}^{\forall\exists p}$ and $\mathrm{FT}^{\forall\exists p}$ have the same collection of theorems. Proofs for $\mathbf{R}^{\forall\exists p}$ and $\mathbf{RM}^{\forall\exists p}$ are just the same, except that account must be taken of the permutation axiom of **R** and the mingle axiom of **RM**. For $\mathbf{E}^{\forall\exists p}$ and $\mathbf{EM}^{\forall\exists p}$, step 17 of the proof above enables us to extend the Lemma of §4.1, as already remarked. We leave details to the reader.

One closing bit of axiom chopping. The following axiom set is equivalent to that presented at the beginning of this section: replace PQ4 by PQ4′ and

PQ8 by the pair PQ8′ and PQ8″, where we have

PQ4′	$A \to \forall p A$	(p not free in A)
PQ8′	$\forall p(A \to B) \to . \exists p A \to \exists p B$	
PQ8″	$\exists p A \to A$	(p not free in A)

This set is more elegant, inasmuch as it is more dual with respect to ∀ and ∃. Its defect is that it emphasizes vacuous quantification, which, on the whole, we don't like.

§33. **Truth values.** Enough has been said by now to indicate that we are in sympathy with the Frege-Church view, according to which sentences in a language are best treated in the same way as names, to the extent that with each sentence, as with each name, this theory associates both a sense (*Sinn*), which is what we grasp when we understand the sentence, and a denotation (*Bedeutung*), which is the truth value (*das Wahre* or *das Falsche*) of the sentence. Eidophobes wince on hearing such talk, but we are inclined to take it seriously, while recognizing that it would be nice to have a clearer formal theory of the matter than presently exists. We believe, in fact, that such a formal theory lurks in the shadows cast by the relevance logics which are the topic of this book, but a large attempt to bring it to light would carry us too far afield. We will, however, push the inquiry a few steps in the direction in which we anticipate that it will go, by offering a theory of truth values, which, although it accords in important particulars with that of Frege and Church, is somewhat more explicit on a few details.

Church 1956 writes:

> Therefore, with Frege, we postulate two abstract objects called *truth-values*, one of them being *truth* and the other one *falsehood* (p. 25),

and adds in a footnote,

> To Frege, as a thoroughgoing Platonic realist, our use of the word "postulate" here would not be acceptable. It would represent his position better to say that the situation indicates that *there are* two such things as truth and falsehood (*das Wahre* and *das Falsche*).

These passages from Church 1956, and those which follow immediately, explain the position we espouse here as clearly as it has been explained in print (to our knowledge), and we refer the reader to §04 of Church's book for further details. In particular, we think of the letters described as *variables* in §30.2 above as *intensional propositional variables* in Church's sense, rather than as *propositional variables* in his sense, which are variables ranging over truth values. We would prefer to call the latter *truth value variables*, and let "propositional variables" mean what the term suggests: variables ranging over propositions taken in intension. We shall indeed follow this practice in the future (when we can remember to do so). But the usage Church adopts is so firmly fixed in the literature that it would be pointless to try to alter it at this late date, and we shall no doubt succumb to habit in the future as we have in the past.

Frege says *there are* truth values, and Church more cautiously postulates their existence. We will propose to side with Frege, and not only say that there are truth values but actually specify what they are. They are certain propositions, notation for which was not available to us before the introduction of propositional quantifiers: to wit, *truth* is the proposition that $\exists pp$, and *falsehood* the proposition that $\forall pp$. It follows that, for each of the sentences "$\exists pp$" and "$\forall pp$," the sense will be identical with the denotation; i.e., these two sentences denote their own senses. This stance of course involves commitments said to be metaphysical, and we shall not pursue them in detail in this book, but simply offer this announcement to warn the reader. We should also add that, from the point of view of formal semantics, the interpretation of $\forall pp$ will be the (possibly infinite) conjunction of all the values of the variable p, and dually for $\exists pp$; so the interpretation of these constants will be fixed by the range of the propositional (note: *propositional*) variables.

In order to motivate these ideas in the context of relevance logics, we first show what happens to them in standard truth value logic, and then consider their place in more sophisticated surroundings.

§33.1. $\mathbf{TV}^{\forall p}$.

To fix ideas we choose the axiom schemata of Church 1956, §27 (or §30).

$$A \supset . B \supset A$$
$$A \supset (B \supset C) \supset . A \supset B \supset . A \supset C$$
$$\sim A \supset \sim B \supset . B \supset A$$

together with the rule: from $A \supset B$ and A to infer B; we call this system "**TV**," to remind us that it is the (two-valued) truth value calculus. The extended "propositional" calculus or theory of "propositions" $\mathbf{TV}^{\forall p}$ may be obtained from this basis by adding notation for universal quantification and also the axiom schemata

$$\forall pA(p) \supset A(B) \text{ and}$$
$$\forall p(A \supset B(p)) \supset . A \supset \forall pB(p) \qquad (p \text{ not free in } A)$$

together with the rule: from A to infer $\forall pA$.

For the purpose of the present discussion only, we revert to the Neanderthal position (which we hope the reader will find, by this time, as difficult as we do) according to which the horseshoe means "if... then —," and there is a sort of fuzzy misunderstanding, occasioned by the term "propositional calculus," about the range of the variables. (These words seem to us accurately to describe the confused frame of mind engendered in students by the practice of using letters to stand for sentences, and then "translating" the sentence "If the maid sees the butler, then he is not in the dining room" into "$M \supset {\sim} D$".) We will try to maintain a straight face while temporarily assuming this peculiar stance (jaw protruding, recessed brow furrowed, somewhat crouched posture, with arms hanging low and a club in one hand).

As is well known, addition of quantifiers to the truth value calculus allows for a certain flexibility and economy in the choice of primitives; in particular Łukasiewicz and Tarski 1930 proposed taking material "implication" and universal quantification as primitive for the extended truth value calculus, defining

$$\begin{aligned} &\boldsymbol{F} =_{\mathrm{df}} \forall pp, \text{ and} \\ &{\sim} A =_{\mathrm{df}} A \supset \boldsymbol{F}. \end{aligned}$$

Suitable axioms for the primitives will guarantee that negation has the right classical properties, and we also define $\boldsymbol{T}$ as ${\sim}\boldsymbol{F}$ or, equivalently, as $\exists pp$.

Now in the resulting $\mathbf{TV}^{\forall p}$ we have as a theorem:

$$A(\boldsymbol{T}) \supset .\ A(\boldsymbol{F}) \supset \forall pA(p),$$

which says roughly (given the amusing fiction that the horseshoe can be read "if . . . then —") that if both *truth* and *falsity* satisfy a condition, then everything does—which says in turn that *truth* and *falsity* are all there is. And in fact this theorem and its consequences show that adding quantifiers to **TV** accomplishes virtually nothing, since $\forall pA(p)$ can be shown to be *concurrent* (Church 1956, p. 152) with $A(\boldsymbol{T})\&A(\boldsymbol{F})$, i.e., always to have the same truth value.

What this means in turn is that $\boldsymbol{T}$ and $\boldsymbol{F}$ can, if taken as primitive, do all the work of quantifiers, with suitable additional axioms. The system described in Church 1956, §10, is built on this approach; if we turn in his rule of substitution for axiom schemata, we get

$$\begin{aligned} &A \supset .\ B \supset A \\ &(A \supset .B \supset C) \supset .\ A \supset B \supset .A \supset C \\ &A \supset \boldsymbol{F} \supset \boldsymbol{F} \supset A, \end{aligned}$$

together with *modus ponens* as sole rule. Then for this system universal quantifiers may be obtained by a kind of notational gimmick:

$$\forall pA(p) =_{\mathrm{df}} A(\boldsymbol{T})\&A(\boldsymbol{F}).$$

And the theorem $\boldsymbol{F} \supset A$ (Church's †122, p. 84) again says in effect that $\boldsymbol{F}$ must have as its interpretation the conjunction of all values of the variable p, to wit, the two values *truth* and *falsehood.*

Now all of this is perfectly satisfactory as far as it goes, but it certainly doesn't go very far. Everyone who can speak any language at all knows perfectly well that there are countless propositions other than T and F, and it is apparent from the foregoing that there is absolutely no way of recognizing their existence within the framework provided by $\mathbf{TV}^{\forall p}$. In the light of the extraordinary restriction of the range of the truth value variables to truth and falsehood, it is not surprising that **TV** and $\mathbf{TV}^{\forall p}$ have some odd properties which are entirely inconsistent with our usual understanding of propositions, and of "if... then —." So long as we keep in mind that $\mathbf{TV}^{\forall p}$ is not *at all* what the name "extended propositional calculus" might suggest to the unwary, we can explain a good bit—even the profoundly misleading doctrine *ex impossibilitate quodlibet.*

§33.2. For every individual x, x is president of the United States between 1850 and 1857. Before turning to this perhaps unexpected topic, we consider those misunderstandings which might lead one to say that a contradiction implies or entails any proposition whatever.

We think that the principal attraction of this aberration lies (irrelevantly) in a persistent, ingrained insistence on forgetting what we are talking about. Having been bamboozled by the prevalent reading of the horseshoe and having noted correctly that free variables *must* mean universality—a feature of the situation which can be made explicit—one then ignores the fact that the only plausible way of adding quantifiers leads *explicitly* to the conclusion that the theory recognizes two and only two "propositions": a false one which "implies" itself and the other, and a true one which is "implied" by itself and the other. Each is moreover equivalent to the denial of the other. Given these severe restrictions, it is *of course* true that "a contradiction implies everything": what is meant is that if we conjoin both the two available "propositions," then either one of them follows from this conjunction; i.e., we have $p\&q\rightarrow p$, and $p\&q\rightarrow q$. Fair enough; but not exciting, exactly.

What makes the claim exciting is simply and purely *muddle*, and nothing else. One hears the old argument from $p\&(\sim p\vee q)\rightarrow q$ to $(p\&\sim p)\rightarrow q$ trotted out again, with the additional claim: "But I *don't* mean this for *just* T and F; I mean it for any propositions whatever—as intensional as you like!" But the reply is obvious. If it is true for all propositions, we would expect to have $\forall p\forall q((p\&\sim p)\rightarrow q)$ which, according to all Laws of God and Man, entails $\forall p((p\&\sim p)\rightarrow\forall qq)$, i.e., $\forall p((p\&\sim p)\rightarrow F)$. Equally, we have $\forall p(F\rightarrow .p\&\sim p)$; if co-entailment guarantees intersubstitutability, we are back at the starting point again, and, if it doesn't, then we don't even know what the theory is about, and, if the proponent of the theory does, we feel that he should let us all in on the secret.

The upshot is that it is *perfectly correct* to say that, for any propositions p and q, the conjunction of p and its denial entails q, and in *just the same* *way* it is *perfectly correct* to assert the sentence that gives this section its title. What gives both statements a strong savor of fatuity is that for both we negligently failed to supply the crucial bit of information that turns the assertions into trivial truths. In both cases we failed to stress the fact that we had *cooked* the range of the variables: in the first case they range over truth and falsehood, and in the second case they range over Millard Fillmore and Franklin Pierce.

§33.3. $\mathbf{E}_{\mathrm{fde}}$ and truth values. This section and the next are devoted to elucidating the position taken in the previous section, by finding formal structures that say the same thing. We first consider the system $\mathbf{E}_{\mathrm{fde}}$ of §15.2, relying for proofs on the fact (there proved) that

$$\vdash_{\mathbf{E}_{\mathrm{fde}}} A \text{ iff } A \text{ is a tautological entailment.}$$

To this system we add a propositional constant T, with the *interpretation* $\exists pp$ (even though we will not have quantifiers), together with the axiom of §27.1.2 and §R2 (true on the interpretation):

$$T1 \qquad A \rightarrow T.$$

The resulting system we call $\mathbf{E}^{T}_{\mathrm{fde}}$. Contraposition, and a definition of F as $\sim T$ yield as a theorem

$$F1 \qquad F \rightarrow A,$$

where F has the interpretation $\forall pp$. We now write $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} A \rightleftarrows B$ to mean $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} A \rightarrow B$ and $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} B \rightarrow A$, and leave it to the reader to verify the following:

$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} \sim T \rightleftarrows F$$
$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} \sim F \rightleftarrows T$$
$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} T \vee T \rightleftarrows T$$
$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} T \vee F \rightleftarrows T$$
$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} F \vee T \rightleftarrows T$$
$$\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} F \vee F \rightleftarrows F$$

(and dually for conjunction). We now define *truth value formula* ("*tvf*") as follows: T and F are tvfs, and if A and B are, so are $\sim A$, $A \vee B$, $A \& B$, (also $A \supset B =_{\mathrm{df}} \sim A \vee B$, if we like), and $A \rightarrow B$. That is, tvfs are formulas with no free variables (all variable-places are filled with T or F). Then, since the condition $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} A \rightleftarrows B$ can easily be shown to give us intersubstitutability in $\mathbf{E}^{T}_{\mathrm{fde}}$, we have, with a minimum of argument, a little

LEMMA. If A is an arrow-free tvf, then either $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} A \rightleftarrows T$ or $\vdash_{\mathbf{E}^{T}_{\mathrm{fde}}} A \rightleftarrows F$.

Where A is a formula of **TV**, we call A' an *associated tvf of A* if A' is got from A by substituting T (or F) throughout for each variable of A. Then it is but a short step to the

THEOREM. Let $A \supset B$ be a formula of **TV**. Then $\vdash_{\mathbf{TV}} A \supset B$ iff $\vdash_{\mathbf{E}^T_{\mathrm{fde}}} A' \rightarrow B'$ for every associated tvf of $A \supset B$.

PROOF. Easy.

COMMENTS. First, we have now completely resuscitated **TV**. We are committed, for example, to provability in $\mathbf{E}^T_{\mathrm{fde}}$ of

$\boldsymbol{T}\&(\sim \boldsymbol{T} \vee \boldsymbol{T}) \rightarrow \boldsymbol{T}$,
$\boldsymbol{T}\&(\sim \boldsymbol{T} \vee \boldsymbol{F}) \rightarrow \boldsymbol{F}$,
$\boldsymbol{F}\&(\sim \boldsymbol{F} \vee \boldsymbol{T}) \rightarrow \boldsymbol{T}$, and
$\boldsymbol{F}\&(\sim \boldsymbol{F} \vee \boldsymbol{F}) \rightarrow \boldsymbol{F}$;

i.e., to $\vdash_{\mathbf{E}^T_{\mathrm{fde}}} A\&(\sim A \vee B) \rightarrow B$ *whenever* $A\&(\sim A \vee B) \supset B$ *is an associated tvf of* $p\&(\sim p \vee q) \supset q$, but *not* of course generally.

Second, this fact sheds light on the canard that "if A and $\sim A \vee B$, then B" *is* true generally. For example, the following deceitful ploy has been recommended to us as a justification for reading the horseshoe as "if . . . then —." We examine the sixteen truth functions of two variables and observe that the function familiarly denoted by the horseshoe *alone* has the property that when the value of the first argument is truth and the value of the function is truth, then the value of the second argument must be *truth*—while the function (unlike, say, conjunction or equivalence) remains relatively noncommittal for other cases, i.e., it does not rule them out.

This is of course persiflage, but it is characteristic of the kind of argument to which one must resort in order to convince the innocent of a falsehood. The attempt is to make it look as if the two premisses A and $\sim A \vee B$ are acting in concert to produce B, as do A and $A \rightarrow B$ when obliging us with B. But a glance at the four formulas above, together with the recollection that we have $\boldsymbol{F} \rightarrow A$ and $A \rightarrow \boldsymbol{T}$, makes the line of argument look as silly as it is. Unlike the case of $A\&(A \rightarrow B)$, where *both* premisses are required for the conclusion, the examples above can be rewritten, after a couple of computations, as a catalogue of certain properties of conjunction:

$\boldsymbol{T}\&\boldsymbol{T} \rightarrow \boldsymbol{T}$,
$\boldsymbol{T}\&\boldsymbol{F} \rightarrow \boldsymbol{F}$,
$\boldsymbol{F}\&\boldsymbol{T} \rightarrow \boldsymbol{T}$, and
$\boldsymbol{F}\&\boldsymbol{T} \rightarrow \boldsymbol{F}$,

or even more honestly as $\boldsymbol{T} \rightarrow \boldsymbol{T}$ and $\boldsymbol{F} \rightarrow \boldsymbol{F}$; better still, what comes to the same thing: $\boldsymbol{T} \rightarrow \boldsymbol{T}$. *This* is all right; as we have said, identity is at the heart of logic. But it is shameful to abuse students by mistreating the truth value calculus as if it were a calculus of propositions. We agree rather with Cervantes 1605: Honesty is the best policy.

Finally, addition of notation for truth values, together with axioms $A \to T$ and $F \to A$ appropriate to the intended interpretation, is also possible for **T**, **E**, and **R**; these systems therefore also satisfy the condition on $\mathbf{E}^{T}_{\mathbf{fde}}$ stated in the theorem above. For $\mathbf{T}^{\forall\exists p}$, $\mathbf{E}^{\forall\exists p}$, and $\mathbf{R}^{\forall\exists p}$, the theorem can be made a bit more perspicuous, as we now see.

§33.4. Truth-value quantifiers. For all three systems $\mathbf{T}^{\forall\exists p}$, $\mathbf{E}^{\forall\exists p}$, and $\mathbf{R}^{\forall\exists p}$, we let

$$F =_{\mathrm{df}} \forall pp$$
$$T =_{\mathrm{df}} \exists pp.$$

Obviously $F \to A$ and $A \to T$ are then theorems of all three systems. We also define (not propositional but) *truth value quantifiers* $[\forall p]$, $[\forall q]$, $[\forall r], \ldots,$ and $[\exists p]$, $[\exists q]$, $[\exists r], \ldots,$ according to the definitions:

$$[\forall p]A(p) =_{\mathrm{df}} A(T) \& A(F)$$
$$[\exists p]A(p) =_{\mathrm{df}} A(T) \vee A(F).$$

We notice then that we can express part of the content of the theorem of the previous section more explicitly. For from

$$((T \& {\sim} T \to T) \& (F \& {\sim} F \to T)) \& ((T \& {\sim} T \to F) \& (F \& {\sim} F \to F))$$

we obtain

$$[\forall q]((T \& {\sim} T \to q) \& (F \& {\sim} F \to q))$$

and then

$$[\forall p][\forall q](p \& {\sim} p \to q).$$

Our notation then suggests that we define a *closed formula* as a formula containing no free variables, and a *truth-value closed* ("tv closed") formula as a closed formula containing only truth value quantifiers (and we then know what is meant by saying that A' is the *closure* {*tv closure*} of A). The theorem of the previous section now generalizes to:

THEOREM. Let $A \supset B$ be a formula of **TV**. Then $\vdash_{\mathbf{TV}} A \supset B$ iff the truth-value closure of $A \to B$ is provable in $\mathbf{T}^{\forall\exists p}$ $\{\mathbf{E}^{\forall\exists p}, \mathbf{R}^{\forall\exists p}\}$.

For example, $\vdash_{\mathbf{TV}}((p\supset q)\supset p)\supset p$, and hence $[\forall p][\forall q](((p\supset q)\supset p)\rightarrow p)$ is provable in all three systems. This theorem explains what was meant all along by calling the horseshoe an implication relation: for *very* specialized cases, it *is* one. So now we have taken care of tv closures of first degree entailments, and we begin generalizing.

§33.5. $\mathbf{R}^{\forall\exists p}$ and TV. For the system $\mathbf{R}^{\forall\exists p}$, the theorem just stated can be generalized considerably; in fact it can be shown that, when the range of variables is restricted to truth values, the arrow of relevant implication simply *is* material "implication"; but it should be remembered that this is *only* because we are treating two very special propositions, each of which has properties of its own.

What comes as a slight surprise is that the lemma of §33.3 can be extended to $\mathbf{R}^{\forall\exists p}$:

LEMMA. If A is a tvf of $\mathbf{R}^{\forall\exists p}$, then either $\vdash_{\mathbf{R}^{\forall\exists p}} A\rightleftarrows \boldsymbol{T}$ or $\vdash_{\mathbf{R}^{\forall\exists p}} A\rightleftarrows \boldsymbol{F}$.

PROOF. This follows from the fact that the following can be added to the list of $\mathbf{E}_{fde}^{\forall\exists p}$-equivalences (hence $\mathbf{R}^{\forall\exists p}$-equivalences) of §33.3, as the reader may check:

$$\vdash_{\mathbf{R}^{\forall\exists p}} \boldsymbol{T}\rightarrow\boldsymbol{T}\rightleftarrows\boldsymbol{T}$$
$$\vdash_{\mathbf{R}^{\forall\exists p}} \boldsymbol{T}\rightarrow\boldsymbol{F}\rightleftarrows\boldsymbol{F}$$
$$\vdash_{\mathbf{R}^{\forall\exists p}} \boldsymbol{F}\rightarrow\boldsymbol{T}\rightleftarrows\boldsymbol{T}$$
$$\vdash_{\mathbf{R}^{\forall\exists p}} \boldsymbol{F}\rightarrow\boldsymbol{F}\rightleftarrows\boldsymbol{T}$$

The lemma then tells us that, if both $\boldsymbol{T}$ and $\boldsymbol{F}$ satisfy a condition, then every tvf of $\mathbf{R}^{\forall\exists p}$ does. So the following is virtually immediate:

THEOREM. Let A be a formula of **TV**, and let A' be obtained from A by replacing all horseshoes with arrows. Then A is a theorem of **TV** iff the tv closure of A' is a theorem of $\mathbf{R}^{\forall\exists p}$.

What this means is that in $\mathbf{R}^{\forall\exists p}$ we have a complete basis for $\mathbf{TV}^{\forall\exists p}$, with quantifiers restricted; e.g., we have as theorems:

$$[\forall p][\forall q](p\rightarrow .q\rightarrow p)$$
$$[\forall p][\forall q][\forall r]((p\rightarrow .q\rightarrow r)\rightarrow .p\rightarrow q\rightarrow .p\rightarrow r)$$
$$[\forall p][\forall q](\sim q\rightarrow \sim p\rightarrow .p\rightarrow q)$$
$$[\forall p]A(p)\rightarrow A(B) \qquad \text{for every tvf } B$$
$$[\forall p](A\rightarrow B)\rightarrow .(A\rightarrow [\forall p]B) \qquad (p \text{ not free in } A)$$

and, as a derived rule:

From $A(T)\&A(F)$ to infer $[\forall p]A[p]$.

With this theorem we feel that we have now done everything that any reasonable person could expect for **TV**. We have shown that it is contained in all the intensional systems of this book, once *tout pur* (§24.1) and a second time (§33.4) in that $\rightarrow$ and $\supset$ act alike if flanked by purely truth-functional tvfs. And now we've shown that $\rightarrow$ and $\supset$ are indistinguishable among tvfs in $\mathbf{R}^{\forall\exists p}$ (if only because $\mathbf{TV}^{\forall\exists p}$ and that fragment of $\mathbf{R}^{\forall\exists p}$ both cheat by surreptitiously *using* properties of truth values). What more could one want for **TV** or $\mathbf{TV}^{\forall\exists p}$?

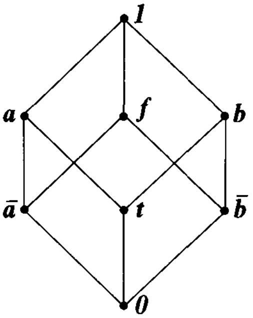
§34. First degree entailments in $\mathbf{E}^{\forall\exists p}$ (by Dorothy L. Grover). Zero degree formulas were defined in §18.6 as those formulas of **E** which contain only truth functional connectives. For $\mathbf{E}^{\forall\exists p}$, we extend this definition to include all formulas containing either truth value connectives or propositional quantifiers; that is, all arrow-free formulas are *zero degree formulas of* $\mathbf{E}^{\forall\exists p}$. *First degree entailments of* $\mathbf{E}^{\forall\exists p}$ are entailments of the form $A\rightarrow B$ where both A and B are zero degree formulas of $\mathbf{E}^{\forall\exists p}$. *Primitive entailments* of $\mathbf{E}^{\forall\exists p}$ are the same as those of **E**.

In this section we prove that the first degree entailment fragment of $\mathbf{E}^{\forall\exists p}$ is decidable. Using a tree construction, we show for each first degree entailment $A\rightarrow B$ that we can obtain a finite set of primitive entailments, S, such that $\vdash A\rightarrow B$ iff each member of S is provable. The tree construction is used to show that if all members of S are provable then $\vdash A\rightarrow B$. In proving the converse, we take a detour through validity, using methods based on §18. As in §18, a lattice-theoretical definition of validity is given for first degree entailments, and, with respect to this definition, the first degree entailment fragment of $\mathbf{E}^{\forall\exists p}$ is shown to be complete. If one of S is not provable then it is not valid; it follows that $A\rightarrow B$ is not valid and, therefore, that $A\rightarrow B$ is not provable.

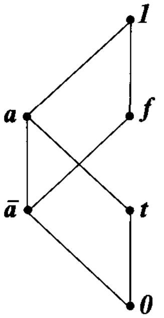
We begin by establishing some algebraic results which we use in developing the semantics of the first degree fragment of $\mathbf{E}^{\forall\exists p}$.

§34.1. The algebra of first degree entailments of $\mathbf{E}^{\forall\exists p}$. The lattices we work with in this section are all De Morgan lattices, and indeed when we refer to "lattice," "sublattice," etc., we shall mean De Morgan lattice, De Morgan sublattice, etc.; such lattices were discussed in §18.3. We use the nontrivial sublattices of M_0 (see §18.4) and the nontrivial sublattices of Smiley's lattice (SL) (see §15.3). We have sketched below the Hasse diagrams of these lattices, partly to make explicit the sublattices of M_0 and SL and partly because we have for mnemonic reasons labeled the elements of the lattices differently. (The "ch" in "4ch" and "3ch" means "chain.")

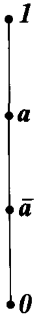
Nontrivial sublattices of M_0



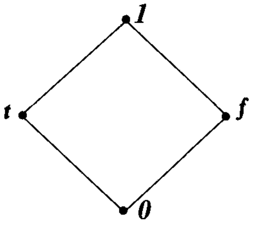
M_0



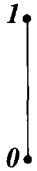
Six



4ch

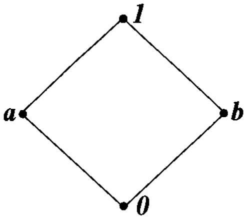


tf

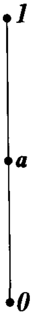


Two

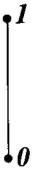
Nontrivial sublattices of SL:



SL



3ch



Two

Negation is defined as follows. In M_0, Six, and 4ch the negation of $\boldsymbol{a}$ is $\bar{\boldsymbol{a}}$ and the negation of $\boldsymbol{b}$ is $\bar{\boldsymbol{b}}$. In SL and 3ch, $\bar{\boldsymbol{a}} = \boldsymbol{a}$ and $\bar{\boldsymbol{b}} = \boldsymbol{b}$. (Note the ambiguous use of the overbar.) In every lattice $\bar{\mathit{1}} = \mathit{0}, \bar{t} = f$. Finally, for all elements x, $\bar{\bar{\mathrm{x}}} = \mathrm{x}$.

We give now some definitions which lead finally to our definition of "valid." Essentially SL determines validity just as it does for E_{fde}, the main difference being that the nontrivial sublattices are taken into account (as the "domains" of the propositional quantifiers), a fact which is reflected in the definitions.

Q is a *model* iff Q is an ordered pair $\langle L, v\rangle$, where L is a finite De Morgan lattice and v is a mapping of the propositional variables into L. Given a model $\langle L, v\rangle$ we define v* as that function mapping all zdfs into L which agrees with v on propositional variables and is such that $v^*({\sim}A) = {\sim}v^*(A)$ and similarly for $\vee$ and &, and where $v^*(\exists pA(p)) = \bigvee\{v'^*(A(p))$: v′ is like v except perhaps at $p\}$, where for X a subset of the elements of any lattice L, $\bigvee$X is the join (where it exists) of the elements in X; and dually for $\forall pA(p)$. Except in one or two delicate cases, we use "v" for "v*."

We have defined "model" in terms of finite lattices because of our interest in obtaining a decision result for the first degree entailment fragment of $\mathbf{E}^{\forall\exists p}$. A more intuitively attractive interpretation would be obtained if the definition of "model" was in terms of complete lattices, but such a definition introduces a few complications we have no need to bother with. Accordingly, in this section, the default assumption is that all De Morgan lattices mentioned are finite.

The *truth set* $\mathrm{T_Q}$ *determined by a model* $\mathrm{Q} = \langle \mathrm{L}, \mathrm{v}\rangle$ is defined as the set of first degree entailments, $B \rightarrow C$, such that $\mathrm{v}(B) \leq \mathrm{v}(C)$. The *falsity set* $\mathrm{F_Q}$ *determined by a model* $\mathrm{Q} = \langle \mathrm{L}, \mathrm{v}\rangle$ is the complement of $\mathrm{T_Q}$ with respect to first degree entailments. A first degree entailment A is *false in a De Morgan lattice* L iff there is a falsity set determined by a model Q defined for L, such that $A \in \mathrm{F_Q}$. A first degree entailment is *true in a lattice* L otherwise. A first degree entailment A is *falsifiable in a De Morgan lattice* L iff there is a model Q defined for a nontrivial sublattice L′ of L such that A is false in L′. A first degree entailment is *falsifiable* just in case it is falsifiable in SL.

A first degree entailment is *valid* just in case it is not falsifiable. That is, A is valid iff it belongs to the truth sets of all models on nontrivial sublattices of SL.

Some falsifiable first degree entailments:

$$\exists pp \rightarrow . \exists p(p \& \sim p) \vee \forall p(p \vee \sim p).$$

This formula is false in 3ch, since for any mapping of the propositional variables into 3ch, $\mathrm{v}(\exists p(p)) = (\boldsymbol{1} \vee \boldsymbol{a} \vee \boldsymbol{0}) = \boldsymbol{1}$, $\mathrm{v}(\exists p(p \& \sim p)) = ((\boldsymbol{1} \wedge \bar{\boldsymbol{1}}) \vee (\boldsymbol{0} \wedge \bar{\boldsymbol{0}}) \vee (\boldsymbol{a} \wedge \bar{\boldsymbol{a}})) = (\boldsymbol{0} \vee \boldsymbol{0} \vee \boldsymbol{a}) = \boldsymbol{a}$, and $\mathrm{v}(\forall p(p \vee \sim p)) = ((\boldsymbol{1} \vee \bar{\boldsymbol{1}}) \wedge (\boldsymbol{0} \vee \bar{\boldsymbol{0}}) \wedge (\boldsymbol{a} \vee \bar{\boldsymbol{a}})) = (\boldsymbol{1} \wedge \boldsymbol{1} \wedge \boldsymbol{a}) = \boldsymbol{a}$; therefore, $\mathrm{v}((\exists p(p \& \sim p)) \vee (\forall p(p \vee \sim p))) = \boldsymbol{a} \vee \boldsymbol{a} = \boldsymbol{a}$. Since $\boldsymbol{1} \nleq \boldsymbol{a}$, the formula belongs to the falsity set determined by the model $\langle 3\mathrm{ch}, \mathrm{v}\rangle$.

$$\exists p(p \& \sim p) \rightarrow \forall p(p \vee \sim p).$$

We use SL. Consider any mapping v of the propositional variables into SL. Then $\mathrm{v}(\exists p(p \& \sim p)) = ((\boldsymbol{1} \wedge \bar{\boldsymbol{1}}) \vee (\boldsymbol{0} \wedge \bar{\boldsymbol{0}}) \vee (\boldsymbol{a} \wedge \bar{\boldsymbol{a}}) \vee (\boldsymbol{b} \wedge \bar{\boldsymbol{b}})) = (\boldsymbol{0} \vee \boldsymbol{0} \vee \boldsymbol{a} \vee \boldsymbol{b})) = \boldsymbol{1}$; and $\mathrm{v}(\forall p(p \vee \sim p)) = ((\boldsymbol{1} \vee \bar{\boldsymbol{1}}) \wedge (\boldsymbol{0} \vee \bar{\boldsymbol{0}}) \wedge (\boldsymbol{a} \vee \bar{\boldsymbol{a}}) \wedge (\boldsymbol{b} \vee \bar{\boldsymbol{b}})) = (\boldsymbol{1} \wedge \boldsymbol{1} \wedge \boldsymbol{a} \wedge \boldsymbol{b}) = \boldsymbol{0}$. Since $\boldsymbol{1} \nleq \boldsymbol{0}$, the formula belongs to the falsity set determined by the model $\langle \mathrm{SL}, \mathrm{v}\rangle$.

$$\forall p(p \vee \sim p) \rightarrow \exists p(p \& \sim p).$$

This formula is false in Two, since for any mapping v of the propositional variables into Two, $\mathrm{v}(\forall p(p \vee \sim p)) = \boldsymbol{1}$ and $\mathrm{v}(\exists p(p \& \sim p)) = \boldsymbol{0}$.

It turns out that if a first degree entailment is falsifiable in a rather small lattice like SL, then it is also falsifiable in $\mathrm{M_0}$; and, in general, if a first degree entailment is falsifiable in a De Morgan lattice L′, then it is falsifiable in any lattice, L, of which L′ is a homomorphic image. (A lattice L′ is a *homomorphic image* of a lattice L just in case there is a homomorphism h, from L onto L′.) This general result is established in the following

FALSIFIABILITY THEOREM. If a first degree entailment $A \rightarrow B$ is falsifiable in a De Morgan lattice L′, and L′ is a homomorphic image of a De Morgan lattice L, then $A \rightarrow B$ is falsifiable in L.

If $A \rightarrow B$ is falsifiable in L′, then there is some nontrivial sublattice L_1' of L′, in which $A \rightarrow B$ is false. It is easily shown for any sublattice L_1' of L′ that there is a sublattice L_1 of L, of which L_1' is the homomorphic image (with respect to the homomorphic mapping of L onto L′); furthermore, if a first degree entailment is false in L_1', then it is false in L_1. The reason for this is, fundamentally, that the value of the quantified formulas is given by (finite) joins or meets, which are of course preserved by homomorphisms. The presence of quantifiers rightly makes everyone nervous, however, and so we state the needed "little lemma." Where h is a homomorphism from L onto L′ and $\langle \mathrm{L}, \mathrm{v} \rangle$ is a model, define v_h as that mapping from variables into L′ such that $\mathrm{v}_\mathrm{h}(p) = \mathrm{h}(\mathrm{v}(p))$; so $\langle \mathrm{L}', \mathrm{v}_\mathrm{h} \rangle$ is also a model. Fixing L, L′, and h, and recalling that v* is the extension of v to all formulas, we show by induction on formulas that, for v such that $\langle \mathrm{L}, \mathrm{v} \rangle$ is a model,

$$\mathrm{v}_\mathrm{h}^*(A) = \mathrm{h}(\mathrm{v}^*(A)).$$

Having established this, we observe that since L_1' is a nontrivial lattice, L_1 is also a nontrivial lattice, and so $A \rightarrow B$ is falsifiable in L (let v′ be the assignment that falsifies $A \rightarrow B$, define $\mathrm{v}(p) \in \mathrm{h}^{-1}(\mathrm{v}'(p))$, note $\mathrm{v}' = \mathrm{v}_\mathrm{h}$, and apply the "little lemma").

We note that SL is a homomorphic image of M_0 since there is a homomorphic mapping h from M_0 onto SL: $\mathrm{h}(\boldsymbol{1}) = \mathrm{h}(\boldsymbol{f}) = \boldsymbol{1}$, $\mathrm{h}(\boldsymbol{a}) = \mathrm{h}(\bar{\boldsymbol{a}}) = \boldsymbol{a}$, $\mathrm{h}(\boldsymbol{b}) = \mathrm{h}(\bar{\boldsymbol{b}}) = \boldsymbol{b}$, $\mathrm{h}(\boldsymbol{0}) = \mathrm{h}(\boldsymbol{t}) = \boldsymbol{0}$.

§34.2. A consistency theorem. We show that if a first degree entailment is provable in $\mathbf{E}^{\forall\exists p}$ then it is not falsifiable in M_0.

To establish this we extend our notion of a model so that all formulas of $\mathbf{E}^{\forall\exists p}$, not just the zero degree formulas, are mapped into M_0. This requires that we define on the lattice an operation that provides an interpretation of the arrow connective; we make use of an operation of §40.4(4.2). Let $\mathrm{T} = \{\boldsymbol{1}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t}\}$ and $\mathrm{F} = \{\boldsymbol{0}, \bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}, \boldsymbol{f}\}$. Then we define the → operation as follows: If $\mathrm{a} \in \mathrm{T}$ then $\mathrm{a} \rightarrow \mathrm{a} = \mathrm{a}$, and if $\mathrm{a} \in \mathrm{F}$ then $\mathrm{a} \rightarrow \mathrm{a} = \bar{\mathrm{a}}$. Also for each a, $\mathrm{a} \rightarrow \boldsymbol{1} = \boldsymbol{1}$, $\boldsymbol{0} \rightarrow \mathrm{a} = \boldsymbol{1}$, $\boldsymbol{t} \rightarrow \mathrm{a} = \mathrm{a}$, $\mathrm{a} \rightarrow \boldsymbol{f} = \bar{\mathrm{a}}$; $\bar{\mathrm{a}} \rightarrow \mathrm{a} = \mathrm{a}$, $\bar{\mathrm{b}} \rightarrow \mathrm{b} = \mathrm{b}$, $\mathrm{a} \rightarrow \bar{\mathrm{a}} = \bar{\mathrm{a}}$, $\mathrm{b} \rightarrow \bar{\mathrm{b}} = \bar{\mathrm{b}}$. Otherwise $\mathrm{a} \rightarrow \mathrm{b} = \boldsymbol{0}$.

S is an $\mathbf{E}^{\forall\exists p}$ *model* iff it is an ordered pair $\langle \mathrm{Q}, \mathrm{I} \rangle$, where Q is a model $\langle \mathrm{L}, \mathrm{v} \rangle$ with L a nontrivial sublattice of M_0, and I a mapping of the formulas of $\mathbf{E}^{\forall\exists p}$ into L such that the conditions on v* just after the definition of "model" in §34.1 hold for I, and furthermore $\mathrm{I}(A \to B) = \mathrm{I}(A) \to \mathrm{I}(B)$. (Sublattices of M_0 are evidently closed under →, once their bottoms {tops} are identified with *0* {*1*}.)

A formula A of $E^{\forall\exists p}$ belongs to the *$E^{\forall\exists p}$ truth set determined by (an* $\mathrm{E}^{\forall\exists p}$ *model)* S iff $\mathrm{v}(A)$ belongs to T, where T is the set defined above; otherwise A belongs to the $\mathrm{E}^{\forall\exists p}$ *falsity set determined by* S. A formula A is $\mathrm{E}^{\forall\exists p}$*-false in a sublattice* L *of* M_0 iff there is an $\mathrm{E}^{\forall\exists p}$ model S associated with L such that A belongs to the falsity set determined by S. A formula is $\mathrm{E}^{\forall\exists p}$*-true in a sublattice* L *of* M_0 otherwise. A formula A is $\mathrm{E}^{\forall\exists p}$*-falsifiable in a lattice* L iff there is some sublattice of L in which A is $\mathrm{E}^{\forall\exists p}$-false.

It is a tedious but simple task to show that if a formula A is provable in $\mathrm{E}^{\forall\exists p}$ then it is not $\mathrm{E}^{\forall\exists p}$-falsifiable in M_0. This is done by first showing that, for each $\mathrm{E}^{\forall\exists p}$ model on a sublattice of M_0, the axioms belong to the $\mathrm{E}^{\forall\exists p}$ truth set determined by the model. The second part is to show that, if the premisses of either of the rules are not $\mathrm{E}^{\forall\exists p}$-falsifiable in M_0, then the conclusion of the rule is not $\mathrm{E}^{\forall\exists p}$-falsifiable.

CONSISTENCY THEOREM. If a first degree entailment $A \to B$ is provable in $\mathrm{E}^{\forall\exists p}$ then it is not falsifiable in M_0.

From the above we have that if $\vdash A \to B$ then, for each $\mathrm{E}^{\forall\exists p}$-model $\mathrm{S} = \langle \mathrm{Q}, \mathrm{I} \rangle$, where $\mathrm{Q} = \langle \mathrm{L}, \mathrm{v} \rangle$ is a model defined for some nontrivial sublattice L of M_0, $A \to B$ belongs to the $\mathrm{E}^{\forall\exists p}$ truth set determined by S. That is, $\mathrm{I}(A \to B) \in \mathrm{T}$. But $\mathrm{I}(A \to B) \in \mathrm{T}$ iff $\mathrm{I}(A) \leq \mathrm{I}(B)$. Since I agrees with v for zero degree formulas, $\mathrm{v}(A) \leq \mathrm{v}(B)$; so $A \to B$ belongs to the truth set determined by Q. For each model Q defined on a sublattice of M_0 there is a corresponding $\mathrm{E}^{\forall\exists p}$-model on the same lattice, so $A \to B$ belongs to all truth sets determined by models on sublattices of M_0. This means that $A \to B$ is not falsifiable in M_0.

§34.3. Provability theorems. At the beginning of §34, we claimed that there is, for each first degree entailment, a set of primitive entailments that are provable iff the first degree entailment is. To establish this result we introduce the notion of a "critical tree for a first degree entailment." By means of these trees it is possible to determine the appropriate set of primitive entailments.

The required primitive entailments turn up at the tips of the tree for $A \to B$, and a first degree entailment $A' \to B'$, where A' and B' are in prenex normal form (§37.1), is at the bottom. If all the tips are provable, then the tree provides a proof of $A' \to B'$. If some tip is not provable, then $A' \to B'$ belongs to the falsity set F_Q determined by a model Q defined on a nontrivial sublattice of SL. The tree is constructed so that, if $A' \to B'$ belongs to a falsity set F_Q, then all entailments below the unprovable primitive entailment also belong to F_Q.

We begin with some definitions of terms we use in our critical-tree description. Let $C \rightarrow D$ be a first degree entailment.

Suppose first, that $C \rightarrow D$ contains at least one free variable, and each free variable in $C \rightarrow D$, as well as each negation of such, occurs as either a disjunctive part (see §24.1.1) of D or a conjunctive part (defined similarly) of C. Then $C \rightarrow D$ is *well sprinkled.* For example, $\forall p(p\&r)\&r \rightarrow {\sim}r$, and $\forall p(p\&r) \rightarrow .\, r \vee {\sim}r$ are both well sprinkled.

Since, at certain stages in building critical trees, we introduce some "special" substitution instances of quantified formulas, we prepare, here, some terminology for talking about these. The special substitution instances of a quantified formula are a subset of its ordinary substitution instances, and they are defined relative to a well-sprinkled first degree entailment in which the quantified formula occurs. Further, as the arrangement of the atoms in this first degree entailment determines the set of special substitution instances, we begin with a classification of the variables in well-sprinkled first degree entailments.

Let $C \rightarrow D$ be a well-sprinkled first degree entailment. A free variable belongs in *category* α if both the variable and its negate are conjuncts of C; it belongs in *category* β if the variable and its negate are disjuncts of D; and it belongs in *category* γ if the variable is a conjunct {disjunct} of C {D} and its negate is a disjunct {conjunct} of D {C}. This is an exhaustive list of the possibilities. The categories are not in general exclusive; but, since the antecedent and consequent of unprovable primitive entailments do not share atoms (see §15), the categories are exclusive for unprovable primitive entailments; i.e., the variables of an unprovable primitive entailment belong to exactly one of the above categories. We use this information in finding countermodels for unprovable primitive entailments. The following definition is for the general case where a variable may belong in more than one category.

The *special substitution instances of* $\exists pA(p)$ and $\forall pA(p)$ *relative to a well-sprinkled first degree entailment* $C \rightarrow D$, are defined as follows:

1. If there are variables in both categories α and β then these variables determine the complete set of special substitution instances of $\exists pA(p)$ and $\forall pA(p)$ relative to $C \rightarrow D$. Let p and q be the alphabetically first variables in categories α and β, respectively. Then $A(p)$, $A(q)$, $A(p\&q)$, and $A(p \vee q)$ are the special substitution instances of $\exists pA(p)$ and $\forall pA(p)$ relative to $C \rightarrow D$.

2. If there are no such variables, then, if there are variables in either α or β—but not both—and there are variables in γ, let r be the alphabetically first variables in either α or β, and s be the first in γ. Then $A(r)$, $A(s)$, and $A({\sim}s)$

constitute the set of special substitution instances of $\exists pA(p)$ and $\forall pA(p)$ relative to $C \rightarrow D$.

3. If there are variables only in γ, in this case, where q is the alphabetically first such variable, $A(q)$ and $A(\sim q)$ are the special substitution instances of $\exists pA(p)$ and $\forall pA(p)$ relative to $C \rightarrow D$.

In motivating this definition, we look forward to our proof that there is a falsifying model for each unprovable formula. (Our explanation at this stage is brief, for we do not have enough information yet to cover all aspects of the problem.) Sublattices of SL provide the falsifying models. It turns out that the lattice used to falsify a first degree entailment when case 1 occurs is SL itself. In this lattice all variables like p and all variables like q are given as values $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Every other formula takes as value one of the elements of SL; and in particular $p \& q$ and $p \vee q$ have the values $\boldsymbol{0}$ and $\boldsymbol{1}$, respectively. For such a valuation on this lattice $\forall pA(p)$ has the same value as $A(p) \& A(q) \& A(p \& q) \& A(p \vee q)$, which is what we want. If case 2 occurs then all variables in category α, say, are assigned the value $\boldsymbol{a}$, and all variables in category γ are assigned the value $\boldsymbol{1}$ or $\boldsymbol{0}$ according as the variable or its negate occurs in C. In this case the lattice used is 3ch, and $\forall pA(p)$ has the same value in such a model as $A(r) \& A(s) \& A(\sim s)$. If case 3 occurs, all variables are in category γ, and so all variables are assigned the value $\boldsymbol{1}$ or $\boldsymbol{0}$ according as the variable or its negate occurs in C, and the lattice used is just Two.

Now the tree definitions. A *tree* is an ordered triple $\langle G, K, R \rangle$, where K is a set of points, G a member of K, and R a relation on those points, such that (i) there is no $H \in K$ such that HRG; (ii) for each $H \in K$, except G, there is a unique H′ such that H′RH. If HRH′, then H′ is said to be *immediately above H. G is the point at the bottom of the tree.* The points of a tree are divided into levels. The *level* of *G* is 1; and the level of all points immediately above those on the *k*th level is $k+1$. A point $H \in K$ is *terminal* (or *a tip*) iff there is no $H' \in K$ such that HRH′. A sequence $\langle H_1, \ldots, H_n \rangle$ in K is a (terminating) *branch* iff H_1 is G, for each i, $(i < n)$, $H_i R H_{i+1}$, and H_n is terminal.

A critical tree for a first degree entailment $A \rightarrow B$ is an ordered pair $\langle\langle G, K, R \rangle, \psi \rangle$, where $\langle G, K, R \rangle$ is a tree and ψ assigns an entailment to each point K, satisfying five conditions to be given below, partly in terms of the notion of "well instantiated," which we now define.

Let $C \rightarrow D$ be an entailment assigned to a point H. Suppose $\forall pC_1(p)$ $\{\exists pC_1(p)\}$ is a conjunctive part of C. Then $\forall pC_1(p)$ $\{\exists pC_1(p)\}$ is *well instantiated in the antecedent at* H if every special substitution instance of $\forall pC_1(p)$ relative to $C \rightarrow D$ {some substitution instance of $\exists pC_1(p)$ such that the formula replacing p is quantifier-free} occurs as a conjunctive part of either C or some antecedent of an entailment assigned below $C \rightarrow D$.

Suppose $\exists pD_1(p)$ $\{\forall pD_1(p)\}$ is a disjunctive part of D. Then, dually, we can define $\exists pD_1(p)$ $\{\forall pD_1(p)\}$ *as well instantiated in the consequent at H.*

Suppose all the quantified conjunctive parts of C are well instantiated in the antecedent at H, and all the quantified disjunctive parts of D are well instantiated in the consequent at H. Then $C \to D$ is *well instantiated at* H.

We give an example to show how this terminology applies. The entailments in the example constitute a piece of a branch of a critical tree for $\forall p(p\&r) \to \forall pp \vee r$.

8 $((r\&r)\&(\sim r\&r))\&\sim r\&\forall p(p\&r) \to \forall pp\vee r$
7 $\forall p(p\&r)\&\sim r \to \forall pp\vee r$
6 $\forall p(p\&r) \to \forall pp\vee r$

The entailment on level 6 is neither well sprinkled nor well instantiated. The entailment on level 7 is well sprinkled, but it is not well instantiated at that point. The entailment on level 8 is both well sprinkled and well instantiated.

Now for the five conditions defining a critical tree for $A \to B$.

1. Let A' and B' be the result of changing A and B into their prenex normal forms (see §37.1) and then, for convenience, replacing the n $(n \geq 0)$ free variables in A and B consistently throughout by the alphabetically first n variables occurring neither bound in A nor bound in B. Then ψ assigns $A' \to B'$ to G.

So at the bottom of a critical tree there is a first degree entailment $A' \to B'$ with all quantifiers in A' and B' initially placed; at the tips of the tree there will be primitive entailments. Clauses 2–5 below show how $A' \to B'$ is to be unpacked (by stages going up the tree) in favor of primitive entailments. The unpacking is carried out in such a way that the tree has two properties: first, if $A' \to B'$ is provable, then the tree provides a proof of $A' \to B'$; second, if $A' \to B'$ is not provable, then the tree provides a falsifying model for $A' \to B'$.

The next two clauses apply for entailments that contain quantifiers. We can assume throughout that all entailments containing quantifiers have as antecedents {consequents} conjunctions {disjunctions} of formulas—including quantifier-free formulas—in prenex normal form, since in going up a tree we add *only* such conjunctive {disjunctive} parts.

2. Let $C \to D$ be an entailment containing quantifiers, and suppose $C \to D$ is either not well sprinkled or not well instantiated at a point H.

(i) If the level of H is $3k$ we sprinkle as follows. Let P be the set of variables that occur free in $C \to D$, or, if there aren't any, then let P be the unit set containing some arbitrarily chosen variable. Let P′ be the set (possibly empty) of variables among those in P which do not occur both negated and unnegated in $C \to D$, as either conjunctive parts of C or disjunctive parts of D. Then, if P′ is nonempty, immediately above $C \to D$ are assigned all possible well-sprinkled entailments of the form $C\&A_1 \to D\vee B_1$, where A_1 is a nonrepeating conjunction of atoms, and B_1 is a nonrepeating disjunction of atoms, such that, for each $p_i \in \mathrm{P}'$, where F is either p_i or $\sim p_i$, F occurs either as a conjunctive part of A_1 or as a disjunctive part of B_1, but not both. If P' is empty, i.e., $C \rightarrow D$ is well sprinkled, then $C \rightarrow D$ is assigned to the point immediately above.

In clause (i) we are preparing to instantiate at the next level either a universally quantified conjunctive part of C or an existentially quantified disjunctive part of D.

If $C \rightarrow D$ is not provable, there is (it has yet to be proved) an assignment to the variables in $C \rightarrow D$ falsifying $C \rightarrow D$ in a sublattice of SL. Each entailment immediately above $C \rightarrow D$ "represents" one of the possible falsifying assignments. We see later that falsifying assignments are defined in terms of the arrangements of variables in unprovable primitive entailments at the tips of critical trees; so what (i) does, in effect, is to bring information contained at the tips down to the places at which instantiation is being carried out. Further, so that we utilize all the information available at any one point, the sprinkling achieved by (i) must be carried out immediately before instantiation.

Therefore, since instantiations of existential and universal quantifiers are treated in separate stages of (ii), the sprinkling of (i) must occur twice in each tree-building cycle.

(ii) The level of H is $6k+1$ $\{6k+4\}$. Suppose at least one universally {existentially} quantified conjunctive {disjunctive} part of C $\{D\}$ is not well instantiated at H. Let $\forall pC_1(p)$ $\{\exists pD_1(p)\}$ be the leftmost of these, and let $C = \theta(\forall pC_1(p))$ $\{D = \varphi(\exists pD_1(p))\}$. (We use $\varphi(A)$ as in §24.1.1 for a formula with A as a disjunctive part, and $\theta(A)$, similarly, for a formula with A as a conjunctive part.) Further, let C'_1 $\{D'_1\}$ be a conjunction {disjunction} of the special substitution instances of $\forall pC_1(p)$ $\{\exists pD_1(p)\}$ relative to $C \rightarrow D$, which are not conjunctive {disjunctive} parts of either C $\{D\}$ or of any antecedent {consequent} below $C \rightarrow D$. Then

$$(\theta(C'_1)\&\forall pC_1(p)) \rightarrow D \ \{C \rightarrow (\varphi(D'_1)\vee\exists pD_1(p))\}$$

is assigned to the point immediately above $C \rightarrow D$.

If C $\{D\}$ does not have at least one such conjunctive {disjunctive} part, then $C \rightarrow D$ is assigned to the point immediately above.

Our reason for keeping the quantified formula is that until we have an entailment that is both well sprinkled and well instantiated at a point, we cannot be sure we have all the substitution instances required for falsifiability to go down the tree.

In (iii) the remaining quantifiers are instantiated.

(iii) The level of H is $6k+2$ $\{6k+5\}$. Suppose at least one existentially {universally} quantified conjunctive {disjunctive} part of C $\{D\}$ is not well instantiated at H. Let $\exists pC_1(p)$ $\{\forall pD_1(p)\}$ be the leftmost of these, and let $C = \theta(\exists pC_1(p))$ $\{D = \varphi(\forall pD_1(p))\}$. Let q be the alphabetically first variable that does not appear in $C\rightarrow D$ or in any entailment below $C\rightarrow D$. Then $\theta(C_1(q))\rightarrow D$ $\{C\rightarrow\varphi(D_1(q))\}$ is assigned to the point immediately above H.

If there is no such conjunctive {disjunctive} part of C $\{D\}$, then $C\rightarrow D$ is assigned to the point immediately above.

3. Let $C\rightarrow D$ be a well-sprinkled, well-instantiated entailment containing quantifiers, at a point H. Let C' be a conjunction whose conjunctive parts are all and only the quantifier-free conjunctive parts of C, and let D' be a disjunction whose disjunctive parts are all and only the quantifier-free disjunctive parts of D. Then $C'\rightarrow D'$ is assigned to the point immediately above H.

4. Suppose $C\rightarrow D$ is quantifier-free but not a primitive entailment. Let $C_1\vee\ldots\vee C_n$ be a disjunctive normal form of C and let $D_1\&\ldots\&D_n$ be a conjunctive normal form of D. Then immediately above $C\rightarrow D$ are assigned all entailments $C_i\rightarrow D_j$ such that $C_i\in\{C_1,\ldots,C_m\}$ and $D_j\in\{D_1,\ldots,D_n\}$.

5. If $C\rightarrow D$ is a primitive entailment at a point H, then H is a terminal point.

CRITICAL TREE THEOREM. Let $A\rightarrow B$ be a first degree entailment. The branches of a critical tree for $A\rightarrow B$ terminate in provable primitive entailments iff $\vdash A\rightarrow B$.

FIRST CRITICAL TREE LEMMA. If the branches of the critical tree for $A\rightarrow B$ terminate in provable primitive entailments then each entailment in the tree is provable.

Let $C\rightarrow D$ be an entailment at a point H. We employ a strong induction on "length of proof," where the *length of proof of an entailment* $C\rightarrow D$ is the distance (counting nodes) $C\rightarrow D$ is from the tip of the longest branch through the point to which $C\rightarrow D$ is assigned.

We treat some cases where $C\rightarrow D$ contains quantifiers.

1. Suppose $C\rightarrow D$ is well sprinkled and well instantiated at H. Let C' be a conjunction of the conjunctive parts of C that are quantifier free, and let D' be a disjunction of the disjunctive parts of D that are quantifier free. Then $C'\rightarrow D'$ is immediately above $C\rightarrow D$. By the inductive hypothesis, $\vdash C'\rightarrow D'$. Since $\vdash C\rightarrow C'$ and $\vdash D'\rightarrow D$, $\vdash C\rightarrow D$.

2. $C\rightarrow D$ is either not well sprinkled or not well instantiated. (i) Suppose H is at a level $3k$. Then immediately above $C\rightarrow D$ are entailments having the form $C\&A_1\rightarrow D\vee B_1$, where A_1 and B_1 are defined as they are in the critical-tree definition. By the inductive hypothesis, all these entailments are provable. $\vdash C\rightarrow D$ follows by Fact 8 of §19.4.

(ii) Suppose H is a point at a level $6k+1$. We consider the case where not all universally quantified conjunctive parts of C are well instantiated in the antecedent at H. Let $\forall pC_1(p)$ be the leftmost of these, and let $C = \theta(\forall pC_1(p))$. Then immediately above $C \to D$ is $(\theta(C_1')\&\forall pC_1(p)) \to D$, where C_1' is a conjunction of some of the special substitution instances of $\forall pC_1(p)$ relative to $C \to D$. Since $\vdash \forall pC_1(p) \to C_1'$ and since, by the inductive hypothesis, $\vdash (\theta(C_1')\&\forall pC_1(p)) \to D$, $\vdash C \to D$.

(iii) Suppose H is at a level $6k+2$. We consider the case where not all the existentially quantified conjunctive parts of C are well instantiated in the antecedent at H. Let $\exists pC_1(p)$ be the leftmost of these, and let $C = \theta(\exists pC_1(p))$. Then immediately above $C \to D$ is $\theta(C_1(q)) \to D$ for some q that does not occur free in $C \to D$. By the inductive hypothesis, $\vdash \theta(C_1(q)) \to D$, which yields by universal generalization $\vdash \forall q(\theta(C_1(q)) \to D)$. Since q is not free in D, $\vdash \exists q\theta(C_1(q)) \to D$, since q is not free in C, $\vdash \theta(\exists qC_1(q)) \to D$; therefore, by change of bound variables, $\vdash C \to D$.

Let $A' \to B'$ be the entailment at the bottom of the critical tree for $A \to B$. Then $\vdash A' \to B'$. Since A' and B' are variable variants of A and B in prenex normal form, $\vdash A \to B$.

SECOND CRITICAL TREE LEMMA. If $\vdash A \to B$, then every branch in the critical tree for $A \to B$ terminates in a provable primitive entailment.

We prove the contrapositive. Suppose some branch of the critical tree for $A \to B$ does not terminate in a provable primitive entailment. Then $A \to B$ is not provable.

It is almost obvious that all critical trees are finite. The proof relies principally on the following two facts. First, there are only three cases in the definition of "special substitution instance" and, accordingly, only three ways in which a quantified statement can fail to be well instantiated and, therefore, at most three occasions on which an instantiation will be called for. Secondly, since each set of special substitution instances is finite, our variables take as values only a finite number of elements. This means that only a finite number of substitution instances are introduced each time a formula is instantiated. As soon as all conjunctive and disjunctive parts are well instantiated, the branch is terminated.

Since the critical tree for $A \to B$ is finite, we can suppose there is a primitive entailment at a tip which is not provable. Let $E \to F$ be such a primitive entailment. Let $A' \to B'$ be the entailment at the bottom of the tree. We show first that $A' \to B'$ is not provable.

Since $E \to F$ is well sprinkled, and E and F fail to share an atom, we can define a mapping v of the propositional variables in $E \to F$, into SL as follows. (α) If a variable p and its negate $\sim p$ occur in E, then $\mathrm{v}(p) = \boldsymbol{a}$. ($\beta$) If a variable p and its negate $\sim p$ occur in F, then $\mathrm{v}(p) = \boldsymbol{b}$. ($\gamma$) If a variable p occurs in E and $\sim p$ occurs in F, then $\mathrm{v}(p) = \boldsymbol{1}$; or, if $\sim p$ occurs in E and p in F, then $\mathrm{v}(p) = \boldsymbol{0}$.

The table below sketches for one variable the details of this mapping.

Category	Occurrences of p and $\sim p$ in E and F		$v(p)$
	E	F	
α	$p, \sim p$		***a***
β		$p, \sim p$	***b***
γ	p	$\sim p$	***1***
	$\sim p$	p	***0***

Let L′ be the set of elements of SL onto which v maps the variables of $E \rightarrow F$. Let L be the smallest sublattice of SL containing L′. As no free variables are lost going up critical trees, v maps the free variables of $A' \rightarrow B'$ into L. Q $= \langle$L, v$\rangle$ is a model. Let $C \rightarrow D$ be an entailment assigned to some point below $E \rightarrow F$. Then we show that $C \rightarrow D \in \mathrm{F_Q}$, i.e., $v(C) \not\leq v(D)$.

1. $C \rightarrow D$ is quantifier free. We treat only the case where C is in disjunctive normal form and D is in conjunctive normal form. Then E is a disjunctive part of C and F is a conjunctive part of D. Therefore $v(E) \leq v(C)$ and $v(D) \leq v(F)$. Since $v(E) \not\leq v(F)$, $v(C) \not\leq v(D)$.

2. $C \rightarrow D$ contains quantifiers. Let $C_1 \rightarrow D_1, \ldots, C_m \rightarrow D_m$ be a list in order of levels, of all the entailments below $E \rightarrow F$ which contain quantifiers. Then $C_m \rightarrow D_m$ is both well sprinkled and well instantiated. For some i, $1 \leq i \leq m$, let C'_i be a conjunctive part of C_i. We show by an induction on the number of quantifiers in C'_i that $v(E) \leq v(C'_i)$.

(i) C'_i is quantifier-free. Let $C' \rightarrow D'$ be the lowest (i.e., at the lowest level) quantifier-free entailment below $E \rightarrow F$. Then C'_i is a conjunctive part of C', and so $v(C') \leq v(C'_i)$. It is easily shown that $v(E) \leq v(C')$, from which it follows that $v(E) \leq v(C'_i)$.

(ii) C'_i is $\forall pH(p)$. Let S $= \{G\colon H(G)$ is a special substitution instance of $\forall pH(p)$ relative to $C_m \rightarrow D_m\}$. Then for each $G \in$ S, there is an i such that $H(G)$ is a conjunctive part of C_i, and by the inductive hypothesis $v(E) \leq v(H(G))$. Since for each element a of L, there is a $G \in$ S, such that $v(G) = $ a, $v(E) \leq v(\forall pH(p))$.

(iii) C'_i is $\exists pH(p)$. $\exists pH(p)$ may be a conjunctive part which is lost from antecedents going up the tree, or it may be a conjunctive part of C_m. In either case, among the conjunctive parts of $C_i, \ldots, C_m$ will be a substitution instance of $\exists pH(p)$ in which the formula substituted for p is quantifier-free. Let $H(G)$ be such a substitution instance of $\exists pH(p)$. Since G is quantifier-free, $H(G)$ has fewer quantifiers than $\exists pH(p)$, and the inductive hypothesis

applies; so $v(E) \leq v(H(G))$. Obviously, $v(H(G)) \leq v(\exists pH(p))$. Therefore, $v(E) \leq v(\exists pH(p))$.

Since, for each conjunctive part C_i' of C_i, $v(E) \leq v(C_i')$, $v(E) \leq v(C_i)$. C is one of the C_i's; so $v(E) \leq v(C)$.

Similarly it can be shown that $v(D) \not\leq v(F)$. It follows, from the fact that $v(E) \not\leq v(F)$, that $v(C) \not\leq v(D)$.

Therefore, $A' \to B' \in \mathrm{F}_{\mathrm{Q}}$.

We have shown that, if a primitive entailment at the tip of a tree for a first degree entailment $A \to B$ is unprovable, then there is a model Q on some nontrivial sublattice of SL such that $A' \to B' \in \mathrm{F}_{\mathrm{Q}}$. That is, $A' \to B'$ is false in L, and therefore $A' \to B'$ is falsifiable. According to the Falsifiability theorem in §34.1, if $A' \to B'$ is falsifiable in SL, then it is falsifiable in M_0, since SL is a homomorphic image of M_0. In §34.2 it was shown that if a first degree entailment is provable then it is not falsifiable in M_0. Therefore, if a primitive entailment at the tip of a critical tree for $A \to B$ is unprovable, then $A' \to B'$, which is the first degree entailment at the bottom of the tree, is unprovable. That is, if $\vdash A' \to B'$ then all the primitive entailments are provable. Since $\vdash A' \to B'$ iff $\vdash A \to B$, $\vdash A \to B$ iff all primitive entailments in the critical tree for $A \to B$ are provable.

§34.4. Completeness and decidability.

COMPLETENESS THEOREM FOR FIRST DEGREE ENTAILMENTS. A first degree entailment is provable iff it is valid.

If $\vdash A \to B$ then $A \to B$ is not falsifiable in M_0 (§34.2), and therefore it is not falsifiable in SL (§34.1); i.e., $A \to B$ is valid. If $A \to B$ is not provable then at least one primitive entailment in the critical tree for $A \to B$ is not provable, and therefore $A \to B$ is falsifiable (§34.3); i.e., $A \to B$ is not valid. Therefore, if $A \to B$ is valid it is provable.

DECIDABILITY THEOREM FOR FIRST DEGREE ENTAILMENTS. There are at least two effective ways of determining whether a first degree entailment is provable in $\mathrm{E}^{\forall\exists p}$. The first is to determine whether it belongs to the truth set of each model on each nontrivial sublattice of SL. The second is to build its finite critical tree and then check whether the primitive entailments at the tips are provable, i.e., whether each primitive entailment is such that its antecedent and consequent share at least one atom.

§35. Enthymemes. In this section we deploy our propositional quantifiers in order to get clear on a classical category of arguments, namely, enthymemes, discussions of which enjoyed a certain vogue toward the end of (what Bocheński has called) the Dark Ages in logic (i.e., from the beginning of the Renaissance up until near the turn of the present century).

§35.1. Intuitionistic enthymemes. We take our text from Jevons 1870 (pp. 153–154):

> A syllogism when incompletely stated is usually called an *enthymeme*, and this name is often supposed to be derived from two Greek words (ἐν, in, and ϑυμός, mind), so as to signify that some knowledge is held by the mind and is supplied in the form of a *tacit*, that is a silent or understood premise.... Of this nature is the following argument: "Comets must be subject to the law of gravitation; for this is true of all bodies which move in elliptic orbits." It is so clearly implied that comets move in elliptic orbits, that it would be tedious to state this as the minor premise in a complete syllogism of the mood Barbara, thus:
>
> | | |
> |---|---|
> | [Major premise (M)] | All bodies moving in elliptic orbits are subject to the law of gravitation; |
> | [Minor premise (m)] | Comets move in elliptic orbits; |
> | [Conclusion (C)] | Therefore comets are subject to the law of gravitation. |

Now everyone agrees that the syllogism as stated in full is valid and that a definition of "validity" that denied validity to the syllogism in Barbara would *ipso facto* be faulty. It is also conceded that corresponding to this form of inference there exists a certain *true* proposition, namely,

If M and m, then C.

The case of the enthymematic inference and the corresponding proposition

If M then C

may, however, occasion some doubt. Asked whether the enthymematic inference is valid, or whether the proposition just above is true, we may want to answer either "No, your premisses are simply insufficient for your conclusion" or "Yes, provided you mean to be using the obviously required premiss m (which we grant that you are, we being in a tolerant mood)."

Now, whichever attitude we may wish to take, it is clear that there is a difference: in the first case there is a robust relation of *logical* consequence between premisses and conclusion; in the second the logical connection is largely, if not entirely, by courtesy. And notice that no one has any difficulty in seeing the clear difference. So much, at any rate, we wish to take for granted; and we now turn to our topic, which is what to make of the situation. In particular, what sort of formal analysis of "if... then —" and "and" can do justice to the clearly felt distinction considered above?

For present purposes, it suffices to treat "and" in a purely truth-functional way: "M and m" is true if and only if M is true and also m is true.

We now ask whether the true "if... then —", corresponding to the syllogism, should be represented by the material "implication":

$$M \& m \supset C.$$

We have argued at length (and somewhat peevishly; see, e.g., §1.1) that material "implication" is simply not a kind of implication. We do not plan to repeat those arguments here; rather we shall offer some fresh arguments for the same thesis. Our point, incidentally, is also telling as against the view that intuitionistic "implication" is a relation of entailment, or logical deducibility.

The present argument is simply this: both views make hash of the distinction between logically valid arguments and enthymemes. For on both theories we have as a "theorem of logic" a principle according to which true premisses aren't really there at all ("true premisses may be suppressed"):

$$\begin{array}{l} m \\ M \& m \supset C \\ \hline M \supset C. \end{array}$$

Suppose now that an argument is valid if and only if the corresponding material or intuitionistic "implication" statement is true. Then, since the argument from M and m to C is valid, $M \& m \supset C$ is true; and since m is true as well and, hence, suppressible, $M \supset C$ is true; hence the argument from M to C is valid, and in precisely the same sense as is the argument from M and m to C. But to say that the argument from M to C is thus valid is in direct contradiction to the doctrine that enthymematic arguments suppress *required* premisses. "Where a necessary premise is missing, without that premise the inference is invalid" (Copi 1953, p. 205).

The conclusion seems inescapable that one cannot both hold that material and intuitionistic "implication" relations answer to the notion of a valid argument *and* try to maintain a distinction between valid arguments and enthymemes.

§35.2. Strict enthymemes. Now it is not unmistakably clear whether or not anyone ever seriously took material implication to explicate "valid argument," but we have no doubt on this score in respect to C. I. Lewis's theory of strict implication. We are told that the meaning of strict implication "is precisely that of ordinary inference and proof" (Lewis 1912, p. 531; see also Lewis and Langford 1932, p. 247).

And in the systems of strict implication, one *can* maintain a distinction between valid arguments and enthymemes, at least those of the sort we have considered above. In particular, if we put hooks for horseshoes, a conjoined premiss may not be dropped simply on the grounds that it is true. "That is, in stating a strict implication one cannot omit a merely true premise which is one of a set of premises which together give the conclusion" (Lewis and Langford 1932, p. 165).

So far so good; but now consider the following argument:

> Copi, in the passage quoted above, is *clearly* correct in holding that valid arguments may be distinguished from enthymemes and that enthymemes as they stand (i.e., without statement of the required supporting premiss) are not valid. Therefore, if all bodies moving in elliptic orbits are subject to the law of gravitation and if comets move in elliptic orbits, then comets are subject to the law of gravitation.

The reader may find the reasoning in the foregoing cooked-up paragraph somewhat difficult to follow, so we forthwith lay bare its logical form. Where A represents "enthymemes may be distinguished from valid arguments" and B represents "enthymemes as they stand are not valid," the argument has the form:

$$\frac{A \text{ and } B}{\text{If } M \text{ and } m \text{ then } C.}$$

When one is asked whether this argument is valid, the initial reaction is likely to be "No," perhaps because the premisses A and B have nothing to do with the conclusion: "If M and m then C." But there have been sophisticated logicians, C. I. Lewis for example, and Bennett 1969, who, on reflection, decided that the argument *is* valid, on the grounds that the consequent, "If M and m then C," is necessary. But this view is debatable (notoriously), and the notion that a necessary proposition is entailed by any old proposition has been heatedly and repeatedly attacked. That is, even apart from us, there have been those who (1) have held that the lack of relevance between premiss and conclusion, just above, reduces to absurdity the claim that strict implication captures intuitive entailment, and those who (2) have defended strict implication on the grounds that the well-known paradoxes do indeed represent "facts about deducibility." We of course side with the first group; the ludicrous argument above cannot really be regarded as valid. But sophisticated logicians must have something in mind, and we shall try (again) later to give such devils their due. Still, the argument above is no more valid as it stands than is the move from "all bodies moving in elliptic orbits are subject to the law of gravitation" to "comets must be subject to the law of gravitation," i.e., the move from M to C.

But now an analogy suggests itself. Couldn't we try to view the odd argument cited above as somehow enthymematic and by this irenic maneuver pacify both parties to the dispute? For *everyone can see* that

$$\frac{A \text{ and } B \text{ and (if } M \text{ and } m \text{ then } C)}{\text{If } M \text{ and } m, \text{ then } C}$$

is valid, and that

$$\frac{A \text{ and } B}{\text{If } M \text{ and } m, \text{ then } C}$$

arises from it by suppressing a true (indeed necessarily true) premiss. And *everyone can see* that there is a difference between these two cases; else there could have been no sensible dispute in the first place.

Of course these latter enthymemes ("strict enthymemes") differ from those of §35.1 (which, for reasons to be brought up later, we shall call "intuitionistic enthymemes") in that the suppressed premiss is *necessarily* true, rather than *simply* true. But the suppression of necessary premisses is, if anything, less objectionable than the supression of contingently true ones; necessarily true premisses are suppressed constantly in arguments in mathematics and in logic, simply because one does not want (ordinarily) to bore one's audience with endless repetition of obvious logical truths.

At any rate we shall take it, for the present, that the sense behind the claim that a necessary proposition is entailed by any old proposition is that, in the alleged entailment, a necessary premiss is suppressed: to wit, the conclusion. And we offer this view as face-saver for both parties. To take another example, the statement "If *A*, then either *A*-and-*B* or *A*-and-not-*B*" can be regarded as *true*, all right; we have just left out an obvious, and obviously required, premiss: "*B* or not-*B*" (see, *per contra*, Bennett 1969).

But if we are asked whether strict implication can be taken as an account of entailment, or logical deducibility, the answer must again, as in the case of material and intuitionistic "implication," be "No." For, as Lewis himself points out, strict implication is enthymematic in the sense we have just described; necessarily true premisses may be suppressed. "The omission of a premise which is *a priori* or logically undeniable does not affect the validity of deduction" (Lewis and Langford 1932, p. 165). How *could* this be so? Can we make *no* distinction between valid and invalid arguments when, as may happen, all the propositions in the argument are necessary? We believe, rather, that the situation is as the citation above from Copi may be misinterpreted as saying: "Where a necessary premise is missing, without that premise the inference is invalid." Necessary premisses are just as necessary as premisses that are not necessary.

§35.3. Enthymematic implication in $\mathbf{E}^{\forall\exists p}$. So now we turn to a discussion of enthymemes within the context provided by the system $\mathbf{E}^{\forall\exists p}$ of entailment with propositional quantifiers. What has been required, classically, of "valid" enthymemes? Two things at least: (1) that it be obvious what the suppressed premiss is; (2) that the suppressed premiss be obviously true. Degrees of obviousness are of course psychological matters, which we dare not touch. But when the psychological requirements of obviousness are stripped away, we are left with a consideration that logic should be able to handle: the suppressed premiss must be *true*.

This leads us to suggest that if we are right in maintaining that $\mathbf{E}^{\forall\exists p}$ correctly captures the intuitive notion of a valid argument then (allowing ourselves propositional quantifiers) the following would be a reasonable definition of an enthymematic "if... then —" (which we symbolize by$\rightarrow *$):

$$A \rightarrow * B =_{\text{df}} \exists r(r \& (r \& A \rightarrow B)).$$

That is, we want to say that the "if... then —" proposition corresponding to any enthymematic argument is true if and only if there is a (suppressed) premiss which is true and which would convert the enthymeme into a valid argument.

Rescher 1962, in reviewing our 1961 paper on enthymemes, points out that our choice of terminology is debatable for the $\rightarrow *$ defined above, for several reasons. For one, we always have, e.g.,

$$A \rightarrow B \rightarrow. (A \rightarrow B) \&. (A \rightarrow B) \& A \rightarrow B,$$

whence

$$A \rightarrow B \rightarrow. A \rightarrow * B,$$

which says that valid entailments are also enthymematic, a use of the latter term which fails to square with the traditional use. Worse, if B is true we have $B \&. (B \& A) \rightarrow B$ for arbitrary A; so apparently fallacies are introduced wholesale from a true B by way of $A \rightarrow * B$. Finally, it seems clear that in traditional accounts it was intended that an enthymeme suppress something that is *required* to be there for validity, a feature left out entirely of $\rightarrow *$. What is suppressed might even be the conclusion; again Jevons 1870:

> It may happen occasionally that the conclusion of a syllogism is left unexpressed, and the enthymeme may then be said to belong to the Third Order (p. 154).

But if it is one of the premisses, presumably the other by itself is not supposed to do the job of guaranteeing the conclusion in an enthymeme.

The situation is made more difficult by the fact that enthymemes are discussed seriously, so far as we have been able to find, *only* in connection with traditional syllogistic logic, and it is simply anyone's guess how to broaden the notion so as to make it amenable to treatment in propositional logic.

We hold no brief for the terminological choice, beyond noting that the etymological remark quoted from Jevons at the beginning of this section is suggestive of what we have in mind when we call $\rightarrow^*$ *enthymematic*. We will continue at any rate to call the inference from A to B when $A\rightarrow^*B$ an enthymeme—a *propositional enthymeme*, if necessary to avoid confusion—and think of it as suppressing premisses which may or may not be required.

When no more than *contingent* truth of the suppressed premiss is required, we have enthymemes of the sort previously referred to as "intuitionistic." We now justify that terminology.

That intuitionistic "implication" is a relation of logical implication, or deducibility, has not been maintained even by some of its strongest proponents. Thus Curry 1959:

> The absoluteness of absolute [i.e., intuitionistic] implication does not depend on any claim to its being a definition of logical consequence. It does not pretend to be anything of the sort (p. 20).

And Heyting 1956 writes:

> The *implication* $p\rightarrow q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q. In other words, a proof of p, together with r, would form a proof of q (p. 98).

One could hardly ask for a clearer or more explicit statement to the effect that intuitionistic "implication" is *enthymematic*; i.e., when $p\rightarrow q$ is intuitionistically true, there is a true premiss r, *which may be required* for the construction of q from p, but which is omitted in the antecedent p.

Now apparently Heyting has in mind a distinction between the sense in which p intuitionistically "implies" q (in the presence of r) and some other sense of "if . . . then —" in which "if p and r then q" is true. Nothing is said about the latter relation, nor indeed can the intuitionistic formalism reflect the intuitive distinction between "implies" and "would automatically effect a construction." But if we are correct in claiming that the definition above captures the notion of an enthymematic "if . . . then —" and if Heyting is correct in saying that intuitionistic "implication" is enthymematic, then we should expect that intuitionistic "implication" and our enthymematic "implication" should exactly coincide.

And this turns out to be the case: if in $\mathbf{E}^{\forall\exists p}$ we define

$$A\supset B =_{\mathrm{df}} \exists r(r\&(r\&A\rightarrow B)),$$

then the set of all theorems of $\mathbf{E}^{\forall\exists p}$ that contain only $\supset$, &, and $\vee$ *coincides exactly* with the positive fragment of intuitionistic logic as formalized by

Heyting. That is, "A intuitionistically 'implies' B" *means* that there is some true proposition r such that the conjunction of r and A *entails* B. (This idea was suggested by Myhill 1953. We originally had a proof only for the positive fragment $\mathbf{E}_{+}^{\forall\exists p}$ of $\mathbf{E}^{\forall\exists p}$, which we present in §36.1; but Meyer improved the result to all of $\mathbf{E}^{\forall\exists p}$, as reported in §36.2 below.)

So we may justifiably (and in conformity with Heyting's own intuitive discussion) think of the intuitionistic horseshoe as expressing a notion of enthymematic (intuitionistically enthymematic) "implication": the antecedent doesn't *really* guarantee the truth of the consequent—but there *are* some true propositions which, together with the antecedent, could really be shown to have the consequent as a logical consequence.

Some say that entailment is a "simply psychological" matter and that the only difference between (say) the law of transitivity and the "law" $A \supset . B \supset A$ is that a nice, warm psychological feeling occurs in the presence of the former and a somewhat sick psychological feeling in the presence of the latter. We think that the result just stated justifies a charge of psychologism in the other direction. The *only* reason anyone could ever have had for thinking that we could justifiably omit required premisses that happened to be true is that *we all knew* that they were true, and the fact just didn't need to be mentioned (in the social context of the argument). At least it is hard to see any other reason why anyone would think (harking back to an earlier example) that the move from M to C was justified at all.

If, moreover, we define intuitionistic negation as follows:

$$\neg A =_{\mathrm{df}} A \supset \forall pp,$$

then the set of theorems of $\mathbf{E}_{+}^{\forall\exists p}$ containing only $\supset$, $\neg$, &, and $\vee$ *coincides exactly* with the full system of intuitionistic propositional calculus. That is, "A is intuitionistically false" *means* that A intuitionistically "implies" that every proposition is true.

We take these facts as evidence that, when people have talked of the horseshoe as a relation of "implication" or "entailment" or "deducibility" or any of a host of equivalents, they have *really* had in mind (at best) a relation of (intuitionistically) *enthymematic* "implication," "entailment," "deducibility," etc.

Turning from intuitionistic enthymemes to strict enthymemes, we observe that the appropriate definition (in $\mathbf{E}^{\forall\exists p}$) of a strictly enthymematic "if... then —" is

$$A \mathbin{-\!\!\!3} B =_{\mathrm{df}} \exists r(\Box r \& (r \& A \rightarrow B)).$$

And it turns out that the set of theorems of $\mathbf{E}^{\forall\exists p}$ that contain only $\mathbin{-\!\!\!3}$, &, and $\vee$ *exactly coincides* with the positive fragment of Lewis's system **S4** of strict implication, in a formulation where the primitives are $\mathbin{-\!\!\!3}$, &, and $\vee$.

That is, "A strictly 'implies' B" (in the sense of the positive part of **S4**) *means* that there is some necessarily true proposition r such that the conjunction of r and A *entails* B. Again, the intuitive considerations are borne out: if we consider only *strict* enthymemes, in which the discarded premiss must be necessary, then we get precisely the (positive) system advocated by the advocates of **S4**. We give some details in §36.2.

Why the limitation here to negation-free **S4**? One does not need to look far to see that the restriction is rooted in relevance considerations, which forbid that $\mathbf{E}^{\forall\exists p}$ contain *all* of **S4** under the definition listed above; for consider that enduring paradigm among the negation-containing irrelevant theorems of **S4**,

$$A\&{\sim}A \mathrel{-\!\!\!3} B.$$

If we tried to explain away this modal curiosity by treating it as a strict enthymeme in $\mathbf{E}^{\forall\exists p}$ in the sense given above, it would come to

$$\exists r(\Box r\&(r\&A\&{\sim}A \rightarrow B)),$$

which has a decidedly irrelevant appearance in $\mathbf{E}^{\forall\exists p}$ (compare §25.1). That is, the strictly positive version of strict enthymematic implication is inappropriate for explaining all the irrelevant curiosities of **S4**, because it does not properly cater to negation. (When we translate the parallel intuitionist negation-based oddity $A\&\neg A\supset B$ into $\mathbf{E}^{\forall\exists p}$, the parallel problem does not arise because in the case of intuitionism, we also translate the negation—as $A\supset\forall pp$!)

We can, however, find a different strict enthymematic "implication" that does the job of embedding all of **S4**, including its negation, in $\mathbf{E}^{\forall\exists p}$: we can show that defining $A \mathrel{-\!\!\!3} B$ by

$$\exists r(\Box r\&(r\&A \rightarrow. B\vee{\sim}r))$$

catches all and only the theorems of **S4** in $\mathbf{E}^{\forall\exists p}$. This result of Meyer 1970a is closely connected to the conjecture of Ackermann's that we discuss in §46.1; it is not, however, as interesting as the foregoing positive results in relation to our particular project, because this second account of the hook is given in terms of only an "implication" and not an implication: in absolute contrast to our positive enthymematic implications, it would be as much a fallacy to suppose that detachment held for the definiens displayed above as it would be to suppose that detachment held for material "implication."

§35.4. Summary. To summarize, we claim in this section to have established (or commented on the establishability of) the following (obvious) truths:

(1) A distinction can be drawn between valid arguments and enthymemes and between enthymemes of two sorts: strict and intuitionistic. This is so

trivial a point that it hardly seems worth arguing for. It *does* need arguing for, but that is only because currently accepted analyses of *implication* are so goofy.

(2) Neither the horseshoe nor the hook will do as a basis for analysis of the notion of a *valid argument* (most elementary and advanced texts to the contrary notwithstanding), since neither theory can distinguish between valid arguments and strict enthymemes.

(3) The formal system $\mathbf{E}^{\forall\exists p}$ of entailment with propositional quantifiers *does* suffice as an explication of the intuitive notion of entailment, or formal deducibility, and *does* account for the differences among valid arguments, strict enthymemes, and intuitionistic enthymemes.

(4) What are we to think about enthymematic arguments? *A priori*, one might say that the notion of an enthymematic argument is sociological (depending on what we can expect our hearers to know) or psychological (depending on what *we* claim to know) and therefore is *a*logical: it is not a part of logic to try to take account of who knows what; and anyway our intuitions concerning enthymemes are altogether too unstable to support any kind of mathematical or logical theory. We on the contrary maintain that there is nothing intrinsically *bad* about sociology and psychology, even from a logician's point of view. Moreover, one can give a precise interpretation to the (admittedly, at the outset, somewhat *vague*) notion of an enthymematic argument. Thus:

(5) If we are sufficiently careless in stating our arguments so that we allow ourselves to suppress any true premiss, then we arrive *precisely* at the intuitionistic theory of "implication."

(6) If we are somewhat more careful, and allow ourselves to suppress only necessarily true premisses, then we arrive *precisely* at strict "implication" (in the sense of $\mathbf{S4}_+$).

(7) If we are *very* careful, and always put down all the premisses we need (i.e., if we argue *logically*), then we arrive *precisely* at the formal system **E** of *logical implication (without quotes, this time), or entailment.*

POSTSCRIPT. It will be recalled from §16 that if Hitler was a military genius (H), then I'm a monkey's uncle (M). We can now give an analysis of the "if... then —" in this hilarious (?) hyperbole. It is a double enthymeme: the suppressed premisses being $H \rightarrow \forall pp$, and the logically valid $\forall pp \rightarrow M$, from which $H \rightarrow M$ follows by transitivity. What makes the assertion so funny is that whereas the speaker wishes to say that H is false ($\sim H$), he doesn't *really* mean to say that it is so obviously false, though of course $\sim H$ does follow from $H \rightarrow \forall pp$ via the rule $\sim$I.

The reader may feel that this analysis takes some of the spontaneity (or whatever it may have been that endeared it to anyone the first time) out of the joke, as in Freud 1905. We nevertheless claim that it is a significant "first." Surely no one before has successfully brought to bear techniques of mathematical logic in order to explain a rhetorical figure of speech, said to be comical in a natural language. (And if the reader finds himself wishing that no one will ever do it again, he stands convicted of harboring an antiscientific spirit.)

§36. Enthymematic implications: Embedding H and S4 in $\mathbf{E}^{\forall\exists p}$. The mathematical results announced in §35 are these:

(1) Let $A \supset B$ be translated $\exists r(r\&.r\&A \to B)$. Let $\neg A$ be translated $A \supset \forall pp$, and let & and $\vee$ translate themselves. Then the system **H** of intuitionism is exactly contained in the system $\mathbf{E}^{\forall\exists p}$ of **E** with propositional quantifiers.

(2) Let $A \prec B$ be translated $\exists r(\Box r\&.r\&A \to B)$, and let & and $\vee$ translate themselves. Then the system $\mathbf{S4}_{+}$ of negation-free strict implication is exactly contained in $\mathbf{E}^{\forall\exists p}$.

We devote this section to establishing these results.

We conjectured (1) almost as soon as a reading of Myhill 1953 suggested it to us, and (2) not long afterward. However, delicate questions of conservative extension of $\mathbf{E}^{\forall\exists p}$ itself made it advisable to consider first the relativizations of the conjecture (1) to the *positive* systems $\mathbf{E}_{+}^{\forall\exists p}$. That is,

(1^{+}) On the translation of (1), **H** is exactly contained in the system $\mathbf{E}_{+}^{\forall\exists p}$ of $\mathbf{E}_{+}$ with propositional quantifiers.

In §36.1 below we give a proof of (1^{+}); as noted toward the end of that section, we rely on a result of Grover 1970, a result that in fact represents a correction of some of our earlier efforts. In §36.2 we provide the extra details, due to Meyer, required for improving (1^{+}) to (1), and we sketch Meyer's proof of (2).

It is with some pleasure that we turn now to purely positive matters, for negation is always a delicate subject. (Hao Wang once remarked in a lecture that negation is the formal logician's original sin; compare Wang 1954.)

§36.1. H in $\mathbf{E}_{+}^{\forall\exists p}$. The system **H** of Heyting 1930 cannot be accused of being a calculus of truth values, for the many reasons with which we will suppose the reader to be familiar; the condescension with which we reconstructed **TV** in §33 is therefore not available to us. So we draw on the motivations just discussed, and in this section present the formal results required to buttress one of the claims made in §35.

We will be concerned with the system $\mathbf{E}_{+}^{\forall\exists p}$, consisting of $\mathbf{E}_{+}$ as in §24.4.1 or §R2, together with propositional quantifiers as in §§30–32. As before we will do our work in the natural deduction equivalent $\mathbf{FE}_{+}^{\forall\exists p}$ (apologies for the increasingly cumbersome notation for systems; we hope that it is memorable). To help recall where we are, we summarize $\mathbf{FE}_{+}^{\forall\exists p}$.

Structural rules

Hyp. A step may be introduced as the hypothesis of a new subproof, and each new hypothesis receives a unit class $\{k\}$ of numerical subscripts, where k is the rank of the new subproof.

Rep. A_{a} may be repeated, retaining the relevance indices a.

Reit. A_{a} may be reiterated (retaining subscripts) into:

- (i) hypothetical subproofs, provided A has the form $\forall p_1 \ldots \forall p_n(A \rightarrow B)$, $0 \leq n$, and
- (ii) categorical subproofs general with respect to p, provided A does not contain p free.

Intensional rules

$\rightarrow$I. From a proof of B_{a} on the hypothesis $A_{\{k\}}$ to infer $A \rightarrow B_{a-\{k\}}$, provided k is in a.

$\rightarrow$E. From A_{a} and $A \rightarrow B_{\mathrm{b}}$ to infer $B_{\mathrm{a} \cup \mathrm{b}}$.

Mixed rules

$\vee$E. From $A \vee B_{\mathrm{a}}$, $A \rightarrow C_{\mathrm{b}}$, and $B \rightarrow C_{\mathrm{b}}$, to infer $C_{\mathrm{a} \cup \mathrm{b}}$,

$\exists$E. From $\exists p A(p)_{\mathrm{a}}$ and $\forall p(A(p) \rightarrow B)_{\mathrm{b}}$ to infer $B_{\mathrm{a} \cup \mathrm{b}}$, provided p is not free in B.

Extensional rules

&I. From A_{a} and B_{a} to infer $A \& B_{\mathrm{a}}$.

&E. From $A \& B_{\mathrm{a}}$ to infer A_{a}. From $A \& B_{\mathrm{a}}$ to infer B_{a}.

$\vee$I. From A_{a} to infer $A \vee B_{\mathrm{a}}$. From B_{a} to infer $A \vee B_{\mathrm{a}}$.

&$\vee$. From $A \& (B \vee C)_{\mathrm{a}}$ to infer $(A \& B) \vee C_{\mathrm{a}}$

$\forall$I. From a categorical subproof general with respect to p, having $A(p)_{\mathrm{a}}$ as final step, to infer $\forall p A(p)_{\mathrm{a}}$.

$\forall$E. From $\forall p A(p)_{\mathrm{a}}$ to infer $A(B)_{\mathrm{a}}$.

$\exists$I. From $A(B)_{\mathrm{a}}$ to infer $\exists p A(p)_{\mathrm{a}}$.

$\forall\vee$. From $\forall p(A \vee B(p))_{\mathrm{a}}$ to infer $A \vee \forall p B(p)_{\mathrm{a}}$, provided p is not free in A.

&$\exists$. From $A \& \exists p B(p)_{\mathrm{a}}$ to infer $\exists p(A \& B(p))_{\mathrm{a}}$, provided p is not free in A.

Our object will be, first, to show that, given a certain understanding of the intuitionistic horseshoe (for implication) and corner (for negation), all the theorems of the intuitionistic propositional calculus are forthcoming in $\mathbf{E}^{\forall\exists p}_{+}$. Secondly, we want to show that where a theorem of $\mathbf{E}^{\forall\exists p}_{+}$ is amenable to intuitionistic scrutiny, it says what an intuitionist can accept.

For the first point, in the case of the horseshoe, we want

$$A \supset B =_{\mathrm{df}} \exists p(p \& .(p \& A) \rightarrow B)$$

(where p is the first variable free in neither A nor B) to be such that intuitionistic theorems like $A\supset(B\supset A)$ will come out to be provable in $E_{+}^{\forall\exists p}$ under translation. This will half convince us that the right-hand side is what intuitionists meant all along by $A\supset B$, and make us feel happy. (Similarly for negation.)

For the second point, we must recognize that there are lots of theorems of $E_{+}^{\forall\exists p}$ which simply won't untranslate themselves into formulas that intuitionists affect to understand. So if an intuitionist sees

$$\exists p(p\&.(p\&A)\rightarrow A),$$

he will say, after a moment's puzzlement, "Oh. You mean that $A\supset A$. Well. I knew that." But if he sees, e.g.,

$$A\rightarrow B\ \rightarrow.\ \forall pp\rightarrow C,$$

which is a theorem of $E_{+}^{\forall\exists p}$, he will say "What?" since this formula does not translate back into his vocabulary. And he will worry that these nontranslatable formulas will somehow allow proof of a formula which he will recognize all right, but of which he will not approve (e.g., $A\vee(A\supset B)$).

We attack these problems one at a time, doing the easiest first. We present a list of formulas in conjunction, disjunction, and intuitionistic negation and implication, which everyone agrees are to intuitionistic tastes, and then show that they are all provable under translation.

§36.1.1. Under translation, $E_{+}^{\forall\exists p}$ contains at least H. The work is straightforward. We need to show that, with the horseshoe defined as above, and

$$\neg A =_{\mathrm{df}} A\supset\forall pp,$$

all the following (taken from Kleene 1952, §§19 and 23) are provable in $E_{+}^{\forall\exists p}$:

H1 $A\supset.\ B\supset A$
H2 $A\supset B\supset.\ (A\supset.B\supset C)\supset.A\supset C$
H3 $A\&B\supset A$
H4 $A\&B\supset B$
H5 $A\supset.\ B\supset.A\&B$
H6 $A\supset.\ A\vee B$
H7 $B\supset.\ A\vee B$
H8 $A\supset C\supset.\ B\supset C\supset.A\vee B\supset C$
H9 $A\supset B\supset.\ A\supset\neg B\supset\neg A$
H10 $\neg A\supset.\ A\supset B$

and that the rule $\supset$E is derivable for the intuitionistic horseshoe so defined.

We attack the last problem first. For the proof to follow we note that the variable p in step 2 below is so chosen that it occurs free in neither A nor

B, which justifies the use of &∃ and ∃E at steps 3 and 10:

1	$A\&(A\supset B)_{\{1\}}$	hyp
2	$A\&\exists p(p\&.(p\&A)\to B)_{\{1\}}$	1 rep (definition)
3	$\exists p(A\&(p\&.(p\&A)\to B))_{\{1\}}$	2 &∃
4	p $A\&(p\&.(p\&A)\to B)_{\{2\}}$	hyp
5	$A_{\{2\}}$	4 &E
6	$p_{\{2\}}$	4 &E (twice)
7	$(p\&A)\to B_{\{2\}}$	4 &E (twice)
8	$p\&A_{\{2\}}$	5 6 &I
9	$B_{\{2\}}$	7 8 →E
10	$B_{\{1\}}$	3 4–9 ∃E
11	$(A\&(A\supset B))\to B$	1–10 →I

This is a striking theorem. Because $A\supset B$ is defined enthymematically, one might have thought that some extra truth might be needed to help one to move from $A\supset B$ and A to B. One was ready, that is, for B to be an enthymematic consequence of the enthymematically defined $A\supset B$ together with A, but the theorem says that something stronger obtains: even though $A\supset B$ is itself only an enthymematic implication, it is nevertheless sufficient when conjoined with A to entail the consequent B without the need for extra help. Recall in contrast that the proposition that A materially "implies" B by no means can be conjoined with A to entail B and that to suggest otherwise would be once more to raise the specter of the disjunctive syllogism. Indeed, one of the results of §25.1 is to the effect that there is not even an enthymematic connection between the conjunction of the material "implication" $\sim A\vee B$ with A on the one hand and B on the other. This theorem, then, emphasizes a dramatic unlikeness between the intuitionist and the classical conditionals insofar as they relate to relevance.

In any event, from this theorem the rule ⊃E follows for $\mathbf{E}^{\forall\exists p}_{+}$ by &I and →E; so that is one item out of the way. Notice that, since step 11 and the corresponding rule are *entailments*, ⊃E can be used in subproofs for →I, provided subscripts on the premisses are the same.

The following helps us prove some results wholesale:

1	$A\to B_{\{1\}}$	hyp
2	$(A\to B)\&A_{\{2\}}$	hyp
3	$A_{\{2\}}$	2 &E
4	$A\to B_{\{1\}}$	1 reit
5	$B_{\{1,2\}}$	3 4 →E
6	$((A\to B)\&A)\to B_{\{1\}}$	2–5 →I
7	$\exists p(p\&.(p\&A)\to B)_{\{1\}}$	1 6 &I ∃I
8	$(A\to B)\to.\ A\supset B$	1–7 →I (definition)

This gives us H3, H4, H6, and H7.

By ∃I and the definition of $\supset$, we also have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} A\&((A\&B)\to C) \to.\ B\supset C,$$

from which it follows easily that, whenever we have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} (A\&B) \to C,$$

we also have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} A\supset.B\supset C.$$

Then $(A\&B)\to A$ gives us H1, and $(A\&B) \to.\ A\&B$ gives us H5.

In fact we can prove something a trifle stronger than the result we just used to get H1 and H5: whenever we have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} (A\&B)\supset C,$$

we also have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} A\supset.B\supset C,$$

because we have

$$\vdash_{\mathbf{E}_{+}^{\forall\exists p}} ((A\&B)\supset C)\&A \to.\ B\supset C,$$

as the following proof shows:

1		$((A\&B)\supset C)\&A_{\{1\}}$	hyp
2		$A\&\exists p(p\&.(p\&(A\&B))\to C)_{\{1\}}$	1 &E &I (definition)
3		$\exists p(A\&(p\&.(p\&(A\&B))\to C))_{\{1\}}$	2 &∃
4	p	$A\&(p\&.(p\&(A\&B))\to C)_{\{2\}}$	hyp
5		$(p\&A)\&.((p\&A)\&B)\to C_{\{2\}}$	4 &E &I
6		$B\supset C_{\{2\}}$	5 ∃I (def)
7		$B\supset C_{\{1\}}$	3 4–6 ∃E
8		$((A\&B)\supset C)\&A \to.\ B\supset C$	1–7 →I

In the presence of $(A\&B)\supset C$, step 8, ∃I, and the definition of the horseshoe, we then have $A\supset.B\supset C$. We call the move from $\vdash_{\mathbf{E}_{+}^{\forall\exists p}} (A\&B)\to C$ or $\vdash_{\mathbf{E}_{+}^{\forall\exists p}} (A\&B)\supset C$ to $\vdash_{\mathbf{E}_{+}^{\forall\exists p}} A\supset.B\supset C$ "exportation" (a word the reader will remember with dread from §22.2.2).

For H2 and H8 it will now suffice to prove $(A\supset B)\&((A\supset.B\supset C)\&A)\supset C$ and $((A\supset C)\&(B\supset C)\&(A\vee B))\supset C$ from which H2 and H8 will follow by exportation. The first is trivial, given $\supset$E; and, for the second, we observe that TE (§23.6) gives us

$$(A\supset C)\&(B\supset C)\&(A\vee B) \to.\ (A\&(A\supset C))\vee(B\&(B\supset C)),$$

from which $\supset$E and properties of disjunction give us

$$(A\supset C)\&(B\supset C)\&(A\vee B)\rightarrow.\ C\vee C,$$

from which TE gives us the required theorem.

An assiduous bookkeeper will notice that we have now taken care of all the positive intuitionistic axioms H1–H8. There remains only negation: H9–H10. Given the definition of the corner ($\neg$) at the beginning of this section, H9 is simply a special case of H2:

$$(A\supset B)\supset((A\supset.B\supset\forall pp)\supset.A\supset\forall pp),$$

and H10 comes from

$$(A\supset\forall pp)\&A\rightarrow B$$

by exportation.

So what the intuitionists say (H1–H10 and $\supset$E) all comes out all right, but there still may be some dispute, since intuitionists elect not to be satisfied even with their own work. Thus Heyting 1956:

> It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof (p. 102).

§36.1.2. Under translation $\mathbf{E}_{+}^{\forall\exists p}$ contains no more than H. The proof proceeds by way of the system $\mathbf{H}^{\forall\exists p}$, got by adding propositional quantifiers to the system **H** of the preceding section. $\mathbf{H}^{\forall\exists p}$ is of course a well-behaved formal system in its own right, and we exploit one of its properties in proving the theorem below. But first we say something about the point of the theorem.

Some of the formulas of $\mathbf{E}_{+}^{\forall\exists p}$ will be translations from intuitionistic propositional terminology according to the preceding definitions of $\supset$ and $\neg$, and some will not; we will be interested only in those which are, and we will call these *translations*. Each translation will then have the form T(C) where C is a formula of **H**, and where T is the translation function from **H** into $\mathbf{E}_{+}^{\forall\exists p}$ obtained from the definitions below by replacing the left side by the right in some routine way that avoids trouble with confused variables.

$$A\supset B=_{\mathrm{df}}\exists p(p\&.(p\&A)\rightarrow B)$$
$$\neg A=_{\mathrm{df}}\exists p(p\&.(p\&A)\rightarrow\forall pp)$$

The Theorem, then, is that if T(C) is provable in $\mathbf{E}_{+}^{\forall\exists p}$ then C is provable in **H**. For its proof we introduce an accessory translation. Let h be that function from $\mathbf{E}_{+}^{\forall\exists p}$ into $\mathbf{H}^{\forall\exists p}$ which replaces arrows by horseshoes; we call h(D), for D in $\mathbf{E}_{+}^{\forall\exists p}$, *the horseshoe analogue of D.* Of particular interest will be horseshoe analogues of translations: h(T(C)) for C in **H**. The argument then proceeds somewhat as follows, for C in **H**:

1. If $\vdash_{\mathbf{E}_{+}^{\forall\exists p}} \mathrm{T}(C)$ then $\vdash_{\mathbf{H}^{\forall\exists p}} \mathrm{h}(\mathrm{T}(C))$.
2. If $\vdash_{\mathbf{H}^{\forall\exists p}} \mathrm{h}(\mathrm{T}(C))$ then $\vdash_{\mathbf{H}^{\forall\exists p}} C$.
3. If $\vdash_{\mathbf{H}^{\forall\exists p}} C$ then $\vdash_{\mathbf{H}} C$.

From which the theorem follows.

We said the argument goes "somewhat" this way. The only hitch in the foregoing is that (for reasons which will emerge in the proof of Lemma 1 below), $\mathbf{H}^{\forall\exists p}$ is not quite strong enough to do the work required, so we use a nonintuitionistic extension $\mathbf{H}^{\forall\exists p'}$ of $\mathbf{H}^{\forall\exists p}$ instead. Otherwise the story is just as outlined, and we now repeat all this, filling in the details.

For $\mathbf{H}^{\forall\exists p}$ we need the axioms H1–H10, and

H11 $\forall pA(p) \supset A(B)$
H12 $\forall p(A \supset B(p)) \supset. A \supset \forall pB(p)$ (p not free in A)
H13 $A(B) \supset \exists pA(p)$
H14 $\forall p(A(p) \supset B) \supset. \exists pA(p) \supset B$ (p not free in B)

In addition to $\supset$E, we require a rule of generalization ("gen"): from A to infer $\forall pA$.

The required extension $\mathbf{H}^{\forall\exists p'}$ of $\mathbf{H}^{\forall\exists p}$ comes by adding the axiom

H15 $\forall p(A \vee B(p)) \supset. A \vee \forall pB(p)$ (p not free in A)

which is not in the spirit of intuitionism. (Explaining why would take us too far afield.) Addition of H15 is just what allows us to prove

LEMMA 1. For D in $\mathbf{E}_{+}^{\forall\exists p}$ (including $D = \mathrm{T}(C)$ as a special case), if D is provable in $\mathbf{E}_{+}^{\forall\exists p}$, then $\mathrm{h}(D)$ is provable in $\mathbf{H}^{\forall\exists p'}$.

PROOF. When we inspect the axioms and rules of $\mathbf{E}^{\forall\exists p}$ (see §32), intellectual intuition reveals that, if we think of the arrow as intuitionistic horseshoe, the only principles involved that are *not* intuitionistically acceptable are E12 (of §21.1 or §R2) and PQ5 (of §32). E12 is not a worry, since none of E12–E14 are part of $\mathbf{E}_{+}$ anyway. And though PQ5 is not acceptable intuitionistically, it is a theorem of $\mathbf{H}^{\forall\exists p'}$, because H15 = PQ5 was just added to $\mathbf{H}^{\forall\exists p}$ for exactly this purpose.

LEMMA 2. For C in **H**, if $\mathrm{h}(\mathrm{T}(C))$ is provable in $\mathbf{H}^{\forall\exists p'}$ then so is C.

PROOF. A second use of intellectual intuition reveals that C can be obtained from $\mathrm{h}(\mathrm{T}(C))$ by making replacements according to the following equivalences, each easy to prove in $\mathbf{H}^{\forall\exists p}$ and hence in $\mathbf{H}^{\forall\exists p'}$:

$$A \supset B \equiv \exists p(p \& .(p \& A) \supset B)$$
$$\neg A \equiv \exists p(p \& .(p \& A) \supset \forall pp)$$

Lemma 3. $\mathbf{H}^{\forall\exists p'}$ is a conservative extension of **H**.

There are two proof strategies available. The first consists in finding a consecution-calculus formulation of $\mathbf{H}^{\forall\exists p'}$ for which the elimination and subformula theorems can be proved, thus showing that any quantifier-free formula has a quantifier-free proof; see Grover 1970, where this has been carried out, using a combination of methods due to Kripke 1965a and Prawitz 1965.

The second is semantic. Suppose we have a family of interpretations F of the quantifier-free formulas with respect to which **H** is *complete*. Suppose further that we can find a way of adding to each member of the family an interpretation of the quantifiers, but without altering the interpretation of the quantifier-free formulas and without changing which elements are undesignated. And suppose finally that $\mathbf{H}^{\forall\exists p'}$ is *consistent* with respect to this modified family F′ of interpretations. Then $\mathbf{H}^{\forall\exists p'}$ must be a conservative extension of **H**; for if a quantifier-free formula is unprovable in **H**, then, by completeness, it will have an undesignated value in some member of the family F. But then it will take the same undesignated value in the corresponding member of F′; so, by consistency, it will be unprovable in $\mathbf{H}^{\forall\exists p'}$.

Details can be accomplished in several ways, of which we mention one. The set of finite pseudo-Boolean algebras constitutes a family with respect to which **H** is complete (Rasiowa and Sikorski 1963, p. 385). Such structures are lattices; so, since they are finite, one can add an easy interpretation of the propositional quantifiers of $\mathbf{H}^{\forall\exists p'}$ in terms of the finite lattice operations; i.e., the interpretation of the quantifiers is a straightforward generalization of the interpretation of & and v. This interpretation obviously does not tinker with which values are designated, does not alter the interpretation of the propositional connectives, and (because of finitude) satisfies all the axioms and rules of $\mathbf{H}^{\forall\exists p'}$. So $\mathbf{H}^{\forall\exists p'}$ is a conservative extension of **H**.

§36.2. H and S4$_+$ in $\mathbf{E}^{\forall\exists p}$. At the beginning of this section we stated the following:

(1) Let $A \supset B$ be translated $\exists r(r\&.r\&A \rightarrow B)$. Let $\neg A$ be translated $A \supset \forall pp$, and let & and v translate themselves. Then the system **H** of intuitionism is exactly contained in the system $\mathbf{E}^{\forall\exists p}$ of **E** with propositional quantifiers.

(2) Let $A \rightarrow\!\!\!\!\!\exists\, B$ be translated $\exists r(\Box r\&.r\&A \rightarrow B)$, and let & and v translate themselves. Then the system **S4**$_+$ of negation-free strict implication is exactly contained in $\mathbf{E}^{\forall\exists p}$.

(It is intended that $\Box r$ have its usual **E** sense of $r \rightarrow r \rightarrow r$, as in §4.3.) In §36.1, however, we proved only a weaker version of (1), a version that relates **H** and $\mathbf{S4}_+$ to the negation-free fragment $\mathbf{E}_+^{\forall\exists p}$ instead of to $\mathbf{E}^{\forall\exists p}$ itself; here, with all credit to Meyer, we improve to the full strength of (1), and we prove (2).

Turning first to (1), that the translation of theorems of **H** are provable in $\mathbf{E}^{\forall\exists p}$ has been proved already in §36.1. The strategy for showing the converse will be similar to that employed in showing that **H** is conservatively extended by $\mathbf{H}^{\forall\exists p'}$, as sketched at the end of §36.1. We first find a family of algebraic interpretations with respect to which **H** is complete; again we use the set of finite pseudo-Boolean algebras, as in Rasiowa and Sikorski 1963, p. 385. Next we identically embed each member of this family (the source) into a target, consisting of an algebraic interpretation for the operations of $\mathbf{E}^{\forall\exists p}$ and a notion of designated value, having the following features: (a) values undesignated in the source are undesignated in the target, (b) every theorem of $\mathbf{E}^{\forall\exists p}$ takes a designated value in the target, and (c) whatever value an **H**-formula takes in the source, its translation in $\mathbf{E}^{\forall\exists p}$ takes the same value in the target. Evidently this suffices to show that if A is unprovable in **H**, then its translation cannot be proved in $\mathbf{E}^{\forall\exists p}$.

This plan can be realized by means of the following recipe. We start with a finite pseudo-Boolean algebra $\mathrm{L} = \langle \mathrm{L}, \supset, \&, \vee, \neg \rangle$, matching the notation with which we picture **H** as endowed, and for convenience we define $\boldsymbol{T} = \mathrm{a} \supset \mathrm{a}$ and $\boldsymbol{F} = \neg \boldsymbol{T}$(a any element of L). We recall that $\{\boldsymbol{T}\}$ exhausts the designated values of a pseudo-Boolean algebra. Let a, b range over L. Let $-\mathrm{a}$ be a one-one function mapping L onto a disjoint set $-\mathrm{L}$. We define an algebraic interpretation $\mathrm{M} = \langle \mathrm{M}, \to, \&, \vee, {\sim} \rangle$ suitable for **E** as follows (recalling that a, b range over L):

$\mathrm{M} = \mathrm{L} \cup -\mathrm{L}$
Designated elements $= \{\boldsymbol{T}\} \cup -\mathrm{L}$
$\mathrm{a} \to \mathrm{b} = \mathrm{a} \supset \mathrm{b}$, $\mathrm{a} \to -\mathrm{b} = \boldsymbol{T}$, $-\mathrm{a} \to \mathrm{b} = \boldsymbol{F}$, $-\mathrm{a} \to -\mathrm{b} = \mathrm{b} \supset \mathrm{a}$
$\mathrm{a} \& \mathrm{b} = \mathrm{a} \& \mathrm{b}$, $\mathrm{a} \& -\mathrm{b} = -\mathrm{b} \& \mathrm{a} = \mathrm{a}$, $-\mathrm{a} \& -\mathrm{b} = -(\mathrm{a} \vee \mathrm{b})$
$\mathrm{a} \vee \mathrm{b} = \mathrm{a} \vee \mathrm{b}$, $\mathrm{a} \vee -\mathrm{b} = -\mathrm{b} \vee \mathrm{a} = -\mathrm{b}$, $-\mathrm{a} \vee -\mathrm{b} = -(\mathrm{a} \& \mathrm{b})$
${\sim}\mathrm{a} = -\mathrm{a}$, ${\sim}-\mathrm{a} = \mathrm{a}$

To complete the definition as required for $\mathbf{E}^{\forall\exists p}$, we need an interpretation of the quantifiers. Noting that M is a finite lattice under & and $\vee$, let the value of $\exists pA$ be the disjunction (in the sense of $\vee$) of all the values of A as p runs through M, and interpret $\forall pA$ similarly in terms of &.

Since, obviously, (a) undesignated values of L are undesignated in M, what needs verification to complete the proof is that (b) every theorem of $\mathbf{E}^{\forall\exists p}$ takes a designated value in the target, M, and that (c) whatever value an **H**-formula A has in L, its translation into $\mathbf{E}^{\forall\exists p}$ has that same value in M. Each of these succumbs to verification, with the following case of (c) being typical: let A and B be **H**-formulas with values a and b in L, and suppose that their translations A' and B' have the same values in M. Since $A \supset B$ has the value $\mathrm{a} \supset \mathrm{b}$ in L, we need to show for (c) that its translation $\exists p(p \& . A' \& p \rightarrow B')$ has that same value in M. That is, we need to show that $\mathrm{a} \supset \mathrm{b}$ is the disjunction in the sense of $\vee$ of all the values $\mathrm{x} \& . \mathrm{a} \& \mathrm{x} \rightarrow \mathrm{b}$, as x chases through M. Choosing x as any value in $-\mathrm{L}$ forces $\mathrm{x} \& . \mathrm{a} \& \mathrm{x} \rightarrow \mathrm{b}$ to $\mathrm{a} \supset \mathrm{b}$ itself; so all that is now required is that $(\mathrm{a} \supset \mathrm{b}) \vee (\mathrm{x} \& . \mathrm{a} \& \mathrm{x} \rightarrow \mathrm{b})$ comes to $\mathrm{a} \supset \mathrm{b}$ when x lies in L, which must be so because L is a pseudo-Boolean algebra. This completes the proof of (1).

With regard to (2), that translations of theorems of $\mathbf{S4}_{+}$ are provable in $\mathbf{E}^{\forall\exists p}$ is a matter of straightforward inductive verification. For the converse, suppose that a translation A' of an $\mathbf{S4}_{+}$ theorem A is provable in $\mathbf{E}^{\forall\exists p}$. By simple containment, the translation A' is then provable in the system $\mathbf{S4}^{\forall\exists p}$ obtained by adding a suitable irrelevant axiom

$$B \rightarrow . A \rightarrow A$$

to $\mathbf{E}^{\forall\exists p}$. Let A'' be the "arrow rewrite" of A obtained by replacing each hook $\mathrel{-\!\!3}$ with an arrow $\rightarrow$; in this system it is possible to show the equivalence of $\exists r(\Box r \& . r \& C \rightarrow B)$ and $C \rightarrow B$; hence it must be that the arrow rewrite A'' of A is provable because the translation A' of A is provable. We may then show that $\mathbf{S4}^{\forall\exists p}$ is a conservative extension of $\mathbf{S4}$ (with the arrow playing the role of strict implication—taken as primitive—so that the connectives are $\rightarrow$, $\&$, $\vee$, and $\sim$) by methods akin to those we described at the end of §36.1 for showing that $\mathbf{H}^{\forall\exists p'}$ is a conservative extension of $\mathbf{H}$, so that A'' must be a theorem of $\mathbf{S4}$. And Hacking 1963 guarantees that $\mathbf{S4}$ so formulated is a conservative extension of $\mathbf{S4}_{+}$ (with the arrow as strict implication), so that it is just a matter of re-rewriting to obtain A as a theorem of $\mathbf{S4}_{+}^{\forall\exists p}$, as needed to complete the proof of (2).

§37. Miscellany. This section is analogous to §8; see the Analytical table of contents for a list of topics.

§37.1. Prenex normal forms (in $\mathbf{T}^{\forall\exists p}$). We discuss prenex normal forms and $\mathbf{T}^{\forall\exists p}$ together, not because they have anything in particular to do with each other, but rather because prenex normal forms have to be discussed somewhere, and they may as well be considered in the context of that implication connective which involves the fewest assumptions about the arrow, since our positive results for $\mathbf{T}^{\forall\exists p}$ will hold *a fortiori* for $\mathbf{E}^{\forall\exists p}$ and $\mathbf{R}^{\forall\exists p}$.

We say that a formula A is in *prenex normal form* if all quantifiers are initially placed, and no quantifiers are *vacuous* (i.e., no quantifier $\forall p$ or $\exists p$ fails to have an occurrence of p in its scope). In view of the provable equivalences $\exists pA \rightleftarrows A$ and $\forall pA \rightleftarrows A$, where p is not free in A, vacuous quantifiers may always be eliminated (here as classically), and we ignore them in the sequel. If A is in prenex normal form, the string of initially placed quantifiers is its *prefix*; the remaining quantifier-free part is its *matrix*.

Classically, we have the equivalences (where p is not free in B and $\supset$ is material):

$$\begin{aligned}
\sim\forall pA &\equiv \exists p\sim A\\
\sim\exists pA &\equiv \forall p\sim A\\
\forall pA(p)\supset B &\equiv \exists p(A(p)\supset B)\\
\exists pA(p)\supset B &\equiv \forall p(A(p)\supset B)\\
B\supset\forall pA(p) &\equiv \forall p(B\supset A(p))\\
B\supset\exists pA(p) &\equiv \exists p(B\supset A(p))
\end{aligned}$$

In a formulation with the horseshoe and tilde as primitive (or any other, if appropriate adjustments are made), these equivalences enable us systematically to drive quantifiers outward and truth-functional connectives inward, so that repeated application to any formula A of $\mathbf{TV}^{\forall\exists p}$ leads to a formula

$$\mathrm{Q}_1 p_1 \ldots \mathrm{Q}_n \mathrm{p}_n A'$$

which is demonstrably "equivalent" (in the sense of $\equiv$) to A, where each Q_i is a quantifier ($\forall$ or $\exists$) and A' is quantifier-free; i.e., A can always be put "equivalently" into prenex normal form.

We would not expect all these equivalences to hold with horseshoes replaced by arrows, since, for example, we would not want

$$\forall pA(p)\rightarrow\forall pA(p) \rightarrow. \exists p(A(p)\rightarrow\forall pA(p)),$$

which is like the third material "equivalence" above, taking B as $\forall pA(p)$. For now taking $A(p)$ as $p\vee\sim p$, the formula would yield

$$\exists p(p\vee\sim p \rightarrow \forall p(p\vee\sim p)),$$

which doesn't sound good. We find it hard to imagine what such an allegedly existent proposition, call it A, would be like. It would have to be a pretty powerful specimen, since both A and $\sim A$ would entail every instance of the law of the excluded middle. (Which of the two do you suppose is the true one?) No; Rational Intuition suggests that there is no such A, and the hunch is confirmed. The reader can use the methods of §34.2 to show that the formula is false in M_0 on the semantics there provided.

Most of the arrow analogues of these equivalences do hold, however, and we tabulate those provable in $\mathbf{T}^{\forall\exists p}$ (where $\leftarrow$ is the coverse of $\rightarrow$). (Note: we omit proofs, relying on the reader to use the very easy Fitch-style methods to construct proofs as desired.)

$$\begin{aligned}
\sim\forall pA(p) &\rightleftarrows \exists p\sim A(p)\\
\sim\exists pA(p) &\rightleftarrows \forall p\sim A(p)\\
\forall pA(p)\rightarrow B &\leftarrow \exists p(A(p)\rightarrow B) && \text{(note: not }\rightarrow\text{)}\\
\exists pA(p)\rightarrow B &\rightleftarrows \forall p(A(p)\rightarrow B)\\
B\rightarrow\forall pA(p) &\rightleftarrows \forall p(B\rightarrow A(p))\\
B\rightarrow\exists pA(p) &\leftarrow \exists p(B\rightarrow A(p)) && \text{(note: not }\rightarrow\text{)}
\end{aligned}$$

We are simply fascinated, astonished, *bouleversés*, by the fact that, though negation in this book is classical and the first two arrow equivalences just above hold, the arrow statements that fail among the last four are *precisely* those which fail for intuitionistic implication and quantifiers (see, e.g., Kleene 1952, §35). True, neither intuitionistic nor relevant implication connectives are definable from $\sim$, $\vee$, and &, but their motivations are so wildly divergent that we are confounded by the coincidence just mentioned, and don't know what to make of it, in spite of the close connections between the two which were discussed in §§35–36.

The situation is not altered in $\mathbf{E}^{\forall\exists p}$ or $\mathbf{R}^{\forall\exists p}$, and therefore this road toward prenex normal forms is blocked, in spite of the fact that appropriate theorems are forthcoming for the two-place extensional connectives; in $\mathbf{T}^{\forall\exists p}$ we have (p still not free in B):

$$\forall pA(p)\&B \rightleftarrows \forall p(A(p)\&B)$$
$$\exists pA(p)\&B \rightleftarrows \exists p(A(p)\&B)$$
$$\forall pA(p)\vee B \rightleftarrows \forall p(A(p)\vee B)$$
$$\exists pA(p)\vee B \rightleftarrows \exists p(A(p)\vee B)$$

These equivalences for truth-functional connectives guarantee that the following is easily proved:

THEOREM. Let A be arrow-free. Then there is a prenex normal form A' of A such that $\vdash_{\mathbf{T}^{\forall\exists p}} A\rightleftarrows A'$.

§37.2. The weak falsehood of $\forall p\forall q(p\to.q\to p)$. Those with stomachs strong enough to have followed us this far will be delighted at the following developments, for which we are indebted to Church 1951. After having condemned $A\to.B\to A$ on all sorts of philosophical grounds, we are happy to find that it can be taken as a *definition* of a sort of falsehood. As we have seen, $\forall pp$ is as false as one can get, and though $\forall p\forall q(p\to.q\to p)$ is not *quite* so false as $\forall pp$, it is false enough to do lots of the damage that falsehood does.

For let f be $\forall p\forall q(p\to.q\to p)$. Then two applications of PQ3 (§32) give us

$$f \to. f\to.q\to f,$$

which in turn yields (even in $\mathbf{T}^{\forall\exists p}_{\to}$, which has contraction as an axiom),

$$f \to. q\to f,$$

or, now with the help of the derived rule $\forall$I of $\mathbf{T}^{\forall\exists p}$ and PQ4 (§32),

$$f \to. \forall q(q\to f).$$

This can be strengthened, since, by PQ7 and PQ8 respectively, we have

$$\forall q(q\to f) \to. \exists pp\to f \text{ and}$$
$$\forall q(q\to f) \to \exists pp.$$

Using "*modus ponens* in the consequent" and putting the result together with the formula above, we get

$$f \rightleftarrows \forall q(q \to f).$$

So this f doesn't say (as $\forall pp$ does), "every proposition!" Rather, this f says "every proposition implies falsehood!" Except perhaps to intuitionists, the latter sounds almost as bad.

It is gratifying to find that a classically valid formula turns out to denote *das Falsche* (in a weak sense); the philosophical point having been made, we now suggest that the reader turn to Church 1951 for further formal details.

§37.3 $\mathbf{R}^{\forall\exists p}$ is not a conservative extension of $\mathbf{R}_+^{\forall\exists p}$. Perhaps it should never be surprising that by adding a connective to a theory with propositional quantifiers one can produce new theorems that involve the quantifiers but do not involve the newly added connective; for one can always imagine that old sentences will receive new quantificational properties by rubbing shoulders with the newly introduced connective or that the new connective will permit definition of new sorts of sentences on which one can then existentially generalize; but in particular cases, it is often difficult to see what is at stake. The present example is due to Meyer (unpublished; see also Meyer 1973). Define F, as usual, by means of

$$F \rightleftarrows \forall pp,$$

and consider

$$(1) \qquad A \to F \lor . A \to F \to F.$$

Meyer's proof of (1) in $\mathbf{R}^{\forall\exists p}$ adduces the following chain of theorems of $\mathbf{R}^{\forall\exists p}$:

$$F \to . \sim F \to F$$
$$A \to F \to . A \to . \sim F \to F$$
$$A \to F \to . \sim F \to . A \to F$$
$$A \to F \to . \sim(A \to F) \to F$$
$$\sim(A \to F) \to . A \to F \to F.$$

The last line yields (1), which is, accordingly, a theorem of $\mathbf{R}^{\forall\exists p}$. That (1) is *not* provable in $\mathbf{R}_+^{\forall\exists p}$ can be seen by the following considerations: (a) obviously the F version is not provable in the intuitionistic system $\mathbf{H}$, where F is intuitionistic absurdity; for (1) would amount to just $\sim A \lor \sim\sim A$; so (b) it is not provable in the system $\mathbf{H}^{\forall\exists p'}$ of §36.1.2, which by Lemma 3 of that section is a conservative extension of $\mathbf{H}$; so (c) neither is the provably equivalent $\forall pp$ version; so (d) since $\mathbf{R}_+^{\forall\exists p}$ is a subsystem of $\mathbf{H}^{\forall\exists p'}$, the $\forall pp$ version is not provable there either; which was to be shown.

Meyer in unpublished work also observes that there is a somewhat more straightforward witness to the reported lack of conservative extension in virtue of the definability in $\mathbf{R}^{\forall\exists p}$ of f by

$$f \rightleftarrows \exists p{\sim}(p\to p)\ \{\text{or } {\sim}\forall p(p\to p)\}.$$

For the following now becomes a negation-free theorem of $\mathbf{R}^{\forall\exists p}$ (by means of existential generalization on f) that is not a theorem of $\mathbf{R}_{+}^{\forall\exists p}$:

$$(2) \qquad \exists q\forall p(p\to q\to q \to p).$$

The parallel question whether $\mathbf{E}^{\forall\exists p}$ is a conservative extension of $\mathbf{E}_{+}^{\forall\exists p}$ is, so far as we know, open; but, as implied in the opening paragraph of this section, we do not attach much philosophical interest thereto.

It is worth noting in closing this section that Meyer 1973 uses the provability of (1) to show in effect that **H** is not translatable into $\mathbf{R}^{\forall\exists p}$ in the way that it was into $\mathbf{E}^{\forall\exists p}$ (see §36). (Actually Meyer 1973 uses sentential constants in the role of $\boldsymbol{t}$ and $\boldsymbol{F}$ so as to obviate the need for propositional quantifiers, but the point is the same.) The trick is to substitute $A\&\boldsymbol{t}$ for A in (1) so that it becomes (upon translation):

$$(3) \qquad \neg A\vee(\neg A\to \boldsymbol{F}),$$

and then weaken the displayed → to a ⊃, getting

$$(4) \qquad \neg A\vee\neg\neg A,$$

which is well-known to be unprovable in **H**.

Meyer's diagnosis is that $\boldsymbol{F}$ is too strong to function as the intuitionistic absurd proposition, since it implies (not just intuitionistically implies) all propositions (not just those statable using intuitionistic connectives and quantifiers).

§37.4. Definitions of connectives in R with propositional quantifiers. In some contexts, propositional quantifiers permit certain connectives to be defined; Russell, for instance, knew that material "implication" and propositional universal quantification suffice for propositionally quantified two-valued logic, and Prawitz 1965 investigates the matter for intuitionism. The following observations are due to Meyer.

For all of propositionally quantified $\mathbf{R}^{\forall\exists p}$, only three logical particles are required: the propositional quantifier ∀, one intensional connective, →, and one extensional connective, &. The remaining connectives of $\mathbf{R}^{\forall\exists p}$ may be defined as follows:

$$\begin{aligned}
A\vee B &\rightleftarrows \forall p((A\to p\&.B\to p)\to p)\\
\exists qA &\rightleftarrows \forall p(\forall q(A\to p\to p)\\
A\circ B &\rightleftarrows \forall p((A\to.B\to p)\to p)\\
f &\rightleftarrows \exists q(q\circ\forall p(p\to q\to q\to p))\\
{\sim}A &\rightleftarrows A\to f
\end{aligned}$$

That *f* and, accordingly, negation are definable in terms of what one would have thought of as purely positive equipment is the most surprising; we omit verification.

It is to be borne in mind that the displayed definitions are available only for **R**, where, because of permutation, there is no ferocious chasm between implications and other propositions. In $\mathbf{E}^{\forall\exists p}$, for instance, because the right sides of the definitions have the form of entailments, which are necessitives, they could not be equivalent to the nonnecessitives listed on the left (see §22.1).

CHAPTER VII

INDIVIDUAL QUANTIFICATION

§38. $\mathbf{R}^{\forall\exists x}$, $\mathbf{E}^{\forall\exists x}$, and $\mathbf{T}^{\forall\exists x}$. The first two sections of this chapter are concerned with natural generalizations of ideas of Chapter IV to first-order quantification theory in the context of relevant systems of logic. Since the results to be proved fall out almost automatically—after a little reflection anyway—we leave proofs almost entirely to the reader. Such philosophical axes as we have to grind have now been largely left behind, since their generalization to individual quantifiers is obvious. Their mathematical generalizations are by no means obvious, and the remaining sections of the chapter are devoted to what is known about these.

We remark that these sections were all locked up many years ago, before the explosion of interest in the interaction of quantifiers and nonextensional connectives (chiefly modal). This explosion was possible because of insights deriving from semantic considerations, whereas this chapter is presemantical both temporally and with respect to the ordering of the material in this book.

The upshot is that what we give is fine, but that its historical conceptual chassis is Model T. For an instructive key to the history, we observe that we were guided by the picture of a universal quantification as a rather large conjunction, and dually for existential quantification, but it is not too early to say by way of anticipation that the situation is more confused than that remark might suggest. Combining the semantics of Chapter IX with either a straightforward domain-and-values account of the quantifiers (with "constant domain") or with a substitutional account as in Dunn and Belnap 1968 leads to a perfectly definite account of validity, at which by hindsight we should perhaps take ourselves to have been aiming; but we learn two things from the work of Fine. In the first place, we learn from §52 below that we missed the mark: although all our theorems and rules are valid (easy), the systems we are about to describe are not complete with respect to the "big conjunction {disjunction}" picture of the universal {existential} quantifier. This does not lead us to believe that our formal systems are somehow inadequate, however, for we also learn from §53 that the formal systems defined in the present section admit a new and subtle semantic conceptualization derived by taking the universal quantification not as a big conjunction, but instead as a statement that its matrix holds for a certain arbitrarily given entity. The "arbitrary entity" semantics of §53, then, is the semantics for which our systems are apt, rather than the "big conjunction" picture that guided us back in the days before we had any semantic analyses at all.

§38.1. Natural deduction formulations.

We summarize these according to the spirit and indeed the letter of §31.5 (except that we drop tiresome reference to **RM** and **EM**), again dividing the rules into three groups. Motivational considerations parallel those of §31.5 sufficiently closely so that we shall not feel obliged to repeat them. We shall, however, spend a minute or two on notation.

We invite the reader to look again at §30.2. Notation is as explained there, except that we will require, instead of propositional variables,

1. *Predicate variables*, each of which has n places, for some n. We use F, G, etc., as ranging over predicate variables, leaving the number of places to context.

2. *Individual variables*, denumerably many, alphabetically ordered. We use x, y, z, etc., as ranging over individual variables.

When it is wanted we make the *parameter-variable* distinction as in §30.2 (parameters never get bound; quantifiers use only variables), and then let x, y, z range over the variables and a, b, c over the parameters. We may occasionally invoke the notion of a *term*, which is either a parameter or a variable, using t, etc., as ranging thereover. (Complex terms involving operators can be added if wanted, otherwise changing nothing; but these need to be supplemented with a theory of identity. See §§72–74 below.)

3. *Individual quantifiers*, $\forall$ and $\exists$ as in §30.2. We take $\exists x$ as defined by $\sim\forall x\sim$, when not primitive, and vice versa.

4. *Formulas*. If F is n-place, $Fx_1 \ldots x_n$ is a formula, and so is $\boldsymbol{t}$ if present. New formulas come from old by connectives as usual, and if A is a formula, so are $\forall xA$ and $\exists xA$. We use A, B, etc., as ranging over formulas, and also Ax, By, etc., with conventions about being *ready for substitution* which exactly parallel those of §30.2. The upshot is that At is defined as the result of putting t for all free x in Ax, after first fixing Ax, if necessary, so that t does not get grabbed by a quantifier. (If it is a parameter, of course, it can't be.) We remind the reader that a *sentence* is a formula without free variables (though it may contain parameters).

With these notational understandings, we go on to state the rules for the natural deduction formulations, almost exactly copying §31.5.

Structural rules

Reit (hypothetical). $\forall x_1 \ldots \forall x_n A_a$ $(n \geq 0)$ may be reiterated into a hypothetical subproof (unrestricted for $\mathbf{FR}^{\forall\exists x}$; for $\mathbf{FE}^{\forall\exists x}$ and $\mathbf{FT}^{\forall\exists x}$, A must have the form $B \rightarrow C$).

Reit (categorical). A_a may be reiterated into categorical subproofs general with respect to x (see §31.1, changing p to x), provided A_a does not contain x free.

Mixed rule

$\exists$E. From $\exists xAx_a$ and $\forall x(Ax\rightarrow B)_b$ to infer $B_{a\cup b}$, provided x is not free in B (and where, for $\mathbf{FT}^{\forall\exists x}$, $\max(b)\leq\max(a)$).

Extensional rules

$\forall$I. From a categorical subproof general with respect to x, having Ax_a as final step, to infer $\forall xAx_a$.

$\forall$E. From $\forall xAx_a$ to infer Ay_a.

$\exists$I. From Ay_a to infer $\exists xAx_a$.

$\forall\vee$. From $\forall x(A\vee Bx)_a$ to infer $A\vee\forall xBx_a$, provided x is not free in

&$\exists$. From $A\&\exists xBx_a$ to infer $\exists x(A\&Bx)_a$, provided x is not free in A.

Adding these rules to a system **FS** or $\mathbf{FS}_+$ yields $\mathbf{FS}^{\forall\exists x}$, or $\mathbf{FS}_+^{\forall\exists x}$. Or, if only one of $\forall$ and $\exists$ is taken as primitive, $\mathbf{FS}^{\forall x}$ or $\mathbf{FS}^{\exists x}$. There is also $\mathbf{FS}_+^{\forall x}$ if anyone wants it.

§38.2. Axiomatic formulations and equivalence. We consider adding the following axioms to those for the systems of §R2.

IQ1	$\forall x(A\rightarrow B)\rightarrow.\ \forall xA\rightarrow\forall xB$	
IQ2	$\forall xA\&\forall xB\rightarrow\forall x(A\&B)$	
IQ3	$\forall xAx\rightarrow Ay$	
IQ4	$\forall x(A\rightarrow B)\rightarrow.\ A\rightarrow\forall xB$	(x not free in A)
IQ5	$\forall x(A\vee B)\rightarrow.\ A\vee\forall xB$	(x not free in A)
IQ6	$\forall x_1\ldots\forall x_n(A\rightarrow A)\rightarrow B\rightarrow B$	($n\geq 0$)
IQ7	$Ay\rightarrow\exists xAx$	
IQ8	$\forall x(A\rightarrow B)\rightarrow.\ \exists xA\rightarrow B$	(x not free in B)
IQ9	$\exists xA\&B\rightarrow\exists x(A\&B)$	(x not free in B)

Axiom clause: if A is an axiom, so is $\forall xA$.
Rules: $\rightarrow$E and &I.

The various systems are now defined in exact parallel to §32: for a system **S** of propositional logic, obtain $\mathbf{S}^{\forall\exists x}$ by adding everything above except IQ6; and add that as well for the distinctively modal systems **E** and **EM**. Then fiddle in the obvious ways to get $\mathbf{S}_+^{\forall\exists x}$, $\mathbf{S}^{\forall x}$, $\mathbf{S}^{\exists x}$, and $\mathbf{S}_+^{\forall x}$.

COMMENT. We note only that IQ6 does the same job here that PQ6 does in §32: both answer to the natural deduction rule allowing reiteration of quantified entailments (see §8.3.3 for help in reconstructing proofs).

The proof of equivalence of the various systems **S** to **FS** proceeds as in §32.

If we are dealing with a formulation whose grammar admits parameters in addition to variables and disallows the free occurrence of variables, then the axioms and rules have to be adjusted, but only a little: we need to point to substitution with parameters rather than with variables by using "Aa" instead of "Ay" in IQ3 and IQ4, and the Axiom clause needs to read, "if Aa is an axiom, so is $\forall xAx$, provided that a does not occur in $\forall xAx$."

§39. Classical results in first-order quantification theory.

Anything labeled "classical" can *only* command respect. We accordingly follow an earlier policy of proving some classical theorems totally independently of the philosophical considerations that motivated our own enterprise and then show later (with a slightly malicious smile) that they fall into place in exactly the way we hoped they would.

In the following two sections we assume familiarity with the classical definitions and elementary properties of *validity*, *satisfiability*, *universal closure*, and related notions, as to be found in, e.g., Church 1956, §43.

§39.1. Gödel completeness theorem.

We recommend another look at §24.1, since what we plan to do now is to offer a generalization of both subsections, again in the spirit of Kanger 1957 and Schütte 1950.

Formulas of *zero degree* will be as before: no arrows, no $\boldsymbol{t}$.

As primitive we take negation, disjunction, and existential quantification, assuming that other truth functions and universal quantification are defined in the usual way.

We extend the notion of an *atom* to include not only propositional variables and their denials, but also formulas $Fx_1 \ldots x_n$ (where F is as under §38.1) and their denials. Nonatomic molecules (introduced in §24.1.1) are those nondisjunctions which are susceptible to analysis into parts, in a way in which atoms are not. They come in the four shapes (a)–(d), to be described in a moment.

Disjunctive parts are as before: (i) A is a disjunctive part (dp) of A, and (ii) if $B \vee C$ is a dp of A, so are B and C. This now leaves us with dps that might be atoms or else (a) double denials, (b) existential quantifications, (c) denials of disjunctions, or (d) denials of existential quantifications. Atoms and disjunctions we need do nothing with, but we will require some way of handling (a)–(d).

Now if we think a minute about how the rules for the wedge of extensional disjunction work (i.e., we want to be sure that *at least one* of the two disjuncts is true), then it is reasonably clear how the rule for the existential quantifier (positive) ought to work. If $\exists xAx$ occurs as a disjunctive part of a formula, we ought to *try each* entity within the range of the variable x and see whether or not one of the Ax's comes out true, since existential quantification is simply a generalization of disjunction.

This suggests (on analogy with §24.1.1) that the rule for (positively occurring) occurrences of existentially quantified disjunctive parts should be something like

$$\frac{\varphi(Ax)}{\varphi(\exists yAy)}$$

But this, though an acceptable rule, doesn't help much in trying to construct trees as in §24.1.1, since, though we might with luck pick the right x as the variable in the premiss, in the course of extending a tree by one step (where by the "right" x we mean one that makes the premiss come out true), we have no guarantee that we will do so. If we pick a wrong one, we want to have a chance to try again. For this reason we put the rule in the form

$$\frac{\varphi(Ax)\vee\exists yAy}{\varphi(\exists yAy).}$$

We observe that the rule in this form involves contraction, in the sense that the conclusion could be viewed as the result of applying two rules, one for the quantifier:

$$\frac{\varphi(Ax)\vee\exists yAy}{\varphi(\exists yAy)\vee\exists yAy,}$$

and one for contracting redundancies:

$$\frac{\varphi(\exists yAy)\vee\exists yAy}{\varphi(\exists yAy).}$$

Adopting the rule as we have above amounts to gobbling up some structural ideas together with the rule for the connective. As we shall see, this procedure leads to economy; we learned it from Kanger 1957.

Surely the rule so put will preserve truth; for (i) if, for a given assignment of values to the variables, at least one disjunctive part of the left-hand disjunct $\varphi(Ax)$ of the premiss is true, then the conclusion $\varphi(\exists yAy)$ will be also, and (ii) if the right-hand disjunctive part $\exists yAy$ is true, then we can disjoin whatever we like, so as to get a true $\varphi(\exists yAy)$. So the rule preserves truth and, hence, validity.

But it is also helpful in another way. For tree construction, what

$$\frac{\varphi(Ax)\vee\exists yAy}{\varphi(\exists yAy)}$$

means is this: we consider a set V of variables $\{v_1, \ldots, v_i, \ldots\}$ (to be specified later) which we want to *try* as candidates for the allegedly existent y in $\varphi(\exists yAy)$ (which is just of the form $(\ldots\vee\exists yAy\vee\ldots)$), trying each in turn until (if possible) we find an axiom. So, for example, if anyone were idiot enough to try to prove $\exists yFy$, he would get out his blackboard and start generating candidates $v_1, \ldots, v_i, \ldots$ with a tree (starting from the bottom) which would look like this:

$$\begin{array}{l} \vdots \\ Fv_1 \vee Fv_2 \vee Fv_3 \vee \exists yFy \\ Fv_1 \vee Fv_2 \vee \exists yFy \\ Fv_1 \vee \exists yFy \\ \exists yFy \end{array}$$

Pretty clearly, this is not going to get one anywhere interesting— a reflection of the fact that $\exists yFy$ has nothing much to recommend it in the way of provability or validity.

On the other hand, if we listen to

$$\exists y(Fy \vee {\sim} Fy)$$

and try a case (throwing the existentially quantified bit, i.e., the whole thing, off to the right so that we can use it again if need be), we get a short tree:

$$\begin{array}{l} Fv_1 \vee {\sim} Fv_1 \vee \exists y(Fy \vee {\sim} Fy) \\ \exists y(Fy \vee {\sim} Fy) \end{array}$$

Formulas, like the one just above, that contain both an atom and its denial as disjunctive parts, we call *axioms*; and we will want to think of this two-node tree as a *proof* of the formula at the bottom.

The point of putting the existentially quantified formula at the extreme right is that we are certain to be able to work on intervening formulas in $\varphi(\exists yAy)$ before we try instantiating $\exists yAy$ again. Similar considerations motivate the decision always to work on the leftmost nonatomic molecule. This guarantees that nobody will be neglected, since, as the leftmost disjunctive parts become atoms, which we simply carry along in constructing trees, we will always have a chance to work at least once on each of the other nonatomic molecules before trying the existential quantifier again.

What about denied existential quantifications? Again we collapse two ideas which are classically distinct: generalization, and distribution of the universal quantifier into that disjunctive part in which quantification is nonvacuous. We explain this enigmatic remark as follows.

Consider a formula $A \vee Bx$, where x is not free in A, but might be free in Bx. Classically the rule of generalization allows us to write

$$\forall x(A \vee Bx)$$

as a consequence and then (with the help of an axiom or a rule)

$$A \vee \forall xBx,$$

since x is not free in A.

We bring these two ideas together and write

$$\frac{\varphi(\overline{Ax})}{\varphi(\overline{\exists yAy})}$$

provided x is not free in the conclusion, relying on the definition of universal quantification to help the reader see the connection between the classical ideas and those expressed here.

Before stating the rules for tree construction, we adopt two conventions which enable us to bypass certain inessential notational complications concerning free and bound occurrences of the same variable in a candidate formula A.

(1) In view of the fact that a formula A is valid in a nonempty domain just in case its universal closure $\forall x_1 \ldots \forall x_n A$ is valid in that domain (see Church 1956, **432, p.229), we agree without loss of generality to restrict attention to universally closed formulas.

(2) Again, for notational convenience, we introduce a new set of variables $\mathrm{V} = \{v_1, \ldots, v_i, \ldots\}$ (as hinted at in the examples above) to be called *parameters* as distinct from the *variables of quantification* $\{x, y, z, \ldots, x_i, y_i, z_i, \ldots\}$, with the understanding that these two styles of variables have no members in common.

We are now in a position to state the rules of tree construction, with a view to getting the two results:

LEMMA A. If every branch terminates in an axiom, the formula proved at the base is valid,

and

LEMMA B. If some branch does not terminate in an axiom, then we can use that branch as a source of a counterexample for the candidate, i.e., as a source of an assignment of values to the parameters which make it come out false.

We first summarize the *rules for tree construction.*

$$(1)\ \frac{\varphi(A)}{\varphi(\bar{\bar{A}})} \qquad (2)\ \frac{\varphi(\bar{A}) \qquad \varphi(\bar{B})}{\varphi(\overline{A \vee B})}$$

$$(3)\ \frac{\varphi(Av_i) \vee \exists xAx}{\varphi(\exists xAx)} \qquad (4)\ \frac{\varphi(\overline{Av_i})}{\varphi(\overline{\exists xAx})}$$

Given a candidate A, we first form its universal closure A_1, and place A_1 at the base of the tree. Then, always working first on the leftmost nonatomic molecule in any formula, we proceed as follows.

The first two rules are as in §24.1.1.

(3) For the third rule we introduce as v_i the *first* parameter *we haven't tried before in that context.* More explicitly, the first time an existential quantification occurs as leftmost nonatomic molecule, we instantiate the quantifier with v_1, throwing the quantified molecule off to the extreme right. The next time we are required to deal with that molecule, we try v_2; and so on.

(4) For the fourth rule, we instantiate with the first v_i that is *entirely new to the branch* (i.e., that has been used *nowhere* downstairs). This requirement will bring together the two ideas alluded to above and guarantee that, if the premiss (with free v_i) is valid, the conclusion will be also.

Then, finally, we count as an *axiom* any formula containing both an atom and its negation, a condition which we can determine by inspection. When we encounter an axiom, we stop the construction of the branch; otherwise we continue according to the four rules stated above.

It is trivial now to prove that, if the tree construction for a formula A winds up with every branch terminating in an axiom, then A is valid, in the sense that *every* assignment of values to its parameters comes out true.

Surely the axioms are true on any assignment of values to the parameters (they amount to nothing more than excluded middles, with perhaps other stuff irrelevantly disjoined).

For rules (1) and (2) the arguments concerning preservation of validity are exactly as in §24.1.1.

For rule (3), we observe that if, for a given assignment of values to the parameters, one of the v_i *does* have the property A, (i.e., if Av_i is a true disjunctive part), then $\exists xAx$ is true, since this is what we want to mean by existential quantification, and that, if truth is carried by some other dp in the premiss, it stays there in the conclusion.

For (4) similar remarks hold; we leave it to the reader to work them out.

This proves Lemma A above: if every branch terminates in an axiom, the formula is valid.

Somewhat more interesting is Lemma B: if some branch fails to terminate in an axiom, then the formula A at the base can be falsified. From this it follows that, if the formula cannot be falsified, i.e., if it is valid, then there is a proof of it, and so the system is *complete*: all valid formulas are provable.

REMARK 1. A formula might fail to terminate in an axiom either because it terminates in a nonaxiom (as must be the case if only rules (1), (2), and (4) apply in constructing its tree) or else because it is infinite (as in the case of the tree for $\exists xFx$ described above). For an unprovable formula, we want to find a falsifying assignment without regard to which of these two cases arises.

REMARK 2. As is so frequently the case for theorems of this kind, we will prove *vastly* more than we really want to know. What we are interested in showing is that the formula A_1 at the base of a tree that has a branch not terminating with an axiom, is falsifiable. What we show in fact is that *every* disjunctive part of *every* formula in a bad branch is falsifiable. Since A_1 is a disjunctive part of itself, and a formula in the bad branch, A_1 will then be falsified. The job is most conveniently divided into a couple of lemmas, but first we describe more explicitly (what Kanger 1957 calls) a *full normal branch.* It is called *normal* because we will know exactly what it looks like; no tricks or ingenuity are involved in its construction. And we call it *full* because it either terminates in an axiom or else exhausts *all* possible proof methods in the vain attempt to construct a good branch. Concisely:

B is a *full normal branch* of a deduction tree for A if A is the first member A_1 of B, and for each A_i in B, (i) if A_i is an axiom or has no nonatomic molecules, then A_i is the terminus of B; (ii) otherwise A_i has the form $\varphi(B)$, where B is the leftmost nonatomic molecule of A_i, and:

(iia) if B is $\bar{\bar{C}}$, then A_{i+1} is $\varphi(C)$;

(iib) if B is $\overline{C\vee D}$, then A_{i+1} is $\varphi(\bar{C})$ or $\varphi(\bar{D})$;

(iic) if B is $\exists yCy$, then A_{i+1} is $\varphi(Cv_j)\vee\exists yCy$, where v_j is the first parameter in V such that Cv_j does not occur as a dp of any of $A_1, \ldots, A_i$; and

(iid) if B is $\overline{\exists yCy}$ then A_{i+1} is $\varphi(\overline{Cv_j})$, where v_j is the first parameter in V not occurring free in $A_1, \ldots, A_i$.

We now enlarge in an obvious way the definition of *disjunctive part*: A is a disjunctive part of *a branch* B just in case A is a disjunctive part of some formula in that branch.

LEMMA 1. If B is a full normal branch for A, and B does not terminate in an axiom, then if a negative atom $\bar{B}$ is a disjunctive part of the branch B, B is not.

PROOF. This follows immediately from the fact that (though we *may* lose pieces of a formula, part of a negated disjunction for example, in the course of traveling up a bad branch), we *never* lose atoms; they always stay around as part of φ. So if an atom $\bar{B}$ cropped up at some step in the branch, and later on B appeared, $\bar{B}$ would still be with us as a dp, and we would have an axiom, *contra* hypothesis.

LEMMA 2. Let B be a full normal branch not terminating in an axiom. Then:

(i) if $B\vee C$ is a disjunctive part of the branch, so are B and C;

(ii) if $\bar{\bar{B}}$ is a dp of the branch, so is B;

(iii) if $\overline{B\vee C}$ is a dp of the branch, then so is either $\bar{B}$ or $\bar{C}$;

(iv) if $\exists yBy$ is a dp of the branch, then so are all formulas Bv_i, for every v_i; and
(v) if $\overline{\exists yBy}$ is a dp of the branch, then so is $\overline{Bv_i}$, for some v_i.

Proof. (i) follows from the definition of disjunctive parts;
(ii) follows from (iia) in the definition of a full normal branch;
(iii) follows from (iib);
(iv) follows from (iic), and the fact that if $\exists yBy$ occurs as a dp of the branch, then we try *every* v_i, so that Bv_i occurs for every v_i (this is the only case in which we get infinite branches);
(v) follows from (iid).

These lemmas now put us in a position to convince ourselves that, if there is a bad branch in a tree, then we can assign values to the parameters in the formulas which will make every disjunctive part of the branch (in particular the candidate formula A) come out false.

Suppose A has a bad branch. As domain we choose the natural numbers. Then to p we assign the value F if it occurs in the branch, and T otherwise. As values for parameters v_i in the branch, we give values from the natural numbers; in particular v_i gets the value i. And as values for n-ary predicate variables F, we give functions of n-tuples of natural numbers which take the n-tuple $\langle i_1, \dots, i_n\rangle$ into F if $Fv_{i_1} \dots v_{i_n}$ is an atom of the branch, and into T otherwise.

So now we have assigned values to all the variables in the bad branch, propositional, individual (i.e., parameters), and predicate. That this choice of domain and assignment does the required work is the content of the following

Theorem. If B is a full normal branch for A, and B does not terminate in an axiom, then the foregoing assignment of values to the variables gives F to every disjunctive part of the branch (and in particular to A).

Proof. We suppose inductively that dp's of B shorter than some fixed length n are all falsified, and proceed to argue by cases that all dp's of B are falsified.

If p or $Fv_{i_1} \dots v_{i_n}$ occurs in the bad branch, it comes out false on the assignment. And if the negations of either of these atoms occurs in the branch, Lemma 1 tells us that the corresponding positive atom B is not in B, hence is true on the assignment, and so its negation in B is false. Upshot: all atoms are falsified. For nonatomic formulas:

(i) $B \vee C$: by Lemma 1 and the hypothesis of induction, we already falsified both disjuncts;
(ii) $\overline{\overline{B}}$: we already falsified B;

(iii) $\overline{B \vee C}$: we already falsified one of $\bar{B}$ and $\bar{C}$; suppose it was $\bar{B}$: then $\bar{B}$ is false, so B is true, so $B \vee C$ is true, so $\overline{B \vee C}$ is false;

(iv) $\exists y By$: we already falsified *all* the shorter formulas Bv_i, which tells us that the claim that *there is* a natural number i such that Bv_i is true must be mistaken; we tried them all (Lemma 2) and they all failed (hypothesis of induction), so $\exists y By$ must be false;

(v) $\overline{\exists y By}$: in this case we know from Lemma 2 that $\overline{Bv_i}$ is false for some v_i; but then Bv_i is true for some v_i, in which case the statement $\exists y By$ is true, and $\overline{\exists y By}$ is false.

And that wraps up the proof. In a bad branch, all the atoms come out wrong, and anything built out of them is equally bad. So every unprovable (bad-branch) formula is falsifiable, from which it follows by contraposition that every unfalsifiable (i.e., valid) formula is provable; and this is just what we wanted: the system is complete.

It is probably worth remarking that most completeness proofs go this way. We want to see that all *valid* formulas are *provable*. But what we prove instead is that any *unprovable* formula is *invalid*. Those of us who (unlike intuitionistic mathematicians) believe that $\sim B \rightarrow \sim A \rightarrow . A \rightarrow B$, are satisfied with the result.

What connection does this have with $\mathbf{E}^{\exists x}$? The answer is that each of the rules for tree construction is mirrored as a theorem of $\mathbf{E}^{\exists x}$:

(1) $\forall x_1 \ldots \forall x_n(\varphi(A) \rightarrow \varphi(\bar{\bar{A}}))$

(2) $\forall x_1 \ldots \forall x_n((\varphi(\bar{A}) \& \varphi(\bar{B})) \rightarrow \varphi(\overline{A \vee B}))$

(3) $\forall x_1 \ldots \forall x_n \forall y((\varphi(Ay) \vee \exists x Ax) \rightarrow \varphi(\exists x Ax))$

(4) $\forall x_1 \ldots \forall x_n(\forall y \varphi(\overline{Ay}) \rightarrow \varphi(\overline{\exists x Ax}))$

where, for (4), in accordance with the fourth rule for tree construction y occurs free in $\varphi(Ay)$ *at most* in $\overline{Ay}$.

We leave most of the work in verifying (1)–(4) to the reader, simply giving some hints about (4).

$\varphi(\overline{Ay})$ looks like $(\ldots \vee \overline{Ay} \vee \ldots)$, where y is not free elsewhere than indicated in the formula. Properties of disjunction ensure that we can rewrite this as $B \vee \overline{Ay}$. Then the (derived) rule of generalization of §38.2 says that we can quantify universally to get $\forall y(B \vee \overline{Ay})$. This leads via Axiom IQ5 of §38.2 to $B \vee \forall y \overline{Ay}$, i.e., $B \vee (\overline{\exists y Ay})$. But now again properties of disjunction let us put the right-hand disjunct back into the position from which it started; so we get $\varphi(\overline{\exists y Ay})$. All this means that

$$\forall y \varphi(\overline{Ay}) \rightarrow \varphi(\overline{\exists x Ax})$$

is provable, provided the stuff in φ does not contain y free. Then the rule of generalization enables us to tack on as many more universal quantifiers $\forall x_i$ as we like.

The extensional fragment of $\mathbf{E}^{\exists x}$ (otherwise known as the first-order predicate calculus) is therefore complete, a fact which pleases us. Again (as in §24.1.2) we observe that the simplicity of the foregoing proof arises in some mysterious way from the fact that the rules correspond to valid entailments (1)–(4) and obviate the use of that clumsy rule (γ).

THEOREM. If any formula of zero degree is classically valid then it is a theorem of $\mathbf{E}^{\exists x}$—and indeed of any of its cousins.

§39.2. Löwenheim-Skolem theorem. Everyone was upset by the theorem of Löwenheim 1915 and by Skolem's generalizations (1920) of it, which we shall prove shortly. But first it might be well to say a few words about why it was so upsetting.

The "fundament of abstract set theory" as Fraenkel 1953, p. 76, called it, is the theorem to the effect that the set of real numbers is not denumerable. This now familiar result of Cantor 1874, together with the paradoxes to which Cantor's naive set theory led, prompted the rise of modern set theory in the logical clothes originally patterned by Cantor himself and developed more rigorously by Zermelo, Russell, von Neumann, Fraenkel, Hilbert, Bernays, Gödel, etc. etc.

One of the principal motivations for this enterprise was to find a set of axioms for the $\in$-relation of set membership which would (a) avoid the familiar paradoxes of Cantor and Russell, and (b) guarantee the existence of non-denumerable sets. The axioms are presumably to be framed within the (classical) first-order predicate calculus, taking a single binary predicate, "$\in$" as primitive.

Now if we have finitely many axioms for the $\in$-relation, it is obvious that we can think of these axioms as really being one: namely the conjunction of whatever axioms we choose.

Löwenheim's startling theorem of 1915 seemed to say that it was impossible to satisfy *both* the conditions (a) and (b) above simultaneously. If (a) is satisfiable (i.e., if the finite set of axioms avoids inconsistency) then the guarantee intended by (b) is lost: there is a *denumerable* interpretation, or model, for the set of axioms.

Indeed this fact follows easily from Gödel's completeness theorem, proved in the preceding section. If a formula A is satisfiable (i.e., if there is an assignment of values to its variables which makes it come out true), then $\bar{A}$ can't be valid (i.e., such that *every* assignment makes $\bar{A}$ come out true). But if $\bar{A}$ is not valid, it is falsifiable, by the argument of the preceding section, where the domain of individuals is the natural numbers. And of course this assignment, which falsifies $\bar{A}$, makes A true—also in the natural numbers.

So, if A is satisfiable at all, it is satisfiable in a domain that is at most denumerable. This seems to say that *categoricity* is out of the question for set theories formulated with finitely many axioms for the membership relation, within the framework of classical first-order logic. Any model of the syntax which has infinite sets at all (the nondenumerable sets being the interesting ones) will also have denumerable models.

Skolem generalized Löwenheim's theorem in a natural way, which quashed one (at the time) reasonable possibility of getting out of Löwenheim's bind. The question can be put as follows. Granted that finitely many axioms for set theory will always have a denumerable model, maybe we could save both (a) and (b) above by enlisting the support of infinitely many axioms. Skolem's contribution was that, in a countable language, this dodge won't work either. We now proceed to prove Skolem's generalization of Löwenheim's theorem. (For a good and careful discussion of the philosophical sense of all this, see Myhill 1953a.)

The fundamental idea is to consider sequences of formulas Γ, which may be *infinite*, and to show by a tree construction like that of §39.1 that if they are satisfiable at all, they are satisfiable in a denumerably infinite domain of individuals (to wit, the natural numbers).

But for this purpose we cannot throw existentially quantified expressions off to the extreme right, as we did in the previous section, since the sequences Γ are too long. On the other hand, we will have to put them *somewhere* in the sequence, in the course of tree construction, since, as before, we may need to try them again. This consideration will lead us to want to toss them some distance to the right: far enough so that we get to work on intervening formulas, but not so far that we lose them altogether.

But now another consideration comes to mind. In the case of the Gödel completeness theorem we were interested in the provability and validity of a formula whose branches *all* came out well. For Skolem's generalization of Löwenheim's theorem we are interested rather in just those branches which do not end badly, where, by saying that they end badly, we mean that they terminate in a contradiction. If a branch has both A and $\bar{A}$ as parts of the sequence, then obviously we cannot assign values to the variables in both A and $\bar{A}$ in such a way that they both turn out to be true; otherwise we *can* satisfy the branch.

This consideration leads us to do everything dually to the Gödel theorem, as follows, where we spend one paragraph introducing notation.

Γ, Δ, Λ are sequences (of type $\leq\omega$) of formulas written in negation conjunction, and universal quantification (other connectives being defined): Γ and Δ are finite or empty, Λ is infinite or empty. If S is the sequence Γ, A, Δ, Λ then (i) A is a *formula of* S, and (ii) if A has the form p, $\bar{p}$, $Fx_1 \ldots x_n$, or $\overline{Fx_1 \ldots x_n}$, then A is an *atom of* S. If S contains atoms A and $\bar{A}$, then S is an *explicit contradiction.*

Again to simplify notation we use parameters $v_1, \ldots, v_i, \ldots$ as in the preceding section, and we rely on the fact that a formula is satisfiable in a nonempty domain just in case its existential closure is satisfiable in that domain (see Church 1956, **433, p. 229). Evidently this result is extendable to a sequence Γ of formulas, which are simultaneously satisfiable just if the sequence of existential closures of members of Γ are simultaneously satisfiable. Again we restrict attention to sequences of closed formulas.

Now let S be a sequence of closed formulas containing as connectives only $\overline{\ }$, &, and $\forall x$. The *satisfiability tree* for S consists of all those satisfiability *branches* B such that S is the first member S_1 of B, and if S_i is the ith member of B, then: (i) if S_i contains only atoms or is an explicit contradiction, then S_i is the terminus of B; (ii) otherwise S_i has the form $\Gamma, A, \Delta, \Lambda$, where A is the leftmost nonatomic formula of S_i, and

(iia) if A is $\bar{\bar{B}}$, then S_{i+1} is $\Gamma, B, \Delta, \Lambda$;
(iib) if A is $B\&C$, then S_{i+1} is $\Gamma, B, C, \Delta, \Lambda$;
(iic) if A is $\overline{B\&C}$, then S_{i+1} is $\Gamma, \bar{B}, \Delta, \Lambda$, or $\Gamma, \bar{C}, \Delta, \Lambda$;
(iid) if A is $\forall xBx$, then S_{i+1} is $\Gamma, Bv_j, \Delta, \forall xBx, \Lambda$, where v_j is the first parameter in V such that Bv_j does not occur as a formula in any of the sequences $S_1, \ldots, S_i$, and where Λ is empty if S_i is finite, and Δ contains i members if S_i is infinite;
(iie) if A is $\overline{\forall xBx}$, then S_{i+1} is $\Gamma, \overline{Bv_j}, \Delta, \Lambda$, where v_j is the first parameter in V not occurring free in $S_1, \ldots, S_i$.

The condition in (iid), requiring us to move the activated molecule $\forall xBx$ i places to the right, is designed to ensure that we keep instantiating $\forall xBx$ as often as necessary, without neglecting any of the other nonatomic molecules in the sequence. We leave it to the reader to state this condition precisely and to show that it achieves its aim.

By an atom or formula of a satisfiability branch B we mean an atom or formula of some S_i in B, and we proceed to three lemmas.

LEMMA 1. If every satisfiability branch for S_1 terminates in an explicit contradiction, then the formulas of S_1 are not simultaneously satisfiable.

PROOF. The tips of all branches are evidently not simultaneously satisfiable, and an inductive proof suffices to show that unsatisfiability passes down the branch.

LEMMA 2. If some satisfiability branch B does not terminate in an explicit contradiction, then if an atom $\bar{A}$ occurs in B, A does not.

PROOF. Otherwise the branch would terminate in an explicit contradiction, since atoms are never lost in moving up the branch.

LEMMA 3. If some satisfiability branch B does not terminate in an explicit contradiction, then

(i) if $\bar{\bar{A}}$ is a formula of B, so is A;
(ii) if $A\&B$ is a formula of B, so are A and B;
(iii) if $\overline{A\&B}$ is a formula of B, so is either $\bar{A}$ or $\bar{B}$;
(iv) if $\forall xAx$ is a formula of B, so is Av_i for every i;
(v) if $\overline{\forall xAx}$ is a formula of B, so is $\overline{Av_i}$ for some i.

PROOF. By rules of satisfiability-branch construction.

Lemmas 2 and 3 now enable us to prove the following inductively:

THEOREM. Let K be a class of simultaneously satisfiable formulas; then K is simultaneously satisfiable in the domain of the natural numbers.

PROOF. Given K, we arrange its members in a sequence S (which can always be done, since the whole calculus has only denumerably many formulas). Then, by Lemma 1, some branch B of the satisfiability tree for S fails to terminate in an explicit contradiction. To the free variables of this branch B assign values as follows: to p give T iff p is an atom of B; to v_i give i; to n-ary F give the function taking $\langle i_1, \ldots, i_n\rangle$ into T iff $Fv_{i_1} \ldots v_{i_n}$ is an atom of B. Then, by an inductive argument dual to that of §39.1, Lemmas 2 and 3 guarantee that every formula A of B (and in particular every formula of S) takes the value T by this assignment.

§39.3. Gentzen's cut elimination theorem. We discuss a new proof of Gentzen's *Hauptsatz*, often called the "Cut theorem," for classical first order logic. The proof is to be found in Dunn and Meyer 1974. We give not much in the way of detail here.

This proof was discovered by analogizing results of Meyer and Dunn and Leblanc 1974 concerning the redundancy or "admissibility" of Ackermann's rule (γ) in relevance logic, specifically for a Hilbert-style (axiomatic) formulation of $\mathbf{R}^{\forall\exists x}$ (see §25 and §42).

We suppose that the proof could be done directly on the formalism of the calculus of sequents of Gentzen 1934. For reasons which are basically stylistic, Dunn and Meyer 1989, then work instead with the formal system $\mathbf{K}_1$, introduced by Schütte 1950 as a variant on Gentzen (see §7.2 and §24.1).

The formation rules for the formulas of $\mathbf{K}_1$ are just as for the system of §39.1 (actually Schütte used a second run of parameters a, b, c, ... for the free variables, but this, and another technical proviso about not allowing overlapping quantifiers with the same variable, are entirely matters of convenience).

The axioms of $\mathbf{K}_1$ are all formulas of the form $A \vee \bar{A}$.

The inference rules divide themselves into two types (as a mnemonic device we have substituted the name of the most nearly similar "Copi rule," from Copi 1954, when the original German name did not translate well):

Structural rules

Interchange *Contraction*

$$\frac{M \vee A \vee B \vee N}{M \vee B \vee A \vee N} \qquad \frac{N \vee A \vee A}{N \vee A}$$

Operational rules

Weakening *De Morgan* (DeM) *Double Negation* (DN)

$$\frac{N}{N \vee B} \qquad \frac{N \vee \bar{A} \vee N \vee \bar{B}}{N \vee \overline{A \vee B}} \qquad \frac{\mathrm{N} \vee A}{N \vee \bar{\bar{A}}}$$

Existential Generalization (EG) *Universal Generalization* (UG)

$$\frac{N \vee A(y)}{N \vee \exists x A(x)} \qquad \frac{N \vee \overline{A(y)}}{N \vee \overline{\exists x \overline{A(x)}}}$$

(UG is subject to the proviso that y, called the *eigenparameter*, is not free in the conclusion.)

In these rules M and N are called the *side formulas*, and the others the *principal formulas*. It is understood in every case but that of *Weakening* that either or both of the side formulas may be missing. Also, there is an understanding in multiple disjunctions that parentheses are to be associated to the right.

Recall that the basic formal objects of Gentzen's *sequenzen* calculus LK, were more complicated, being the *sequents* $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$, where A_is and B_js are formulas (any or all of which might be missing). Such a sequent may be interpreted as a statement to the effect that either one of the A_is is false or one of the B_js is true. To every such sequent there corresponds what we might call its "right-handed counterpart", $\vdash \bar{A}_1, \ldots, \bar{A}_m, B_1, \ldots, B_n$. In a straightforward fashion it is possible to develop a calculus parallel to Gentzen's using only "right-handed" sequents, i.e., those with empty left side. This is in effect what Schütte did, but with one further trick. Instead of working with a right-handed sequent $\vdash A_1, \ldots, A_m$, which can be thought of as a sequence of formulas, he in effect replaced it with the single formula $A_1 \vee \ldots \vee A_m$. The reasons for using Schütte's formalism in preference to Gentzen's have something to do with the fact that theories will be constructed out of such disjunctive formulas, with talk of some such being deducible from others, etc., all in analogy with situations in Hilbert-style formalisms for relevance logic, where the appropriate formal objects are indeed just plain old formulas (*not* sequents).

In Meyer and Dunn and Leblanc 1974 it was said that the rule *Cut* is just (γ) "in peculiar notation." In the context of Schütte's formalism the notation is not even so different. Thus:

$$\frac{M \vee A \qquad \bar{A} \vee N}{M \vee N}\,Cut \qquad\qquad \frac{A \qquad \bar{A} \vee B}{B}\,(\gamma)$$

Since it is understood by Schütte that either M or N may be missing, obviously (γ) is just a special case of *Cut* (and, conversely, a few manipulations will get Cut from (γ) in the context of the other rules of $\mathbf{K}_1$).

We shall not regard the rule *Cut* as a part of the primitive basis of $\mathbf{K}_1$. This is unlike Schütte, who (following Gentzen) did regard the rule of Cut as primitive. This will produce a superficial difference in the "Cut Theorems." Thus Schütte's (Gentzen's) says that *Cut* is actually redundant, whereas we shall say that Cut would be redundant if it were added; i.e., it is *admissible* in the sense that adding it would produce no new theorems.

The reader should have no trouble in seeing the system of §39.1 as a variant of $\mathbf{K}_1$, with *Interchange* and *Weakening* dispensed with in favor of more general forms of *excluded middle* as axioms and with contraction built into the rule for the existential quantifier. Although these differences are matters of taste and convenience when the systems are regarded as theorem-producing devices, the explicit presence of the structural rules in $\mathbf{K}_1$ seems essential to adapting the (γ)-proof.

Of course the rule of *Cut* is admissible for the system of §39.1. This follows not only because of its equivalence (as a theorem producer) to the system $\mathbf{K}_1$, but also because of its completeness (and soundness) as shown in §39.1. It is obvious that *Cut* preserves validity. The interest in the present section is, then, not in the result (which is already implicit in §39.1, blurring the distinctions between systems), but in the method of proof (which, in the terminology of Smullyan 1968, is "synthetic" rather than the usual "analytic"—see §42 below).

The basic novelty in the proof is to take the rules of the system (without *Cut*) to be rules of deducibility and not just rules of proof. Because of the "subformula property" (see §7.2) of Gentzen-style rules (other than *Cut*), there are strong connections of relevance between the premisses and conclusion. This means that if one defines an appropriate notion of "deducibility" based upon these rules, it will be nonclassical in ways basically familiar to students of relevance logic. In particular, not every formula is "deducible" from a contradiction. In fact, the rules all correspond to provable entailments in the system **E** (see §39.1).

Where Γ is a set of formulas, we define a *deduction* of A from Γ as a finite tree of formulas, with A as its origin, with members of Γ or axioms as its end points, and such that each point that is not an end point follows from its successors by one of the rules, but where it is required that, if the rule is that for the negated existential quantifier, then the subtree having as origin the conclusion be such that the "eigenparameter" occurs at the endpoints only in axioms. We say A *is deducible from* Γ (in symbols, "$\Gamma \vdash A$") iff there is a deduction of A from Γ. We shall call a set of formulas closed under deducibility a *theory*. Here we get more vague. The basic idea of the proof is to assume that A and $\bar{A} \vee B$ are theorems (deducible from the null set) and that B is not. Then, using Lindenbaum-Henkin methods, one builds up a complete theory T which is maximal with respect to not containing B and which contains all the theorems and has some other nice properties respecting disjunction and existential quantification. (Here it is important to mimic the original (γ)-proof of Meyer and Dunn and Leblanc 1974, and not the prettier "symmetric" construction of §42. Deducibility as defined in this section does not have all the properties required by the symmetric construction.)

This theory T will be highly inconsistent, since (because of the subformula property) it will often be the case that both a formula and its negation can be added (neither one being a subformula of B). One next chooses one's favorite way of shrinking T to a complete, *consistent* theory T'. (Meyer calls this "the Converse Lindenbaum Lemma"; §42 calls it "The Way Down.") Dunn and Meyer 1989 happen to use the "metavaluation" approach of Meyer 197+ (again see §42).

We now take the occasion briefly to describe some other results (unpublished) that have similar suggestions of a connection between relevance logic and (classical) proof theory.

The first result is due to Meyer 1976f, where he shows the admissibility of the rule (γ) for a wide variety of higher-order relevance logics (any order $\leq \omega$, and any reasonable choice of instances of the comprehension scheme). Back in 1976 (unpublished), Dunn and Meyer with E. P. Martin extended and analogized this proof, in much the same way as the (γ)-argument for first-order logic has been analogized, so as to apply it to a Gentzen-style formalism for classical higher-order logic and thus to obtain a new proof of Takeuti's theorem (cut elimination for simple type theory). This proof dualizes the proofs of Takahashi and Prawitz (see Prawitz 1968) in the same way that the proof here dualizes the usual semantical proofs of cut elimination for classical first-order logic. This dualization is vividly described by saying that, in place of "Schütte's lemma" that every semivaluation may be extended to a (total) valuation, there is instead the "Converse Schütte lemma" that every "ambivaluation" (sometimes assigns a sentence both the values *0*, *1*) may be restricted to a (consistent) valuation.

§40. Algebra and semantics for first degree formulas with quantifiers.

The business of this section is to extend the results of §§18 and 19, which we presuppose, to *complete* intensional lattices and to first degree formulas (in the sense of §19) involving quantifiers.

In §40.1 we develop some algebraic properties of complete intensional lattices in analogy to §18. The reader interested only in completeness and consistency may skip all but the first few paragraphs (up to the first theorem) of this section; for it is in §§40.2–7 that we develop those results immediately relevant to completeness and consistency.

Establishing the consistency and completeness of the first degree fragments *of the intensional logics* $\mathbf{E}^{\exists x}$, $\mathbf{E}$, $\mathbf{R}^{\exists x}$, *and* $\mathbf{R}$ *will require*, besides (1) the definition and general properties of complete intensional lattices, the introduction of the following notions: (2) some special facts about complete intensional lattices, (3) the theory of propositions, (4) intensional models, (5) full normal branches and trees, and (6) a certain kind of critical model determined by a full normal branch. Our strategy will be to introduce these notions section by section in the order and with the numbering given above, demonstrating as we go along those required lemmas which can be stated and proved in terms of the concepts so far introduced. Then (7) we shall draw together the previously developed machinery in order to give brief proofs of our main theorems. The reader may wish to consult §40.7 for a preview.

We assume $\mathbf{E}^{\exists x}$ and $\mathbf{R}^{\exists x}$ formulated with $\rightarrow$, $\bar{\ }$, $\vee$, and $\exists$ as primitive.

§40.1. Complete intensional lattices. Intensional lattices (i.l.) as defined in §18.2 suffice for the semantics of the quantifier-free systems **E** and **R**. For the systems $\mathbf{E}^{\exists x}$ and $\mathbf{R}^{\exists x}$ involving quantifiers, however, we need the effect not only of finite meets and joins but also of infinite meets and joins to be used in connection with the universal and existential quantifiers; introducing infinite meets and joins causes us also suitably to modify the distributive and truth-filter conditions.

An intensional lattice $\langle L, \leq, \bar{\ }, T\rangle$ is, accordingly, said to be *complete, completely distributive*, and with *complete truth filter*, provided

(C) For every subset G of L, there exist in L unique elements $\bigwedge G$ and $\bigvee G$ such that for every $a \in L$, $a \leq \bigwedge G$ iff $a \leq b$ for every $b \in G$, and $\bigvee G \leq a$ iff $b \leq a$ for every $b \in G$.

Where $\{a_x\}_{x\in X}$ is an indexed set, we also write $\bigwedge_{x\in X} a_x$ for $\bigwedge\{a_x\}_{x\in X}$ and $\bigvee_{x\in X} a_x$ for $\bigvee\{a_x\}_{x\in X}$.

(CD) $\bigwedge_{x\in X}\bigvee_{y\in Y} a_{x,y} = \bigvee_{f\in S}\bigwedge_{x\in X} a_{x,f(x)}$, and dually. Here S is the set of all functions from X into Y.

(CT) For every $G \subseteq L$, $\bigwedge G \in T$ iff $G \subseteq T$. Since in the sequel we do not consider intensional lattices which are complete in the sense of satisfying (C) but which fail to satisfy the further conditions (CD) and (CT), we shall permit ourselves to mean by a complete intensional lattice one satisfying all of (C), (CD), and (CT).

(CD) has a lot of subscripts, but the idea is not so bad geometrically. *Picture* the left side as a giant conjunction of giant disjunctions. Now, without blinking, run through all the disjunctions, picking one member from each. Collect these together as a conjunction, and set it aside. Now run through again, and in a new way, pick one member from each disjunction, make a conjunction out of the result, and set it aside. Having done this twice, *very quickly* proceed to do it in all possible ways. Lastly, take all these newly made conjunctions and put them together disjunctively—getting the right side of (CD).

The following are some easily demonstrated and pervasively applicable properties of complete intensional lattices; the finite analogues, of course, also hold.

(1) $a \le \wedge G$ iff $a \le b$ for all $b \in G$, $\vee G \le a$ iff $b \le a$ for all $b \in G$, and $\vee G \le \wedge H$ iff $b \le c$ for $b \in G$, $c \in H$. Also $\bigwedge_{x\in X} \bigvee_{y\in Y} a_{x,y} \le b$ iff $\bigwedge_{x\in X} a_{x,s(x)} \le b$ for all functions s from X into Y, and $a \le \bigvee_{x\in X} \bigwedge_{y\in Y} b_{x,y}$ iff $a \le \bigvee_{x\in X} b_{x,s(x)}$ for all functions s from X into Y.

(2) $G \subseteq H$ implies $\wedge H \le \wedge G$ and $\vee G \le \vee H$; and if G and H share some element, then $\wedge G \le \vee H$.

(3) $\overline{\bigwedge_{x\in X} a_x} = \bigvee_{x\in X} \overline{a_x}$ and $\overline{\bigvee_{x\in X} a_x} = \bigwedge_{x\in X} \overline{a_x}$.

(4) Generalized associativity and commutativity hold for any combination of finite and infinite meets and for any combination of finite and infinite joins.

(5) If a set G generates a complete intensional lattice L, then any $a \in L$ can be represented in a meet-normal form $a = \bigwedge_{x\in X} \bigvee_{y\in Y} a_{x,y}$, where for $x \in X$, $y \in Y$, $a_{x,y} \in G \cup \{\bar{g} : g \in G\}$, and also in a join-normal form $\bigvee_{x\in X} \bigwedge_{y\in Y} a_{x,y}$.

Some more definitions. A set G contained in a {complete} intensional lattice L is said to be {*completely*} *independent* if (i) for all $a \in L$, if $a \in G$ then $\bar{a} \notin G$, and (ii) for every pair of finite {and infinite} subsets $\{a_x\}_{x\in X}$ and $\{a_y\}_{y\in Y}$ of $G \cup \{\bar{c} : c \in G\}$ if $\bigwedge_{x\in X} a_x \le \bigvee_{y\in Y} a_y$, then for some $x \in X$ and $y \in Y$, $a_x = a_y$.

A {complete} intensional lattice $\langle L', \le', ^{-\prime}, T' \rangle$ is a {complete} sublattice of a {complete} intensional lattice $\langle L, \le, ^{-}, T \rangle$ iff (i) $L' \subseteq L$; (ii) with respect to L′, $\le'$ agrees with $\le$, $^{-\prime}$ agrees with $^{-}$, and T′ agrees with T; and (iii) meets and joins of members of L′ {and of subsets of L′} are the same with respect to L′ and L; which is to say that L′ is closed under joins and meets in the sense of $\langle L, \le, ^{-}, T \rangle$.

A mapping h from a {complete} intensional lattice L into a {complete} intensional lattice L′ is a {*complete*} T-homomorphism from L into L′ iff, for $a, b \in L$, and $\{a_x\}_{x\in X} \subseteq L$, $a \le b$ implies $h(a) \le h(b)$, $h(\bar{a}) = \overline{h(a)}$, $h(a \vee b) = h(a) \vee h(b)$ $\{h(\bigvee_{x\in X} a_x) = \bigvee_{x\in X} h(a_x)\}$ (and dually for $\wedge$), and $a \in T$ implies $h(a) \in T'$, where T and T′ are the {complete} truth filters of L and L′, respectively. (After this section we often say "homomorphism" for "T-homomorphism.") The first condition follows from the others, as does the dual condition for $\wedge$, and the fact that, for all $a \in L$, $a \in F = (L - T)$ implies $h(a) \in F' = (L' - T')$. If h is in addition one-one, then it is a {complete} *isomorphism*, and if h is onto L′, then L and L′ are {completely} *isomorphic.*

That G be a {completely} independent set of generators for a {complete} intensional lattice L is a necessary and sufficient condition for G's being a set of *free generators* for L, in the sense that if f is any mapping of G into a {complete} intensional lattice L′ having the property that for all $a \in G$, $a \in T$ implies $f(a) \in T'$ and $a \in F$ implies $f(a) \in F'$, then f can be extended to a {complete} homomorphism of L into L′.

If a {complete} intensional lattice is generated by some {completely} independent set of generators, then the lattice itself is said to be {completely} independent. (What is here called a "completely independent complete intensional lattice" was called an "atomic frame" in Anderson and Belnap 1963; that terminology was chosen because of the relevance of the concept to the philosophical notion of an atomic proposition, but we here modify our usage in order to conform to that of algebraists.)

The rest of this section is devoted to presenting a set of illuminating facts about complete intensional lattices.

One more definition, and we may state a theorem. A filter P is *completely prime* iff, if $\bigvee_{x \in X} a_x \in P$, then some $a_x \in P$. Note that, if P is a complete and completely prime filter, then $-P$ is a complete and completely prime ideal (defined dually).

THEOREM 1. Every complete and completely prime filter P of a complete i.l. (intensional lattice) $\langle L, \leq, \bar{\ }, T\rangle$ determines a complete T-homomorphism h of L into M_0 that satisfies conditions 1, 2, and 3 of Theorem §18.4(1) (where F_i is the principal filter generated by i in M_0):

1 $h(a) \in F_{-1}$ iff $a \in P$,

2 $h(a) \in F_{-2}$ iff $\bar{a} \in -P$ (where $-P$ is the set-theoretical complement of P relative to L),

3 $h(a) \in F_{+0}$ iff $a \in T$.

In view of Theorem §18.4(1) and the remarks concerning the strategy of its proof, it suffices to show that $h(\bigwedge_{x \in X} a_x) \in F_i$ iff $\bigwedge_{x \in X} h(a_x) \in F_i$ ($i = -1, -2, +0$). Again we treat $i = -1$ and $i = +0$ together, observing first that F_{-1} $\{F_{+0}\}$ is complete (since M_0 is finite) and that P {T} is complete. Thus, $h(\bigwedge_{x \in X} a_x) \in F_{-1}$ $\{F_{+0}\}$, iff $\bigwedge_{x \in X} a_x \in P$ {T}, iff $\{a_x\}_{x \in X} \subseteq P$ {T}, iff $\{h(a_x)\}_{x \in X} \subseteq F_{-1}$ $\{F_{+0}\}$, iff $\bigwedge_{x \in X} h(a_x) \in F_{-1}$ $\{F_{+0}\}$. We treat $i = -2$ by utilizing the fact that $-P$ is a complete and completely prime ideal, together with a De Morgan property of $\bar{\ }$ over $\wedge$. Thus, $h(\bigwedge_{x \in X} a_x) \in F_{-2}$, iff $\overline{\bigwedge_{x \in X} a_x} = \bigvee_{x \in X} \bar{a}_x \in -P$ iff $\{\bar{a}_x\}_{x \in X} \subseteq -P$, iff $\{h(a_x)\}_{x \in X} \subseteq F_{-2}$, iff $\bigwedge_{x \in X} h(a_x) \in F_{-2}$.

We again also have the converse theorem:

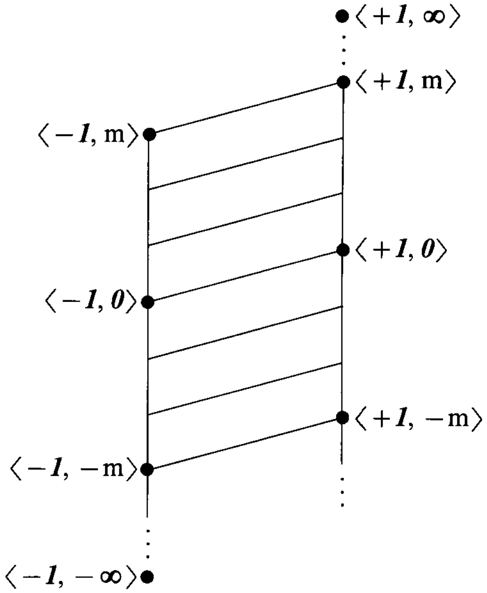
THEOREM 2. Every complete T-homomorphism h of a complete i.l. $\langle L, \leq, \bar{\ }, T\rangle$ into M_0 determines a complete and completely prime filter P of $\langle L, \leq, \bar{\ }, T\rangle$ in accord with conditions (1) and (2) of Theorem §18.4(1).

That condition (1) determines P to be a complete and completely prime filter follows immediately from the easily verified fact that if h is a complete homomorphism into a complete and completely prime filter (in this case F_{-1}), then the inverse image of that complete and completely prime filter under h (in this case P) is a complete and completely prime filter also. The proof that P as determined by (1) also satisfies (2) is the same as in Theorem §18.4(2).

We remark that by combining Theorems 1 and 2 we obtain a natural one-to-one correspondence between complete, completely prime filters and complete T-homomorphisms into M_0. The justification of this remark rests on the same argument as that after Theorem §18.4(2).

Let us say that a structure that is like a complete intensional lattice except that it is not required that there exist any truth filter, is a *complete De Morgan lattice.*

An example due to Spencer (see Belnap and Spencer 1966) shows that there are De Morgan lattices that satisfy (C) and such that $^-$ has no fixed point, and yet still have no truth filter T that satisfies (CT). For let $\langle L, \leq, ^-\rangle$ be such that (i) the elements of L are $\langle 1, \infty\rangle$ and $\langle -1, -\infty\rangle$ together with all ordered pairs $\langle a, b\rangle$ where $a = \pm 1$ and b is an integer; (ii) $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$; and (iii) $\overline{\langle a, b\rangle} = \langle -a, -b\rangle$. It is easily shown that $\langle L, \leq, ^-\rangle$ is a complete distributive lattice satisfying (N_1)–(N_3) of §§18.2–3.



Suppose now that there is a complete, consistent, and exhaustive filter T in L. Then, since $\langle -\mathit{1}, \mathit{0}\rangle \leq \overline{\langle -\mathit{1}, \mathit{0}\rangle}$, we shall have to have $\langle -\mathit{1}, \mathit{0}\rangle \notin \mathrm{T}$, on pain of inconsistency of T; hence $\overline{\langle -\mathit{1}, \mathit{0}\rangle} \in \mathrm{T}$, since T is exhaustive. However, for all $\mathrm{a} \geq \mathit{0}$, $\langle -\mathit{1}, \mathrm{a}\rangle \wedge \overline{\langle -\mathit{1}, \mathit{0}\rangle} = \langle -\mathit{1}, \mathit{0}\rangle$; so the filterhood of T implies that $\langle -\mathit{1}, \mathrm{a}\rangle \notin \mathrm{T}$ for all such a. But T is exhaustive; so the set X of all pairs $\overline{\langle -\mathit{1}, \mathrm{a}\rangle}$ $(= \langle \mathit{1}, -\mathrm{a}\rangle)$, for $\mathrm{a} \geq \mathit{0}$, is a subset of T; and, since T is a complete filter, the meet of X is in T. But $\langle -\mathit{1}, -\infty\rangle$ is evidently the meet of X; so $\langle -\mathit{1}, -\infty\rangle \in \mathrm{T}$—from which it follows immediately that T is inconsistent, contrary to assumption.

We now prove that adding the condition (CD) makes a material difference in this matter.

THEOREM 3. A necessary and sufficient condition for a complete De Morgan lattice $\langle \mathrm{L}, \leq, {}^{-}\rangle$ to be a complete i.l. is that, for all $\mathrm{a} \in \mathrm{L}$, $\mathrm{a} \neq \bar{\mathrm{a}}$.

The necessity is obvious, because, for all $\mathrm{a} \in \mathrm{A}$, T must contain exactly one of a and $\bar{\mathrm{a}}$.

To prove sufficiency, we establish first a related lemma of some independent interest. First, where $\langle \mathrm{L}, \leq, {}^{-}\rangle$ is a complete De Morgan lattice, we define $\boldsymbol{t} = \bigwedge \{\mathrm{a} \vee \bar{\mathrm{a}} \colon \mathrm{a} \in \mathrm{L}\}$. Dually, we define $\boldsymbol{f}$ as the disjunction of all contradictions; the formal definition is $\boldsymbol{f} = \bigvee \{\mathrm{a} \wedge \bar{\mathrm{a}} \colon \mathrm{a} \in \mathrm{L}\}$. Note that, by complete De Morgan properties, $\boldsymbol{t} = \bar{\boldsymbol{f}}$.

LEMMA. Let $\langle \mathrm{L}, \leq, {}^{-}\rangle$ be a complete De Morgan lattice. Then a necessary and sufficient condition that L have a fixed point (with respect to negation) is that $\boldsymbol{t} \leq \boldsymbol{f}$, where $\boldsymbol{t}, \boldsymbol{f}$ are as just defined. That is, $\boldsymbol{t} \leq \boldsymbol{f}$ iff there exists an $\mathrm{a} \in \mathrm{L}$ such that $\mathrm{a} = \bar{\mathrm{a}}$.

We prove the lemma first and then complete the proof of Theorem 3. Again, half of the lemma is trivial. For, if there is an $\mathrm{a} \in \mathrm{L}$ such that $\mathrm{a} = \bar{\mathrm{a}}$, then clearly $\boldsymbol{t} \leq \mathrm{a} \vee \bar{\mathrm{a}} = \mathrm{a} = \bar{\mathrm{a}} = \mathrm{a} \wedge \bar{\mathrm{a}} \leq \boldsymbol{f}$, by definitions of $\boldsymbol{t}$ and $\boldsymbol{f}$ and lattice properties. We turn to the converse. This proof using the Maximal Chain lemma is Meyer's (unpublished); for a proof using well-ordering, see Dunn and Belnap 1968a.

Suppose, then, that $\boldsymbol{t} \leq \boldsymbol{f}$. We wish to find an $\mathrm{a} \in \mathrm{L}$ such that $\mathrm{a} = \bar{\mathrm{a}}$. We consider the set S of elements b of L such that $\boldsymbol{t} \leq \mathrm{b} \leq \bar{\mathrm{b}}$, and we note that S is partially ordered by $\leq$. By the Maximal Chain lemma, a well-known equivalent of the Axiom of Choice, there is a maximal chain $\mathrm{M} \subseteq \mathrm{S}$ such that $\boldsymbol{t} \in \mathrm{M}$. Let

(1) $\mathrm{m} = \bigvee \mathrm{M}$.

We show that $\mathrm{m} = \bar{\mathrm{m}}$, which will establish the lemma. Easily,

(2) $\mathrm{m} \leq \bar{\mathrm{m}}$,

which follows quickly from complete De Morgan properties, once it is noticed that $a \leq \bar{b}$ for all a, $b \in M$; which in turn requires invoking that M is a chain and that $M \subseteq S$. Accordingly

(3) $m \in M$

since M is a maximal chain in S and, by 2, $m \in S$. Note next that

(4) if $m \leq a \leq \bar{a}$, then $a = m$,

for all $a \in L$. For otherwise we could extend our maximal chain M by tacking on a further member a of S, contradicting maximality. We can now establish the converse of 2, $\bar{m} \leq m$. Starting on a somewhat mysterious note, observe first that

(5) $m \vee (\bar{m} \wedge b \wedge \bar{b}) \leq \bar{m} \wedge (m \vee b \vee \bar{b}) = \overline{m \vee (\bar{m} \wedge b \wedge \bar{b})}$

for all elements $b \in L$, by 2 and De Morgan lattice properties. So, by 4 and lattice properties, we have

(6) $\bar{m} \wedge b \wedge \bar{b} \leq m$, for all $b \in L$.

By that (weaker—see below and §40.3) form of (CD) which distributes finite $\wedge$ over generalized $\bigvee$, and recalling the definition of $\boldsymbol{f}$, (6) yields

(7) $\bar{m} \wedge \boldsymbol{f} \leq m$.

But, since $\boldsymbol{t} \leq m$, the left side of (6) is, by De Morgan principles, just $\bar{m}$ itself, which completes the proof that m is our desired fixed point. Observe that, although we promised the reader only one fixed point when $\boldsymbol{t} \leq \boldsymbol{f}$, our proof procedures make it clear that every upward passage from $\boldsymbol{t}$ to $\boldsymbol{f}$ in L passes through a fixed point. Thus, e.g., the lattice SL of §34.1 has two fixed points.

Finally, to complete the proof of Theorem 3, suppose that the complete De Morgan lattice L has no fixed point with respect to negation. We must find a complete truth filter T such that $T \subseteq L$. If L has no fixed point, then, by the lemma, $\boldsymbol{t} = \bigwedge\{p \vee \bar{p}\colon p \in L\} \not\leq \bigvee\{p \wedge \bar{p}\colon p \in L\} = \boldsymbol{f}$. It is then easy to see, by complete distribution and lattice properties, that there is a consistent and exhaustive set T (i.e., $p \in T$ iff $\bar{p} \notin T$) such that $\bigwedge T \not\leq \overline{\bigwedge T}$; for otherwise we should have $\boldsymbol{t} \leq \boldsymbol{f}$ after all. The principal filter determined by $\bigwedge T$ is then the desired complete truth filter, ending the proof of Theorem 3.

We also have a complete generalization of Theorem §18.4(4).

THEOREM 4. A necessary and sufficient condition for a complete De Morgan lattice $\langle L, \leq, {}^{-} \rangle$ to be a complete i.l. is that there be a complete homomorphism h of L into M_0.

The necessity of this condition is an immediate corollary of Theorem 1, with T playing the double role of the truth filter and the complete and completely prime filter. The sufficiency follows by letting T be the inverse image of F_{+0} under the assumed complete homomorphism h and by noting, as has already been remarked, that the inverse image of a complete, completely prime filter under a complete homomorphism is itself a complete, completely prime filter. Thus T so defined is complete, and that T is a truth filter follows from the proof of Theorem §18.4(4).

At this point a proof of a complete generalization of the embedding theorem of §18.5 would be appropriate, since it can be easily proved that the product of complete i.l.'s (as defined in §18.5) is a complete i.l. and since the analogue holds in the Boolean case. But such a generalization cannot be proved without restriction. Let us look at the proof of the theorem of §18.5 to see where it breaks down if we try to modify it by substituting a complete and completely prime filter in the role of the prime filter. The key assumption then would be the analogue to §18.4(3): for any two elements a, b such that $a \not\leq b$, there exists a complete and completely prime filter P such that $a \in P$ and $b \notin P$. We find that this assumption is unjustified in the case of complete intensional lattices. Thus, let L be the union of the two closed intervals $[0, 1/3]$ and $[2/3, 1]$ of the real line with the usual ordering. It is easily seen that this is a complete and completely distributive lattice and that, if $\bar{a}$ is defined as $1-a$, then $^-$ is antitone and of period two, and $[2/3, 1]$ becomes a complete truth filter. It is easy to see that there is no complete and completely prime filter that is a subset of the half-open interval $(2/3, 1]$; for suppose there is such a filter P. Since P is complete, for some $a \in (2/3, 1]$, P is the principal filter generated by a. But then consider $-P = \{b \in L: b < a\}$. It is well known from analysis that $a = \bigvee -P$; but, since P and $-P$ are disjoint, P is not completely prime. Thus, no two points in $(2/3, 1]$ can be separated by a complete and completely prime filter.

If we assume, however, that a complete i.l. $\langle L, \leq, {}^-, T\rangle$ is such that any two elements of L can be separated by a complete and completely prime filter P then the proof of the complete generalization of the theorem of §18.5 would go through as for the original theorem, with the obvious association of a complete and completely prime filter P_x, with each pair $(a, b)_x$. The complete T-homomorphisms $\{h_x\}_{x \in X}$ thereby associated would then define a T-isomorphism h of $\langle L, \leq, {}^-, T\rangle$ into M^c, just as before. And obviously h preserves $\wedge$, since each h_x preserves $\bigwedge$, and $\bigwedge$ in M^c is defined componentwise. Thus we have

LEMMA 1. Let L be a complete i.l. $\langle L, \leq, -, T\rangle$ of cardinality d such that, *for every pair of elements a, b with $a \not\leq b$, there exists a complete and completely prime filter P such that $a \in P$ and $b \notin P$. Then L is completely T-isomorphic to a complete sublattice of M^c for some cardinal $c \leq d^2$.*

We also have a converse form of Lemma 1, namely,

LEMMA 2. Every complete i.l. $\langle L, \leq, {}^{-}, T\rangle$ that is completely T-isomorphic to a complete sublattice of M^c for some cardinal c is such that, for every pair of elements a, b with $a \not\leq b$, there exists a complete and completely prime filter P such that $a \in P$ and $b \notin P$.

In view of the given T-isomorphism of L onto a complete sublattice L′ of M^c, it suffices to show that, for $a', b' \in L'$ with $a' \not\leq b'$, there exists a complete and completely prime filter P′ such that $a' \in P'$ and $b' \notin P'$. And, to show this, it suffices to show that, for $\{a_x\}_{x<c}, \{b_x\}_{x<c} \in M^c$ with $\{a_x\}_{x<c} \not\leq \{b_x\}_{x<c}$, there exists a complete and completely prime filter Q of M^c such that $\{a_x\}_{x<c} \in Q$ and $\{b_x\}_{x<c} \notin Q$. For let Q be a complete and completely prime filter of M^c such that $a' \in Q$ and $b' \notin Q$. Then it is easy to see that $Q \cap L'$ is a complete and completely prime filter of L′ such that $a' \in Q \cap L'$ and $b' \notin Q \cap L'$. So consider $\{a_x\}_{x<c}, \{b_x\}_{x<c} \in M^c$ with $\{a_x\}_{x<c} \not\leq \{b_x\}_{x<c}$. Then for some $x_0 < c$, $a_{x_0} \not\leq b_{x_0}$. Then there exists a complete and completely prime filter Q_{x_0} of M_{x_0} such that $a_{x_0} \in Q_{x_0}$ and $b_{x_0} \notin Q_{x_0}$, as inspection of M_{x_0} shows. Now define functions f and g on $\{x: x < c\}$ such that $f(x_0) = Q_{x_0} \cap F_{+0}$ and for $x \neq x_0$, $f(x) = F_{+0}$, and such that $g(x_0) = Q_{x_0} \cap \overline{F_{+0}}$ and for $x \neq x_0$, $g(x) = \overline{F_{+0}}$. Then let $Q = \times_{x<c} f(x) \cup \times_{x<c} g(x)$. Componentwise considerations now show that Q is the desired complete and completely prime filter of M^c.

It is interesting to note that the condition that every pair of distinct elements be separated by a complete and completely prime filter is equivalent to a rather simple condition on the elements.

LEMMA 3. Given a complete and completely distributive lattice $\langle L, \leq\rangle$, a necessary and sufficient condition for any pair of distinct elements' being separated by a complete and completely prime filter is that every element be a generalized join of completely join-irreducible elements, where an element a is said to be completely join-irreducible if for no set $\{a_x\}_{x \in X}$ such that each $a_x < a$, does $a = \bigvee_{x \in X} a_x$.

The sufficiency is easy; for suppose that $a \not\leq b$, and say $a = \bigvee_{x \in X} a_x$, where each a_x is completely join-irreducible. Now choose some $a_x \not\leq b$. (There must be one; for if all $a_x \not\leq b$, then $a = \bigvee_{x \in X} a_x \leq b$, contrary to assumption.) Since $a_x \not\leq b$, the principal filter P generated by that a_x contains a but does not contain b. And it is trivial that P is complete, as are all principal filters in a complete lattice. It remains to show that P is completely prime. But this follows immediately from a complete generalization of a lemma in Birkhoff 1948, namely that, in a complete and completely distributive lattice, if an element p is completely join-irreducible, then $p \leq \bigvee_{x \in X} q_x$ implies $p \leq q_x$

for some $x \in X$. For proof of the generalization, note that $p \leq \bigvee_{x \in X} q_x$ implies $p = p \wedge (\bigvee_{x \in X} q_x) = \bigvee_{x \in X} p \wedge q_x$. But then, since p is completely join-irreducible, $p = p \wedge q_x$; i.e., $p \leq q_x$, for some $x \in X$.

The necessity follows by letting $\{p_x\}_{x \in X}$ be the set of all completely join-irreducible elements p_x such that $p_x \leq a$ for some arbitrary element a. This set is nonempty, since $\wedge L$ is completely join-irreducible. Now if $a = \bigvee_{x \in X} P_x$, we are through. So assume $a \nleq \bigvee_{x \in X} p_x$. But then there is, by assumption, a complete and completely prime filter P that contains a and does not contain $\bigvee_{x \in X} p_x$. But $\wedge P$ is completely join-irreducible; for otherwise P, being completely prime, would contain some element $q < \wedge P$. Thus $\wedge P = p_x$ for some $x \in X$, and $\wedge P \leq \bigvee_{x \in X} p_x$, contrary to the hypothesis that P does not contain $\bigvee_{x \in X} p_x$.

We remark in passing that Lemma 3, although handy for our purposes, is not stated in the best possible form; for in its proof we make use of the condition of complete distributivity only in the proof of the complete generalization of Birkhoff's lemma, and there only in a weakened form. Further, we could do without any condition on distributivity by defining a completely join-prime element as one satisfying the consequent of the generalization of Birkhoff's lemma, and formulating the necessary and sufficient condition of Lemma 3 in terms of completely-join-primeness instead of completely-join-irreducibility.

We now collect our lemmas together in the following theorem:

THEOREM 7. For a complete i.l. $\langle L, \leq, \bar{\ }, T \rangle$, the following conditions are equivalent:

(i) There exists a complete T-isomorphism of L into M^c, for some cardinal c.

(ii) For any pair of elements a, b with $a \nleq b$ there exists a complete and completely prime filter P such that $a \in P$ and $b \notin P$.

(iii) Every element a is a generalized join of completely join-irreducible elements.

§40.2. Some special facts about complete intensional lattices. Here we set forth three special properties of intensional lattices and complete intensional lattices; these are explicitly used in the sequel.

(1) M_0 is a complete intensional lattice (as are all finite intensional lattices)—as may readily be verified.

Let L be an intensional lattice and let $a, b \in L$ be such that (i) $a, b \in T$, (ii) $\bar{a} \leq a$, $\bar{b} \leq b$, (iii) $\bar{a} \wedge b \leq \bar{b}$, $\bar{b} \wedge a \leq \bar{a}$, and (iv) $\bar{a} \nleq b$. Then the sublattice generated by $\{a, b\}$ is isomorphic to M_0 under the natural extension of the mapping $f(a) = +\mathit{1}$, $f(b) = +\mathit{2}$.

(2) Let $\{\langle L_x, \leq_x, {}^{-x}, T_x\rangle\}_{x\in X}$ be an indexed set of {complete} intensional lattices, and let $\Pi_{x\in X}L_x$ be defined as that ordered quadruple $\langle L, \leq, {}^{-}, T\rangle$ such that the following hold: $T = \times_{x\in X}T_x$, $F = \times_{x\in X}F_x$, and $L = T\cup F$; also, for $a, b \in L$, if $a = \{a_x\}_{x\in X}$ and $b = \{b_x\}_{x\in X}$, then $a \leq b$ iff $a_x \leq_x b_x$ for all $x\in X$, and $\bar{a} = \{\overline{a_x}^{x}\}_{x\in X}$. Then $\Pi_{x\in X}L_x$ is also a {complete} intensional lattice.

Next we seek conditions under which we can find an isomorphic copy of $\Pi_{x\in X}L_x$ *inside a given complete intensional lattice.*

(3) Let L be a complete intensional lattice and let $\{\langle L_x, \leq, {}^{-}, T_x\rangle\}_{x\in X}$ be an indexed set of complete sublattices of L. (Note that the ordering and complementation is that of the parent lattice.) Define $T^* = \{\bigwedge B: B \in \times_{x\in X}T_x\}$, let $F^* = \{\bigvee B: B \in \times_{x\in X}F_x\}$, and let $L^* = T^*\cup F^*$. Then $\langle L^*, \leq, {}^{-}, T^*\rangle$ is a complete intensional lattice completely isomorphic to $\Pi_{x\in X}L_x$ (as defined in (2)) provided the various sublattices L_x are "sufficiently independent" in the following sense: (i) for $x, y \in X$, if $x\neq y$, if $t_y \in T_y$, and if $f_x \in F_x$, then $f_x \leq t_y$; (ii) for $x, y \in X$, if $x\neq y$, if $t_x, t'_x \in T_x$, and if $t_y \in T_y$, then $t_x \leq t'_x \vee t_y$, and (by contraposition) dually.

We remark (a) that the conditions (i) and (ii) turn out to be exactly what is needed for the isomorphism and appear to have no other interest, and (b) that each member of T^* $\{F^*\}$ is a meet {join} of a lot of elements a_x, one from each T_x $\{F_x\}$.

The reader who wants to see why (3) is true should spell it out for a very small case, say $X = \{1, 2, 3\}$. For the reader who *instead* wants a proof, we provide one. Regrettably it requires throwing around a lot of subscripts; it will occupy the rest of this subsection.

The isomorphism we have in mind is h, where, for $\{a_x\}_{x\in X} \in \Pi_{x\in X}L_x$, we define $h(\{a_x\}_{x\in X})$ as $\bigwedge_{x\in X}\{a_x\}$ or $\bigvee_{x\in X}\{a_x\}$ according as $\{a_x\}_{x\in X} \in T$ or $\{a_x\}_{x\in X} \in F$.

Then, in the first place, $h(\overline{\{a_x\}_{x\in X}}) = h(\{\overline{a_x}\}_{x\in X}) = \overline{h(\{a_x\}_{x\in X})}$ by the nature of $\Pi_{x\in X}L_x$ and by generalized De Morgan, which takes care of ${}^{-}$.

Next, set $a = h(\bigvee_{y\in Y}\{a_{x,y}\}_{x\in X})$; for homomorphism it suffices to show that $a = \bigvee_{y\in Y}h(\{a_{x,y}\}_{x\in X})$. We observe in the first place that, by the nature of $\Pi_{x\in X}L_x$,

$$a = h(\{\bigvee_{y\in Y}a_{x,y}\}_{x\in X}).$$

Let Y_T be the set of all $y\in Y$ such that $a_{x,y} \in T_x$ for all $x\in X$, and let $Y_F = (Y - Y_T)$ be the set of all $y\in Y$ such that $a_{x,y} \in F_x$ for $x\in X$. Then, if Y_T is empty, we will have $\bigvee_{y\in Y}a_{x,y} \in F_x$ for each $x\in X$, and so, by the definition of h,

$$a = \bigvee_{x\in X}\bigvee_{y\in Y}a_{x,y}.$$

Now

$$\bigvee_{x\in X}\bigvee_{y\in Y}a_{x,y} = \bigvee_{y\in Y}\bigvee_{x\in X}a_{x,y}$$

by lattice properties; so, since $a_{x,y} \in F_x$ for each $y \in Y$, $x \in X$, the definition of h leads to

$$a = \bigvee_{y\in Y} h(\{a_{x,y}\}_{x\in X}),$$

as required.

Suppose on the other hand that Y_T is nonempty, so that (since then, for each $x\in X$, $\bigvee_{y\in Y} a_{x,y} \in T_x$) the definition of h leads to

$$a = \bigwedge_{x\in X} \bigvee_{y\in Y} a_{x,y}.$$

Using distribution and letting S be the set of all functions s from X into Y, we have then

$$a = \bigvee_{s\in S} \bigwedge_{x\in X} a_{x,s(x)}.$$

Let S_T be that subset of S whose members s have the property that, for all $x\in X$, $s(x) \in Y_T$. Then

$$a = \Big(\bigvee_{s\in S_T} \bigwedge_{x\in X} a_{x,s(x)}\Big) \vee \Big(\bigvee_{s\in(S-S_T)} \bigwedge_{x\in X} a_{x,s(x)}\Big),$$

where here and below the right term is to be deleted if Y_F is empty. If Y_F is nonempty, however, we show that

$$\bigvee_{s\in(S-S_T)} \bigwedge_{x\in X} a_{x,s(x)} = \bigvee_{y\in Y_F} \bigvee_{x\in X} a_{x,y}.$$

That $\bigvee_{s\in(S-S_T)} \bigwedge_{x\in X} a_{x,s(x)} \le \bigvee_{y\in Y_F} \bigvee_{x\in X} a_{x,y}$ is trivial, since, for every $s \in (S-S_T)$, $s(x) = y$ for some $y \in Y_F$ and $x\in X$, so that, for that choice of x and y, $a_{x,s(x)} = a_{x,y}$. For the converse, we need to show that

$$a_{x,y} \le \bigvee_{s\in(S-S_T)} a_{q(s),s(q(s))}$$

where $x\in X$, $y \in Y_F$, and q is a function from $S-S_T$ into X. But since Y_T contains some member, say y', it follows that there is bound to be some member s of $S-S_T$ such that $s(x) = y$ and for all $x' \neq x$, $s(x') = y'$. Hence, for that s we shall have

$$a_{x,y} \le a_{q(s),s(q(s))}$$

either trivially (if $q(s) = x$) or by condition (i) of the hypothesis (if $q(s) \neq x$); so $a_{x,y} \le \bigvee_{s\in(S-S_T)} a_{q(s),s(q(s))}$ follows at once by lattice properties.

Turning attention now to the left term, we show that

$$\bigvee_{s\in S_T} \bigwedge_{x\in X} a_{x,s(x)} = \bigvee_{y\in Y_T} \bigwedge_{x\in X} a_{x,y}.$$

That $\bigvee_{y\in Y_T} \bigwedge_{x\in X} a_{x,y} \le \bigvee_{s\in S_T} \bigwedge_{x\in X} a_{x,s(x)}$ is trivial, since, for each $y \in Y_T$, there is an $s \in S_T$ such that, for all $x\in X$, $s(x) = y$. For the converse we need to show that

$$\bigwedge_{x\in X} a_{x,s(x)} \le \bigvee_{y\in Y_T} a_{q(y),y}$$

for every $s \in S_T$ and every function q from Y_T into X. That the relation holds is trivial if, for some $x\in X$, $q(y) = x$ for all $y \in Y_T$; if on the other hand q

assumes at least two distinct values x and x′, then the relation obtains in virtue of condition (ii) of the hypothesis.
Consequently,

$$a = (\bigvee_{y\in Y_T} \bigwedge_{x\in X} a_{x,y}) \vee (\bigvee_{y\in Y_F} \bigvee_{x\in X} a_{x,y});$$

that is,

$$\begin{aligned} a &= (\bigvee_{y\in Y_T} h(\{a_{x,y}\}_{x\in X})) \vee (\bigvee_{y\in Y_F} h(\{a_{x,y}\}_{x\in X})) \\ &= \bigvee_{y\in Y} h(\{a_{x,y}\}_{x\in X}). \end{aligned}$$

That $h(\bigwedge_{y\in Y}\{a_{x,y}\}_{x\in X}) = \bigwedge_{y\in Y} h(\{a_{x,y}\}_{x\in X})$ follows by an exactly dual argument, and that $\{a_x\}_{x\in X} \in T$ implies $h(\{a_x\}_{x\in X}) \in T^*$ is immediate from the definitions of h and T*. So h is a complete homomorphism of $\Pi_{x\in X} L_x$ into L*.

Finally we must show that h is one-one. Evidently T* and F* are disjoint; so it suffices to show that if $\bigwedge_{x\in X}\{t'_x\} \leq \bigwedge_{x\in X}\{t_x\}$ then $t'_y \leq t_y$ for each $y\in X$—i.e., we treat only the case where everything in sight is in the appropriate T, T*, T_x. Assume the former; so anyhow

$$\bigwedge_{x\in X}\{t'_x\} \leq t_y,$$

whence $\bigwedge_{x\in X}\{t'_x\} \vee t_y \leq t_y$, and so, by complete distribution,

$$\bigwedge_{x\in X}\{t'_x \vee t_y\} \leq t_y.$$

To finish, we need only show $t'_y \leq t'_x \vee t_y$ for each $x\in X$. This is trivial for $x = y$, and a consequence of condition (ii) of the hypothesis of (3) when $x \neq y$.

§40.3. The theory of propositions. In this section we adumbrate that portion of the theory of propositions which is essential for the semantics of relevance logic and go on to prove a lemma required later. In order to make it quite clear just what it is to which we are committing ourselves, we render this theory in the form of a definition together with a set of rigorously framed assumptions which we believe to be true. As preliminary informal explanation, we remark that we take a proposition to be the sort of thing that can serve as the (extralinguistic) *logical* content of a sentence, so that, although there can be unexpressed propositions, still two sentences will express the same proposition just in case the sentences play the same inferential role. In addition to the notion of a proposition, the primitives of the theory are propositional negation, the (intensional) relation of logical entailment or implication, and the notion of propositional truth. (For the remainder of §40, we provide numbered but untitled subsubsections and number lemmas accordingly, so as to make cross-reference easy; e.g., Lemma 3.2 is to be found in §40.3.2.)

§40.3.1. Our assumptions will make sense only for sets of propositions that are closed in certain ways; we therefore begin with two definitions.

Definition. A set of propositions P is {completely} closed iff P is closed under propositional negation and constitutes a {complete} lattice with respect to the relation of logical entailment as between propositions.

Definition. If P is {completely} closed, $\leq$ is the relation of propositional entailment on P, $^-$ is the operation of propositional negation on P, and T is the set of all true propositions in P, we say that the ordered quadruple $\langle P, \leq, ^-, T\rangle$ is a {complete} propositional lattice.

Assumption 1. Every {complete} propositional lattice is an intensionally complemented {completely} distributive {complete} lattice with {complete} truth filter.

For propositional a and b, $a \wedge b$ is the proposition that both a obtains and b obtains, $a \vee b$ is the proposition that either a obtains or b obtains, and, for sets of propositions A, $\bigwedge A$ is the proposition that every member of A obtains, and $\bigvee A$ is the proposition that some member of A obtains. $\bar{a}$ is, of course, the proposition that a fails to obtain. (For this use of "obtains," see the Grammatical propaedeutic, especially §A3.4.)

Assumption 2. For every finite *n*, there is a propositional lattice generated by *n* independent generators.

These two assumptions when taken together throw us into conflict with the orthodox theory of propositions, since, classically, every propositional lattice is a Boolean algebra, whereas of course the free intensional lattice with *n* generators is no such thing.

Assumption 3. There is a complete propositional lattice generated by a completely independent set (of propositions) of cardinality $\aleph_0$.

The third assumption of course implies the second; we separate them in order to remark that only the second is required for the completeness of the quantifier-free systems **E** and **R**. A similar remark applies to Assumption 1: the part referring to complete propositional lattices is unnecessary for the consistency of the quantifier-free systems. Furthermore, even keeping in mind the consistency of the quantificational systems, it is likely that one could weaken the sort of distributivity postulated for complete propositional lattices: $a \vee (\bigwedge_{x \in X} b_x) = \bigwedge_{x \in X}(a \vee b_x)$ would probably suffice, while the restriction of (CD) to denumerable sets would surely do so. But we think that the stronger assumption of complete distributivity is just as true for complete propositional lattices as it is for complete fields of sets.

Remarks. There are two very different kinds of theories about the numbers. One kind is like Peano's—it takes the concept of "number" as primitive and gives us a theory about what numbers there are and how the concept of number connects with the other primitive concepts of successor and zero. But Peano never attempts to get inside his numbers to see what makes them tick. He does not attempt to analyze them in terms of something else. In contrast stand the theories of numbers of Frege or Whitehead-Russell or von Neumann, etc. In these theories we are told what the numbers "really" (at least in the case of Frege) are. Such "analyses" of the numbers are useful, but it is to be observed that they do not reduce the importance of theories of the Peano kind; indeed, if a set-theoretical (say) reconstruction of the numbers does not yield Peano's theories, one is apt to throw out the reconstruction rather than the analysis—though we do not at all mean to imply that things should always go this way.

By analogy, there are two kinds of theories of propositions. The kind we have *not* offered would tell us what propositions "really" are, would render them as complex entities of some sort—perhaps as sets of possible or impossible worlds, or as equivalence classes of sentences, or as types of speech acts, or as certain cerebral dispositions. Instead, we have offered a theory like Peano's, which says something about propositions and about how that primitive idea is connected to the primitive concepts of negation, propositional implication, and truth. And, also by analogy, we think that if a candidate analysis of propositions does not yield this theory then it is by so much suspect—our theory stands as a kind of "condition of adequacy" in Carnap's sense for any analysis of what a proposition really is.

The second remark is spoken with forked tongue. Jabbing to the right, we hereby permit any eidophobe having a distaste for the philosophical notion of proposition as adumbrated in the introductory remarks to the theory, to read "propositional lattice" as if it were an uninterpreted phrase. Then the mathematical import of those lemmas and theorems mentioning propositional lattices—directly or indirectly—will lie in the fact that we make very weak assumptions about which {complete} propositional lattices exist.

But, jabbing now to the left, we really think that such an attitude is not correct. What makes our results *semantic* and not merely *algebraic* is that the interpretations we have in mind really do involve propositions, and not just elements of some abstract algebra. Analogy: in the semantics of **TV** we must at some point postulate that there are two truth values. For purely mathematical purposes, we just need any two entities, say **0** and **1**. But not for semantics—the intended interpretation of **TV** is not given unless the two entities are the truth values; Millard Fillmore and Franklin Pierce, though sufficing for the limited purposes of §33.2, will not do here.

But, you may ask, what is the cash value of the difference between our jab to the right and our jab to the left? Alas, we are inclined to agree with Church when he was addressing himself to a similar point: "nothing can be said about this different significance except to postulate it as different" (Church 1956, n. 143).

§40.3.2. On the basis of these assumptions, we now establish a lemma which is required below in showing the various systems of relevance logic to be semantically complete with respect to their intended propositional interpretation. Following the spirit though not quite the letter of the definition of M^c in §18.5, for c a cardinal larger than 0 and possibly as big as $\aleph_0$, we define M^c as $\Pi_{x:x<c}L_x$, where for each $x < c$, L_x is M_0.

LEMMA 3.2. For every cardinal $0 < c \leq \aleph_0$, there is a propositional lattice P^c isomorphic to M^c.

PROOF. By Assumption 3 of §40.3.1, there is a propositional lattice $P = \langle P, \leq, \bar{\ }, T\rangle$ generated by a set G of 2c independent generators, where, for disjoint sets $\{r_i\}$ and $\{s_i\}$, $c > i > 0$, $G = \{r_i\}\cup\{s_i\}$. (The finite case evidently needs only assumption 2 of §40.3.1.) We show that P has a sublattice $P^c = \langle P^c, \leq, \bar{\ }, T^c\rangle$ isomorphic to M^c. We may assume that $G \subseteq T$, since, if not, we can find an independent set of generators G' such that $G' \subseteq T$ by choosing $G' = \{f(a)\colon a\epsilon G\}$, where $f(a) = \bar{a}$ if $a\notin T$ and $f(a) = a$ if $a\epsilon T$. Where $\bar{G} = \{\bar{g}\colon g\epsilon G\}$, we make for each $i < c$ the following definitions:

$$\begin{aligned}
U_i &= \{a\colon (a\epsilon\bar{G} \text{ and } a\neq\overline{r_i}) \text{ or } a=s_i\},\\
V_i &= \{a\colon (a\epsilon\bar{G} \text{ and } a\neq\overline{s_i}) \text{ or } a=r_i\},\\
u_i &= \vee U_i, \overline{u_i} = \wedge\overline{U}_i = \wedge\{a\colon (a\epsilon G \text{ and } a\neq r_i) \text{ or } a=\overline{s_i}\},\\
v_i &= \vee V_i, \overline{v_i} = \wedge\overline{V}_i = \wedge\{a\colon (a\epsilon G \text{ and } a\neq s_i) \text{ or } a=\overline{r_i}\},\\
+0_i &= (u_i\wedge v_i),\\
+1_i &= (u_i\wedge v_i)\vee\overline{v_i}=(u_i\vee\overline{v_i})\wedge v_i,\\
+2_i &= (u_i\wedge v_i)\vee\overline{u_i}=(v_i\vee\overline{u_i})\wedge u_i,\\
+3_i &= (u_i\wedge v_i)\vee\overline{v_i}\vee u_i=(u_i\vee\overline{v_i})\wedge(v_i\vee\overline{u_i}),\\
-0_i &= \overline{+0_i},\ -1_i=\overline{+1_i},\ -2_i=\overline{+2_i},\ -3_i=\overline{+3_i},\\
T_i &= \{+0_i,\ +1_i,\ +2_i,\ +3_i\}, F_i = \{-0_i,\ -1_i,\ -2_i,\ -3_i\},\\
P_i &= (T_i\cup F_i),\\
T^c &= \{\wedge B\colon B \in \times_{i<c}T_i\}\\
F^c &= \{\vee B\colon B \in \times_{i<c}F_i\}\\
P^c &= T^c\cup F^c.
\end{aligned}$$

We need to show that $\langle P^c, \leq, \bar{\ }, T^c\rangle$ is isomorphic to M^c.

In the first place, it can be easily seen that the elements $+1_i$ and $+2_i$ serve as generators for P_i; indeed, $+0_i = (+1_i\wedge +2_i)$ while $+3_i = (+1_i\vee + 2_i)$. Consequently, in order to establish that P_i is isomorphic to M_0 for every i, it suffices to establish the conditions (i)–(iv) set forth in §40.2(1): (1) Since by assumption s_i, $r_i \in T$, it follows by the filterhood of T that u_i and v_i and, accordingly, all members of T_i are in T. (ii) That $-1_i \leq +1_i$ and $-2_i \leq +2_i$

follows lattice-theoretically from the fact that $\overline{u_i} \leq u_i$ (since U_i and $\overline{U_i}$ share s_i and indeed $\overline{s_i}$ as well) and similarly for v_i, as does (iii) the fact that $-1_i \wedge +2_i \leq -2_i$ and $-2_i \wedge +1_i \leq -1_i$. That $-1_i \nless +2_i$ follows lattice-theoretically from the fact that $v_i \nless u_i$, which in turn is due to the independence of G and the fact that $\overline{V_i}$ and U_i have no common members.

Since $P^c = T^c \cup F^c$ is defined consonantly with §40.2(3), there remains only to show that P^c satisfies conditions (i)–(ii) of §40.2(3), from which it will follow immediately that P^c is isomorphic to M^c. (i) That $f_i \leq t_k$ for $i \neq k$, where $f_i \in F_i$ and $t_k \in T_k$, follows lattice-theoretically from the fact that $\overline{u_i} \leq v_k$ and $\overline{v_i} \leq u_k$ for $i \neq k$, as does (ii) that for $i \neq k$, $t_k \leq t'_k \vee t_i$, where $t_k, t'_k \in T_k$ and $t_i \in T_i$.

§40.4. Intensional models. In this section we define a notion of "model" appropriate to quantified first degree intensional logic, which in turn gives rise to definitions of the related notions required for supplying an adequate semantics for first degree formulas. We also establish a consistency result relating the various intensional logics with certain critical models.

§40.4.1. The following notion of model is appropriate for quantificational first degree formulas; a much simpler concept of model would suffice for propositional logic. For brevity, however, we shall not carry along propositional logic as a separate case, having tended to it from another point of view in §19.

Q is an *intensional model* iff Q is an ordered triple $\langle L, I, s \rangle$, where L is a complete intensional lattice, I is a nonempty domain of individuals, and s is an *assignment function* for L and I, i.e., a function such that, for each individual variable x, $s(x) \in I$; for each n-ary predicate variable F, $s(F)$ is a function from I^n into L; and, for each propositional variable p, $s(p) \in L$.

Given an intensional model $Q = \langle L, I, s \rangle$, we define a (zero degree) *valuation* determined by Q and a (first degree) *truth set* determined by Q. (Valuations give us actual elements of a lattice for zero degree formulas (zdfs), whereas formulas with arrows are only divided into the good guys and the bad guys.)

We say that v_Q is the (zero degree) *valuation determined by* $Q = \langle L, I, s \rangle$ iff v_Q is a function, defined for every zero degree formula and having values in L, determined as follows: for all zero degree formulas (zdfs) A,

if A is a propositional variable, then $v_Q(A) = s(A)$;
if A has the form $Fx_1 \ldots x_n$, then $v_Q(A)$ is the value in L of the
    function $s(F)$ for the argument $\langle s(x_1), \ldots, s(x_n) \rangle$;
if A has the form $\bar{B}$, then $v_Q(A) = \overline{v_Q(B)}$;
if A has the form $B \vee C$, then $v_Q(A) = v_Q(B) \vee v_Q(C)$;
if A has the form $\exists xB$, then $v_Q(A) = \bigvee C$, where C is that set
    such that $c \in C$ iff $v_{Q'}(B) = c$ for some intensional model
    $Q' = \langle L, I, s' \rangle$ such that s' is like s except perhaps at x.

Turning now to the notion of truth set, we say that T_Q is the (first degree) *truth set determined by* Q = ⟨L, I, s⟩ iff, for all first degree formulas (fdf) A and where L is the complete intensional lattice L = ⟨L, $\leq$, $\bar{}$, T⟩,

if A is a zdf, then $A \in T_Q$ iff $v_Q(A) \in T$;
if A has the form $B \rightarrow C$, then $A \in T_Q$ iff $v_Q(B) \leq v_Q(C)$;
if A has the form $\bar{B}$, then $A \in T_Q$ iff $B \notin T_Q$;
if A has the form $B \vee C$, then $A \in T_Q$ iff either $B \in T_Q$ or $C \in T_Q$;
if A has the form $\exists xB$, then $A \in T_Q$ iff $B \in T_{Q'}$, for some intensional model Q′ = ⟨L, I, s′⟩, where s′ is like s except perhaps at x.

We also define F_Q, the *falsity set determined by* Q, as the complement of T_Q relative to the set of fdfs. Then an fdf A is said to be *true in* a model Q if $A \in T_Q$, and *false in* Q if $A \in F_Q$. We leave it to the reader to determine that the terminology is well chosen, at least to the extent of satisfying the condition that an fdf A is true in Q exactly when $\bar{A}$ is false.

We also define an fdf as *valid in* (a complete intensional lattice) L if it is true in every model Q = ⟨L, I, s⟩, and *falsifiable* in L if it is false in some model Q = ⟨L, I, s⟩.

By an *intensional frame* we mean an ordered pair ⟨L, I⟩, where L is a complete intensional lattice and I is a nonempty domain of individuals. We shall say that an fdf A is *valid* in a frame ⟨L, I⟩ or *falsifiable* in a frame ⟨L, I⟩ according as A is true in every intensional model ⟨L, I, s⟩ or false in some intensional model ⟨L, I, s⟩.

If L is a complete propositional lattice, then a frame ⟨L, I⟩ is a *propositional frame*, and an intensional model ⟨L, I, s⟩ is a *propositional model.*

This leads us to our last and most crucial semantic definition: an fdf is *valid* (without qualification) iff it is valid in every propositional frame, hence iff true in every propositional model; and it is *falsifiable* iff falsifiable in some propositional frame, hence iff false in some propositional model. Of course the justification for these last definitions lies in the intended interpretation of our intensional logics. (See second remark at end of §40.3.1.)

§40.4.2. At this point we can give a rigorous statement of part of our chief result: for first degree formulas validity is a necessary and sufficient condition for provability in $\mathbf{E}^{\exists x}$ and also for provability in $\mathbf{R}^{\exists x}$ and $\mathbf{T}^{\exists x}$. As a lemma required for this result, we establish a connection between provability in $\mathbf{R}^{\exists x}$ and validity in the intensional lattices M^c.

LEMMA 4.2. Every fdf provable in $\mathbf{R}^{\exists x}$ is valid in M^c for all c. (The same holds a fortiori for $\mathbf{E}^{\exists x}$, **E**, and **R**.)

For proof we need to introduce the notion of an implicative intensional lattice and then to extend the notions of valuation and truth set accordingly.

The idea is that proofs of fdfs in $\mathbf{R}^{\exists x}$ can take detours through formulas that are *not* first degree, a situation which we meet by finding a suitable interpretation of these non-fdfs which is an extension of the interpretation we have for the fdfs. (Alas, the interpretation is partly *ad hoc*; otherwise we should not have to wait for §48 in order to have a viable semantics for the whole of these relevance logics rather than just for their first degree fragments.)

A {*complete*} *implicative intensional lattice* is an ordered quintuple $\langle \mathrm{L}, \leq, {}^{-}, \mathrm{T}, \rightarrow \rangle$, where $\langle \mathrm{L}, \leq, {}^{-}, \mathrm{T} \rangle$ is a {complete} intensional lattice and where $\rightarrow$ is a two-place operation on L such that, for a, b $\in$ L, $(\mathrm{a} \rightarrow \mathrm{b}) \in \mathrm{T}$ iff $\mathrm{a} \leq \mathrm{b}$. A {complete} implicative intensional lattice $\langle \mathrm{L}, \leq, {}^{-}, \mathrm{T}, \rightarrow \rangle$ is said to be an implicative extension of $\langle \mathrm{L}, \leq, {}^{-}, \mathrm{T} \rangle$. Given L, we also write the extension as $\langle \mathrm{L}, \rightarrow \rangle$.

Let $\mathrm{Q} = \langle \mathrm{L}, \mathrm{I}, \mathrm{s} \rangle$ be an intensional model determining a valuation v_{Q} and truth set T_{Q}, and let $\langle \mathrm{L}, \rightarrow \rangle$ be an implicative extension of L. Then $\mathrm{Q}' = \langle \langle \mathrm{L}, \rightarrow \rangle, \mathrm{I}, \mathrm{s} \rangle$ is an *implicative intensional model* determining an *implicative valuation* $\mathrm{v}_{\mathrm{Q}'}$, provided for every formula A, if A is a zdf, then $\mathrm{v}_{\mathrm{Q}'}(A) = \mathrm{v}_{\mathrm{Q}}(A)$; if A has the form $B \rightarrow C$, then $\mathrm{v}_{\mathrm{Q}'}(A) = \mathrm{v}_{\mathrm{Q}'}(B) \rightarrow \mathrm{v}_{\mathrm{Q}'}(C)$; if A is a non-zdf having one of the forms $\bar{B}$, $B \vee C$, or $\exists x B$, then $\mathrm{v}_{\mathrm{Q}'}(A)$ has one of the values, respectively, $\overline{\mathrm{v}_{\mathrm{Q}'}(B)}$, $\mathrm{v}_{\mathrm{Q}'}(B) \vee \mathrm{v}_{\mathrm{Q}'}(C)$, or $\bigvee \mathrm{C}$, where $\mathrm{c} \in \mathrm{C}$ iff $\mathrm{v}_{\mathrm{Q}'}(B) = \mathrm{c}$ for some implicative intensional model $\mathrm{Q}'' = \langle \langle \mathrm{L}, \rightarrow \rangle, \mathrm{I}, \mathrm{s}'' \rangle$, where s″ is like s except perhaps at x.

A formula A is said to be *valid* in an implicative intensional lattice $\langle \mathrm{L}, \rightarrow \rangle$ iff $\mathrm{v}_{\mathrm{Q}'}(A) \in \mathrm{T}$ for every implicative intensional model $\langle \langle \mathrm{L}, \rightarrow \rangle, \mathrm{I}, \mathrm{s} \rangle$.

It is readily seen via the filterhood of T that, for fdfs, validity in an implicative extension $\langle \mathrm{L}, \rightarrow \rangle$ of L is equivalent to validity in L. Furthermore, where we use T^{c} for the truth filter of M^{c} and F^{c} for its complement relative to M^{c}, it is easily seen that for each c, $(\bigwedge \mathrm{T}^{\mathrm{c}}) \wedge (\bigvee \mathrm{F}^{\mathrm{c}}) = \bigwedge \mathrm{M}^{\mathrm{c}}$; so Lemma 4.2 is a consequence of the following.

Let $\mathrm{L} = \langle \mathrm{L}, \leq, {}^{-}, \mathrm{T} \rangle$ be a complete intensional lattice with the property that, where $\mathrm{F} = (\mathrm{L} - \mathrm{T})$, $(\bigwedge \mathrm{T}) \wedge (\bigvee \mathrm{F}) = \bigwedge \mathrm{L}$; then there is an implicative extension $\langle \mathrm{L}, \rightarrow \rangle$ of L in which every theorem of $\mathbf{R}^{\exists x}$ is valid.

Indeed, let $\rightarrow$ be defined by cases as follows: (I) if $\mathrm{a} \in \mathrm{F}$ and $\mathrm{b} \in \mathrm{T}$, then (i) $\mathrm{a} \rightarrow \mathrm{b} = \bigwedge \mathrm{L}$ if $\mathrm{a} \nleq \mathrm{b}$, (ii) $\mathrm{a} \rightarrow \mathrm{b} = \bigvee \mathrm{L}$ if $\mathrm{a} = \bigwedge \mathrm{L}$ or $\mathrm{b} = \bigvee \mathrm{L}$, (iii) $\mathrm{a} \rightarrow \mathrm{b} = (\bar{\mathrm{a}} \wedge \mathrm{b})$ otherwise; (II) if $\mathrm{a}, \mathrm{b} \in \mathrm{F}$, then $\mathrm{a} \rightarrow \mathrm{b} = \bar{\mathrm{a}} \wedge (\mathrm{a} \rightarrow (\mathrm{b} \vee (\bigwedge \mathrm{T})))$; (III) if $\mathrm{a}, \mathrm{b} \in \mathrm{T}$, then $\mathrm{a} \rightarrow \mathrm{b} = \mathrm{b} \wedge ((\mathrm{a} \wedge (\bigvee - \mathrm{F})) \rightarrow \mathrm{b})$; and (IV) if $\mathrm{a} \in \mathrm{T}$ and $\mathrm{b} \in \mathrm{F}$, then $\mathrm{a} \rightarrow \mathrm{b} = \bar{\mathrm{a}} \wedge \mathrm{b} \wedge (\mathrm{a} \wedge (\bigvee \mathrm{F}) \rightarrow \mathrm{b} \vee (\bigwedge \mathrm{T}))$.

The argument that $\langle \mathrm{L}, \rightarrow \rangle$ is implicative is sufficiently straightforward once one observes that the fact that $(\bigwedge \mathrm{T}) \wedge (\bigvee \mathrm{F}) = \bigwedge \mathrm{L}$, postulated for L, implies (for case (II)) that if $\mathrm{a}, \mathrm{b} \in \mathrm{F}$, then $\mathrm{a} \leq \mathrm{b}$ iff $\mathrm{a} \leq (\mathrm{b} \vee (\bigwedge \mathrm{T}))$, and also (for case (III)) that if $\mathrm{a}, \mathrm{b} \in \mathrm{T}$, then $\mathrm{a} \leq \mathrm{b}$ iff $(\mathrm{a} \wedge (\bigvee \mathrm{F})) \leq \mathrm{b}$. Indeed, suppose that $\mathrm{a} \leq (\mathrm{b} \vee (\bigwedge \mathrm{T}))$; then $\mathrm{a} \leq (\mathrm{b} \vee (\bigwedge \mathrm{T})) \wedge \mathrm{a}$ (by lattice considerations) $\leq \mathrm{b} \vee ((\bigwedge \mathrm{T}) \wedge \mathrm{a})$ (by distribution) $\leq \mathrm{b} \vee ((\bigwedge \mathrm{T}) \wedge (\bigvee \mathrm{F}))$ (since $\mathrm{a} \in \mathrm{F}$ implies $\mathrm{a} \leq \bigvee \mathrm{F}$) $\leq \mathrm{b} \vee (\bigwedge \mathrm{L}) \leq \mathrm{b}$. The other case is dual. To complete the argument one requires only the equally straightforward (though tedious and case-ridden) verification that each of the axioms of $\mathbf{R}^{\exists x}$ is valid in $\langle L, \rightarrow\rangle$ and that the rules preserve validity.

§40.4.3. We here append a few extra remarks about implicative extensions.

(1) Given an intensional lattice in which $(\wedge T)\wedge(\vee F) = \wedge L$ as above, an alternative implicative extension is given by the following: (I) $(\wedge T\rightarrow a) = a$, $(a\rightarrow \vee F) = \bar{a}$, $(a\rightarrow \vee L) = \vee L$, and $(\wedge L\rightarrow a) = \vee L$; (II) otherwise, $(a\rightarrow b) = \wedge T$ or $(a\rightarrow b) = \wedge L$ according as $a \leq b$ or $a \nleq b$.

(2) There are complete intensional lattices L which have no implicative extension $\langle L, \rightarrow\rangle$ such that every theorem of **R** is valid in $\langle L, \rightarrow\rangle$; e.g., $\langle L, \leq, \bar{\ }, T\rangle$, where L, T, and $\bar{\ }$ are as in M_0, and $\leq$ is as follows: if $a\in F$ and $b\in T$, then $a \leq b$; if $b\in T$, then $+\boldsymbol{0} \leq b$; $a \leq +\boldsymbol{3}$ for all a; and $a \leq b$ iff $\bar{b} \leq \bar{a}$. But, as in effect pointed out in §24.3, for every complete intensional lattice there is an implicative extension in which every theorem of **E** is valid.

(3) If $\langle L, \rightarrow\rangle$ is any implicative intensional lattice in which all the theorems of **E** are valid, then so also is $\langle \Pi_{i<n}L_i, \rightarrow\rangle$, for n finite, $L_i = L$ for $i < n$, and where, for elements $a = \{a_i\}_{i<n}$ and $b = \{b_i\}_{i<n}$ of $\Pi_{i<n}L_i$, $(a\rightarrow b)$ is defined as $\{c_j\}_{j<n}$, where all the c_js are alike and each $c_j = \wedge_{i<n}(a_i\rightarrow b_i)$. That this is so can be seen by observing that theoremhood in **E** is preserved under the sort of simultaneous substitution suggested by the definition of $\langle \Pi_{i<n}L_i, \rightarrow\rangle$. An example of some interest is the case in which L is M_0 and $\langle L, \rightarrow\rangle$ is as in 4.2; in such implicative intensional lattices we find that every theorem of **E** is valid (since all are valid in M_0), but that the theorem $A\rightarrow((A\rightarrow A)\rightarrow A)$ of **R** fails when $n > 1$.

The simultaneous substitution definition is as follows: given a formula A of **E**, let $s^n(A) = s_1^n(A)\& \ldots \& s_n^n(A)$, where each function s_i^n, defined for all well-formed parts of A and having formulas as values, is given by the following.

(i) If p is the alphabetically jth variable in A, then $s_i^n(p)$ is the $((j\times n)+(i-1))$th variable not occurring in A.

(ii) $s_i^n(A\vee B) = s_i^n(A)\vee s_i^n(B)$

(iii) $s_i^n(A\&B) = s_i^n(A)\&s_i^n(B)$

(iv) $s_i^n(\bar{A}) = \overline{s_i^n(A)}$

(v) $s_i^n(A\rightarrow B) = (s_1^n(A)\rightarrow s_1^n(B))\& \ldots \&(s_n^n(A)\rightarrow s_n^n(B))$.

Then we have obviously that, if $s^n(A)$ is valid in $\langle L, \rightarrow\rangle$, then A is valid in $\langle \Pi_{i<n}L_i, \rightarrow\rangle$.

What is more interesting is that the move from A to $s^n(A)$ preserves theoremhood in **E** but not theoremhood in **R**.

(4) Implicative intensional lattices are not in general implicative lattices in the sense of Curry 1963, p. 143; nor is the converse true. Indeed, although

defining $(a \rightarrow b)$ in $\langle L, \rightarrow \rangle$ as $\bigvee\{c\colon c \in L \text{ and } a \wedge c \leq b\}$ guarantees that if $a \leq b$ then $(a \rightarrow b) \in T$ (since then $(a \rightarrow b) = \bigvee L$), there is no guarantee that if $(a \rightarrow b) \in T$ then $a \leq b$; for $b \in T$ is by itself sufficient for $(a \rightarrow b) \in T$, but by no means sufficient for $a \leq b$. Exactly the same remark holds for the definition of $a \rightarrow b$ as the material "implication" $\bar{a} \vee b$.

§40.5. Branches and trees. In this section we define a number of concepts leading up to the notions "full normal branch" and "full normal tree" used in establishing the connection between provability and validity. The general plan of attack is similar to that in certain versions of Gödel's completeness proof for the first-order functional calculus: given a candidate first degree formula, one tries to build a dendriform "proof" of the candidate by working upward from "conclusion" to "premisses"; if every branch of the tree thus constructed terminates in a certain recognizable sort of "axiom," the candidate is provable; if some branch fails to terminate, we can use that branch in order to define a model in which the candidate is false (see §39.1).

§40.5.1. We begin with a number of definitions useful in defining the sort of branches and trees we require, as well as central to the method of proof we employ.

(1) By an *atom* we mean an fdf that is either a propositional variable p, a primitive formula $Fx_1 \ldots x_n$, or the negation of one of these. Atoms are *positive* or *negative* according as they do not or do begin with the sign of negation.

An *entailment* is a formula having the form $A \rightarrow B$; a *negated entailment* has the form $\overline{A \rightarrow B}$.

A formula A is a *disjunctive part* of B iff either A is B itself or else there is a formula $A \vee C$ or $C \vee A$ which is a disjunctive part of B.

$\varphi(A)$ is a formula of which A is a disjunctive part, and $\varphi(B)$ is the result of replacing A in $\varphi(A)$ by B.

If A does not have the form $B \vee C$, then A is a *minimal disjunctive part* of $\varphi(A)$.

(2) Where X is a set of fdfs, C is an fdf, and Z is a sequence of variables $x_1, \ldots, x_i, \ldots$, we define by cases: X is a *direct reduction set* for C relative to Z iff

(i) $X = \{A\}, C = \bar{\bar{A}}$;
(ii) $X = \{A, B\}, C = (A \vee B)$;
(iii) $X = \{\bar{A}\}$ or $X = \{\bar{B}\}, C = \overline{A \vee B}$;
(iv) $X = \{\overline{Ax_i}\}$ for some $x_i \in Z$, $C = \exists x Ax$;
(v) $X = \{Ax_i\}_{x_i \in Z}, C = \exists x Ax$.

If C has one of the forms (i)–(v) then C is said to be *reducible*; but if C is an atom, an entailment, or a negated entailment, then C has no direct reduction set and is said to be *irreducible*.

(3) A set X of fdfs is *completely reduced* relative to Z iff, for any reducible $A \in X$, there is $Y \subseteq X$ such that Y is a direct reduction set for A relative to Z.

(4) The following play roles similar to that of a direct reduction set. A negated entailment $\overline{A \to B}$ is *directly represented* in a set X iff either $A \in X$ or $\bar{B} \in X$. A negated entailment $\overline{A \to B}$ is *pairwise represented* in a pair of sets $\langle X, Y\rangle$ iff either at least one of A and $\bar{B}$ is in both of X and Y, or both of A and $\bar{B}$ are in at least one of X and Y.

§40.5.2. Now we turn to the notion of a "full normal branch." We remark in explanation that the nodes of the branches will each be a sequence of fdfs, to be thought of as a kind of multiple disjunction of members of the sequence. In order to handle the version of the Löwenheim-Skolem theorem we want to prove, we shall allow the sequences to be infinitely long; and, in order to avoid some technical troubles, we shall consider only "regular" sequences, where a sequence is said to be *regular* iff (i) every constituent is an fdf, (ii) every entailment occurring as a well-formed part of any constituent has the form $\bar{A} \to B$, (iii) no variable occurs both bound and free in it, and (iv) infinitely many variables fail to occur in it. We therefore define the *regularization* of S as the result of first replacing each formula $A \to B$ in S by $\bar{\bar{A}} \to B$, and then, for each i, replacing each bound occurrence of the ith individual variable by the $(3 \times i)$th variable and each free occurrence of the ith individual variable by the $((3 \times i) + 1)$th variable. The regularization of S is provable in the sense of §40.5.4 in any one of $\mathbf{E}^{\exists x}$, $\mathbf{E}$, $\mathbf{R}^{\exists x}$, $\mathbf{R}$ iff S is, and also valid or falsifiable in a frame $\langle L, I\rangle$ according as S is valid or falsifiable in $\langle L, I\rangle$.

In order to be able to refer to certain formulas and sets of formulas involved in our "full normal branches," we introduce a bit of notation: suppose that $Br = S_1, \ldots, S_j, \ldots$, is a sequence of regular sequences of fdfs; relative to Br we define:

Z is the set of variables not bound in S_1.
D_j is the set of all fdfs belonging to some $S_{j'}$ with $j' \leq j$.
D is the union of all the D_j.

$E_{(i,j)}$ is the $(i + 1)$th entailment in S_j which is such that every constituent to its left is either irreducible or a zdf. (Note that $E_{(i,j)}$ may be undefined. The intuitive idea is that, for constant i, all the $E_{(i,j)}$ will form a chain within the branch; a sort of branch-within-a-branch, with $E_{(i,j+1)}$ serving as a "premiss" for $E_{(i,j)}$ in just the way that S_{j+1} serves as a "premiss" for S_j. The conditions are imposed so that, as we build our branch upwards, the numbering of the entailments won't become confused by the production of new entailments to the left of old ones.)

$L_{(i,j)}$ is the set of all A such that some $E_{(i,j')}$, with $j' \leq j$, has the form $\overline{\varphi(A)} \to B$. (See §40.5.1(1) for our use of "φ".)

$R_{(i,j)}$ is the set of all A such that some $E_{(i,j')}$, with $j' \leq j$, has the form $B \to \varphi(A)$.

L_i is the union of all the $\mathrm{L}_{(i,j)}$.

R_i is the union of all the $\mathrm{R}_{(i,j)}$.

"L" and "R" are meant to suggest "left" and "right." We may remark in this connection that we consider formulas $\bar{A}\to B$ instead of the more general case so that we can render exactly similar—instead of dual—treatments of the left side and the right side of entailments.

We can now specify the conditions under which a sequence $\mathrm{Br} = \mathrm{S}_1, \ldots, \mathrm{S}_j, \ldots$, is a *full normal branch* for a sequence S of fdfs.

(A) S_1 is the regularization of S.

(B) S_j is the terminus of Br if *explicitly tautological*, in the sense that there is a pair of atoms A and $\bar{A}$ such that at least one of the following holds: both A and $\bar{A}$ are constituents of S_j; there is a constituent $\bar{B}\to C$ of S_j such that either A is a disjunctive part of B and $\bar{A}$ of C, or $\bar{A}$ is a disjunctive part of B and A of C.

(C) S_j is the terminus of Br if all the following hold: (1) Each of D_j, $\mathrm{L}_{(i,j)}$, and $\mathrm{R}_{(i,j)}$ (for every i such that $\mathrm{E}_{(i,j)}$ is defined) is completely reduced relative to Z; (2) every negated entailment in D_j is directly represented in D_j; and (3) for each i such that $\mathrm{E}_{(i,j)}$ is defined, every negated entailment in D_j is pairwise represented in $\langle \mathrm{L}_{(i,j)}, \mathrm{R}_{(i,j)}\rangle$.

(D) Otherwise:

(0) If $j = 5n$: then if every constituent of S_j is irreducible, $\mathrm{S}_{j+1} = \mathrm{S}_j$; otherwise S_j has the form

$$A_1, \ldots, A_{q-1}, A_q, A_{q+1}, \ldots, A_{q+k}, \ldots$$

where A_q is the leftmost reducible constituent of S_j, and where $k = j$ if S_j is infinite, and S_j has $q + k$ constituents if S_j is finite. If A_q has the form (i) $\bar{\bar{B}}$, (ii) $B\vee C$, (iii) $\overline{B\vee C}$, (iv) $\overline{\exists xBx}$, or (v) $\exists xBx$, then S_{j+1} is the result of replacing A_q in S_j, respectively, by (i) B, (ii) the sequence B, C, (iii) either $\bar{B}$ or $\bar{C}$, (iv) $\overline{Bx_i}$, where x_i is the first variable in Z not occurring free in S_j, or (v) Bx_i, where x_i is the first variable in Z such that Bx_i does not occur in D_j. In Case (v), to obtain S_{j+1} one must also insert the formula $A_q = \exists xBx$ to the right of A_{q+k}.

(1) {(2)} If $j = 5n + 1$ {if $j = 5n + 2$}: let f be some function defined on the positive integers and having each positive integer as value infinitely many times; then if $\mathrm{E}_{(\mathrm{f}(n),j)}$ is undefined, or has the form $\bar{B}\to C$ where every minimal disjunctive part of B {of C} is irreducible, then $\mathrm{S}_{j+1} = \mathrm{S}_j$. Otherwise, $\mathrm{E}_{(\mathrm{f}(n),j)}$ is such that its antecedent has the form $\overline{\varphi(A)}$ {its consequent has the form $\varphi(A)$}, where A is the leftmost reducible minimal disjunctive part of $\varphi(A)$.

If A has the form (i) $\bar{\bar{B}}$, (ii) $\overline{B\vee C}$, (iii) $\overline{\exists xBx}$, or (iv) $\exists xBx$, then S_{j+1} is the result of replacing $\varphi(A)$ in S_j by (i) $\varphi(B)$, (ii) $\varphi(\bar{B})$ or $\varphi(\bar{C})$, (iii) $\varphi(\overline{Bx_i})$, where x_i is the first variable in Z not free in S_j, or (iv) $(\varphi(Bx_i)\vee\exists xBx)$, where x_i is the first variable in Z such that Bx_i does not occur in $\mathrm{L}_{(\mathrm{f}(n),j)}$ {does not occur in $\mathrm{R}_{(\mathrm{f}(n),j)}$}.

(3) If $j = 5n+3$: if every negated entailment that is a constituent of S_j is directly represented in D_j, then $S_{j+1} = S_j$; otherwise, where $\overline{A\to B}$ is the leftmost negated entailment in S_j not directly represented in D_j, S_{j+1} is the result of inserting either A or $\bar{B}$ immediately to the right of $\overline{A\to B}$ in S_j.

(4) If $j = 5n+4$: if, for every i such that $E_{(i,j)}$ is defined, every negated entailment that is a constituent of S_j is pairwise represented in $\langle L_{(i,j)}, R_{(i,j)}\rangle$, then $S_{j+1} = S_j$. Otherwise, let $\langle k, i\rangle$ be the first pair of integers (in some ordering) such that, where $\overline{A\to B}$ is the alphabetically kth of all negated entailments, $\overline{A\to B}$ is a constituent of S_j, $E_{(i,j)}$ is defined, and $\overline{A\to B}$ is not pairwise represented in $\langle L_{(i,j)}, R_{(i,j)}\rangle$. Then, where $E_{(i,j)}$ has the form $\bar{C}\to D$, S_{j+1} is the result of replacing $E_{(i,j)}$ in S_j by one of the following four:

$$\overline{C\vee A}\to D\vee A,\ \overline{C\vee \bar{B}}\to D\vee \bar{B},\ \overline{C\vee A\vee \bar{B}}\to D,\ \bar{C}\to D\vee A\vee \bar{B}.$$

This completes our definition of "Br is a full normal branch for S."

§40.5.3. We will say that a full normal branch for S is *tautological* or *nontautological* according to whether it does or does not terminate in accordance with §40.5.2(B) in an explicitly tautological sequence. For use in §40.6, we observe that, if a full normal branch Br is nontautological, then (using the notation of §40.5.2)

(1) D is completely reduced relative to Z, and the set of zdfs in D is also completely reduced relative to Z.
(2) If $A\to B$ occurs in D, then some $E_{(i,j)}$ has the form $A\to B$;
(3) for each i, both L_i and R_i are completely reduced relative to Z;
(4) every negated entailment in D is directly represented in D;
(5) for each i such that $E_{(i,j)}$ is defined for some j, every negated entailment in D is pairwise represented in $\langle L_i, R_i\rangle$;
(6) there is no pair of atoms A and $\bar{A}$ such that both occur in D;
(7) for no i and for no pair of atoms A and $\bar{A}$ do either of the following occur: $A \in L_i$ and $\bar{A} \in R_i$, or $\bar{A} \in L_i$ and $A \in R_i$;
(8) if S is quantifier-free and finite, then Br is finite.

§40.5.4. We prove a lemma needed in showing that the relevance logics are complete.

Let us say that a (possibly infinite) sequence of formulas $A_1, \ldots, A_i, \ldots$, is *provable* in $\mathbf{E}^{\exists x}$ $\{\mathbf{E}, \mathbf{R}^{\exists x}, \mathbf{R}\}$ iff there is a finite n such that $\sim\exists x_1\exists x_2 \ldots \exists x_k \sim (A_1\vee \ldots \vee A_n)$ is a theorem of $\mathbf{E}^{\exists x}$ $\{\mathbf{E}, \mathbf{R}^{\exists x}, \mathbf{R}\}$ (Of course $k = 0$ when we are speaking of a quantifier-free system.)

Then we have

LEMMA 5.4. If every full normal branch for a sequence S of fdfs is tautological then it is possible to find constructively a proof of S in $\mathbf{E}^{\exists x}$, and indeed in $\mathbf{E}$ if S is quantifier-free.

Indeed, the full normal branches for a sequence S may be arranged into a tree—the *full normal tree* for S—beginning with the regularization of S as the bottom node, with two-way forkings when a node satisfies one of (0-iii), (1-ii), (2-ii), and (3), and with four-way forkings when a node satisfies (4) of the definition in §40.5.2 of "full normal branch." By König's lemma (see §13.3), a finitely forking tree, every branch of which terminates, will be finite; so we may use induction, the hypothesis of the induction being that the theorem is true for all sequences in the tree below fewer than *n* nodes. Suppose S_j is below *n* nodes in the full normal tree for S. If S_j satisfies §40.5.2(B), then S_j is provable by §§19.2 and 24.3. If S_j satisfies any of (0)–(4) of §40.5.2(D) then it can be shown, by methods of §19 extended to quantification in a straightforward way and relying especially on Fact 8 of §19.4, that a suitable conjunction of disjunctions answering to initial segments of the nodes immediately above S_j (together with $B \to B$, $C \to C$, and $D \to D$ in case (4)) entails a disjunction answering to an initial segment of S_j. Since the former disjunctions are theorems of $E^{\exists x}$ by the hypothesis of the induction and since $E^{\exists x}$ has both a rule of adjunction and a rule of *modus ponens*, S_j is provable in $E^{\exists x}$—and it now follows by induction that the bottom node is provable in $E^{\exists x}$. Since the bottom node is the regularization of S, it follows that S is also provable in $E^{\exists x}$.

The second part of the Lemma can be proved merely by observing that the quantificational machinery of $E^{\exists x}$ is not needed when S_j is quantifier-free.

§40.5.5. Needed for the consistency part of the main theorem is a lemma the statement of which requires the extension of the semantical notions to sequences of formulas: given a complete intensional model $Q = \langle L, I, s \rangle$, we say that a sequence S of fdfs is *true in* Q iff some member of S is true in Q; i.e., if some member belongs to T_Q. Then S is *valid in* L iff true in every $\langle L, I, s \rangle$.

LEMMA 5.5. If every full normal branch for a sequence S is tautological, then S is valid in every complete intensional lattice.

For proof, observe that S_1, which is the regularization of S, is valid in an intensional lattice iff S is. From this point on a straightforward argument by cases suffices, using an induction like that in §40.5.4.

§40.6. Critical models.

We devote this section to a lemma connecting certain complete intensional models with nontautological full normal branches. The lemma is central in that it is used as a stepping stone in both the consistency and the completeness parts of the main theorem.

§40.6.1. For each nontautological branch Br, we are going to define an intensional model in which the members of Br are simultaneously falsified.

For this purpose we first baptize certain subsets of M_0 which will play an important role in the proceedings:

$$I_{+\mathit{1}} = \{+\mathit{1}, -\mathit{1}, +\mathit{0}, -\mathit{3}\} \qquad I_{+\mathit{2}} = \{+\mathit{2}, -\mathit{2}, +\mathit{0}, -\mathit{3}\}$$

and

$$F_0 = \{-\mathit{0}, -\mathit{1}, -\mathit{2}, -\mathit{3}).$$

The intensional model determined by a nontautological full normal branch Br is defined as the complete intensional model $Q = \langle M^c, I, s\rangle$, where c is the least cardinal greater than any i such that $E_{(i,j)}$ is defined for some j (and M^c is as in §40.3.2), where I is the set of positive integers and where the assignment function s is defined in terms of Br as follows (notation as in §40.5.2):

Where x is the ith variable in Z, $s(x) = i$, and if $x \notin Z$, $s(x) = 1$.

For each n-ary predicate variable F, $s(F)$ is that function from n-tuples of members of I into M^c which takes the n-tuple $\langle j_1, \ldots, j_n\rangle$ into $a = \{a_i\}_{i<c}$, where, letting x_{j_k} $(1 \leq k \leq n)$ be the j_k^{th} variable in Z and setting $Fx_{j_1} \ldots x_{j_n} = A$, for each $i < c$, a_i is the numerically least member of M_0 satisfying (1)–(6) below:

(1) If $A \in L_i$, then $a_i \in I_{+\mathit{1}}$; (2) if $\bar{A} \in L_i$, then $\bar{a}_i \in I_{+\mathit{1}}$; (3) if $A \in R_i$, then $a_i \in I_{+\mathit{2}}$; (4) if $\bar{A} \in R_i$, then $\bar{a}_i \in I_{+\mathit{2}}$; (5) if $A \in D$, then $a_i \in F_0$; and (6) if $\bar{A} \in D$ then $\bar{a}_i \in F_0$.

Evidently $c \leq \aleph_0$.

There is clearly at most one such model determined by any nontautological Br; that there is also at least one follows from the fact that the only way the directions (1)–(6) for assigning an ith coordinate to $s(A)$ could conflict would be if either $A \in L_i$ and $\bar{A} \in R_i$, or else $\bar{A} \in L_i$ and $A \in R_i$, or else $A \in D$ and $\bar{A} \in D$. But these possibilities are ruled out by §40.5.3(6) and §40.5.3(7).

We will let $v_Q^i(A)$ be the ith coordinate (in M_0) of $v_Q(A)$ (in M^c).

§40.6.2. Preparatory to proving that the model determined by a nontautological Br has the right properties, we introduce a couple of new notions and make an observation.

A set Y of fdfs is *hereditary* relative to Z iff, for all fdfs A and sets of fdfs X, if X is a direct reduction set for A relative to Z (see §40.5.1(2)) and if $X \subseteq Y$, then $A \in Y$.

The following, which expresses the form of a good many arguments of the sort presented here, is a consequence of our definitions: if (i) every irreducible fdf in X (see §40.5.1(2)) is also in Y, if (ii) X is completely reduced relative to Z (see §40.5.1(3)), and if (iii) Y is hereditary relative to Z, then $X \subseteq Y$. (This follows by induction on the length of formulas, noting that each member of a direct reduction set for A is shorter than A.)

We shall use this fact in several places in the course of the proof, establishing that (i)–(iii) hold for a certain X and Y and then concluding by the above that $X \subseteq Y$.

We will later make use of the fact that certain sets are hereditary. Let $Q = \langle L, I, s\rangle$ be an intensional model and let Z be a set of variables such that for all $i \in I$, there is an $x \in Z$ such that $s(x) = i$. Let B be a *complete ideal* in L, in the sense that if $C \subseteq L$ then $C \subseteq B$ iff $\bigvee C \in B$. Then the set of zdfs A such that $v_Q(A) \in B$ is hereditary relative to Z. It follows immediately that where $Q = \langle M^c, I, s\rangle$ is determined by a nontautological branch Br as in §40.6.1, the following sets are hereditary relative to Z: the set of zdfs A such that $v^i_Q(A) \in I_{+1}$; such that $v^i_Q(A) \in I_{+2}$; such that $v^i_Q(A) \in F_0$. For each of I_{+1}, I_{+2}, and F_0 is a complete ideal in M_0. And it is equally easy to see that F_Q is hereditary relative to Z.

LEMMA 6.3. Let Br be a nontautological full normal branch for a sequence S, and let $Q = \langle M^c, I, s\rangle$ be the intensional model determined by Br; then $D \subseteq F_Q$—i.e., the members of all the S_j in Br are simultaneously falsified in Q.

We abbreviate the hypotheses of the lemma by "(H)," and continue using the notation of §40.5.2.

(1) If (H) and if A is a zdf in D, then, for each $i < c$, $v^i_Q(A) \in F_0$. Indeed, that the property holds for atoms can be read off immediately from §40.6.2(5) and §40.6.2(6). Now since by §40.5.3(1) the set of zdfs in D is completely reduced and since by §40.6.2 the property in question is hereditary relative to Z, the conclusion follows by §40.6.2.

(2) If (H) then, for all $i < c$, $A \in L_i$ implies $v^i_Q(A) \in I_{+1}$ and $A \in R_i$ implies $v^i_Q(A) \in I_{+2}$. That this is so for atoms is immediate from §40.6.1.(1)–(4). Then since by §40.5.3(3) both L_i and R_i are completely reduced relative to Z and since, by §40.6.2, the properties are hereditary relative to Z, the conclusion follows by §40.6.2.

(3) If (H) then, for each $i < c$ and for each j such that $E_{(i,j)}$ is defined, $E_{(i,j)} \in F_Q$. To begin with, each $E_{(i,j)}$ will have the form $\bar{A} \rightarrow B$. Since, then, $A \in L_i$ and $B \in R_i$, we have by (2) above that $v^i_Q(A) \in I_{+1}$ and $v^i_Q(B) \in I_{+2}$. Then a check of the properties of M_0 shows that $v^i_Q(\bar{A}) \not\leq v^i_Q(B)$, so that in turn $v_Q(\bar{A}) \not\leq v_Q(B)$. Hence $(\bar{A} \rightarrow B) = E_{(i,j)} \in F_Q$.

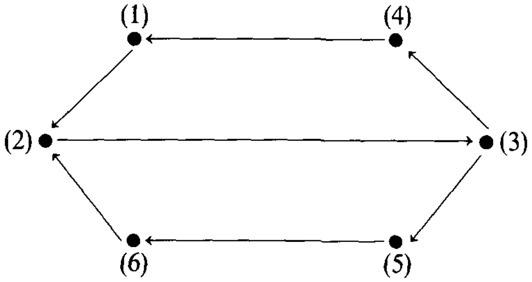
(4) If (H) and if $\overline{A \rightarrow B} \in D$, then $\overline{A \rightarrow B} \in F_Q$. It clearly suffices to show that $(A \rightarrow B) \in T_Q$, which can be established by showing that for each $i \leq c$, $v^i_Q(A) \leq v^i_Q(B)$. By §40.5.3(4) and (5), $\overline{A \rightarrow B}$ must be directly represented in D and pairwise represented in $\langle L_i, R_i\rangle$ for $i < c$; by considering for each $i < c$ the eight different ways (see §40.5.1(4)) in which this can happen, we can show that $v^i_Q(A) \leq v^i_Q(B)$. Taking one case by way of example, suppose that $A \in D$ and that $\bar{B}$ is in both L_i and R_i. Then, by (2) above, $v^i_Q(\bar{B}) \in (I_{+1} \cap I_{+2})$, and so

$v^i_Q(B) \in \{-\mathit{0}, +\mathit{3}\}$. Also, by (1) above, $v^i_Q(A) \in F_0$; so establishing that $v^i_Q(A) \leq v^i_Q(B)$ requires only a check of the properties of M_0 to see that $a \in F_0$ and $b \in \{-\mathit{0}, +\mathit{3}\}$ suffices for $a \leq b$.

Returning now to the proof of the lemma, suppose (H); then, by (4), all negated entailments in D are in F_Q, and, by (1), all zdfs in D are in F_Q. Further, (3) above, together with §40.5.3(2), implies that every entailment in D is in F_Q, so all irreducible fdfs in D are in F_Q. So since D is completely reduced relative to Z (by §40.5.3(1)) and since F_Q is, by §40.6.2, hereditary relative to Z, it follows by §40.6.2 that $D \subseteq F_Q$.

§40.7. Main theorems. This brief section is devoted to drawing together the various lemmas we have proved along the way in order to show six properties equivalent.

§40.7.1. Quantificational sequences. The first theorem states the chief result in the form of a general equivalence theorem. The pattern of proof of equivalence of the properties numbered below in the statement of the theorem is exhibited in the following diagram; note the twice-used implication from (2) to (3), which is the content of Lemma 6.3:



THEOREM. Given an at most denumerable sequence S of first degree formulas, the following statements are equivalent:

(1) S is valid.
(2) S is valid in every intensional frame $\langle M^c, I\rangle$, with $c \leq \aleph_0$ and with I the positive integers.
(3) S has no nontautological full normal branch.
(4) S is valid in every complete intensional frame.
(5) S is provable in $\mathbf{E}^{\exists x}$.
(6) S is provable in $\mathbf{R}^{\exists x}$.

PROOF. Since for every M^c with $c \leq \aleph_0$ there is, by Lemma 3.2, a propositional lattice P^c isomorphic to M^c, if S were falsifiable in $\langle M^c, I\rangle$ then it would also be falsifiable in $\langle P^c, I\rangle$; so, by contraposition, (1) implies (2). That

(2) implies (3) can be seen by contraposing Lemma 6.3 and recalling that the regularization of S is falsifiable in a frame iff S is. That (3) implies (4) is the content of Lemma 5.5. And that (4) implies (1) is guaranteed by Assumption 1 of the theory of propositions (§40.3.1). So (1)–(2)–(3)–(4) are all equivalent.

As we remarked above, (2) implies (3) by Lemma 6.3, and, according to Lemma 5.4, (3) implies (5). That (5) implies (6) is trivial, since $\mathbf{R}^{\exists x}$ is obtained by adding an axiom to $\mathbf{E}^{\exists x}$; and that (6) implies (2) is the content of Lemma 4.2. So, by a second circular argument, (2)–(3)–(5)–(6) are all equivalent; and hence all of (1)–(6) are equivalent.

That (1) implies (5) and (6) is the *completeness* of the first degree fragments of $\mathbf{E}^{\exists x}$ and $\mathbf{R}^{\exists x}$; that (5) and (6) each imply (1) is their *consistency.*

That the denial of (1) (or perhaps (4)) implies the denial of (2) amounts to an appropriate version of the *Löwenheim-Skolem theorem* for first degree relevance logic: i.e., if a set of fdfs is falsifiable then it is so in a complete intensional frame $\langle L, I\rangle$, with I denumerable and with L generated by a denumerable set.

§40.7.2. Quantifier-free sequences. The second main theorem is strictly parallel to the first, except that it concerns quantifier-free sequences of fdfs. The main corollary is the decidability of provability in **E** of finite quantifier-free sequences. Of course we already know this from §19 by a normal-form argument, and so the corollary, though using new methods, is not news.

Theorem 2. Given a sequence S of quantifier-free first degree formulas, the following statements are equivalent:

(1) S is valid.
(2) S is valid in M^c, with c the number of occurrences of the sign $\rightarrow$ in S.
(3) S has no nontautological full normal branch.
(4) S is valid in every complete intensional lattice.
(5) S is provable in **E**.
(6) S is provable in **R**.

Before proceeding to the proof, we remark that, for quantifier-free formulas, the notion of intensional model can be generalized so that, in an intensional model $Q = \langle L, I, s\rangle$, L need not be complete; for of course the requirement of completeness is used only in connection with the quantifiers. Hence, for quantifier-free S, we could drop the word "complete" in (4), yielding

(4*) S is valid in every intensional lattice,

and in effect change (1) in the same way to

(1*) S is valid in every propositional lattice.

PROOF. The argument that (1) or (1*) implies (2) is the same as in the proof of Theorem 1. We show that (2) implies (3) by considering the contrapositive. Suppose the denial of (3), i.e., that some full normal branch for S is nontautological; let it be Br. Then S is falsifiable in the intensional model $Q = \langle M^c, I, s\rangle$ determined by Br, where c is, by §40.6.1, the least upper bound on i in all defined $E_{(i,j)}$ as j varies. Since $E_{(i,j)}$ is always defined as an entailment, if it is defined at all, for fixed j the least upper bound on i in all defined $E_{(i,j)}$ must be no greater than the total number of occurrences of the sign → in S_j. And an examination of the process of regularization and of the branch-generating rules in §40.5.2 shows that, for quantifier-free S, the number of occurrences of the sign → is never increased as one passes from S to S_1 and from S_j to S_{j+1} in a full normal branch for S; hence, the number of occurrences of the sign → in S is an upper bound for c. But, since falsifiability of S in M^c obviously implies falsifiability of S in $M^{c'}$, if c′ is not less than c, the desired denial of (2) is secured. (However, if S contains quantifiers, case (v) of §40.5.2(0) can lead to an increase in the number of occurrences of the sign → in passing from S_j to S_{j+1}; see the last sentence of §40.5.2(0). Note that it is only quantified entailments, not entailments between quantifiers, which lead to increases.)

That (3) implies (4) is the content of Lemma 5.5; a slight modification of Lemma 5.5 ensures that (3) also implies (4*). And that (4) implies (1) and (4*) implies (1*) is again guaranteed by Assumption 1; so (1)–(1*)–(2)–(3)–(4)–(4*) are all equivalent.

We showed above that (2) implies (3). That (3) implies (5) is part of Lemma 5.4. That (5) implies (6) is trivial, since **R** is obtained from **E** by adding an axiom; and that (6) implies (2) again follows by Lemma 4.2. So (2)–(3)–(5)–(6) and accordingly all of (1)–(6) are equivalent.

That (1) or (1*) implies (5) and (6) is the *completeness* of the first degree fragments of **E** and **R**; that (5) or (6) implies (1) and (1*) is their *consistency*.

That the denial of (1) (or perhaps (4), or one of the starred versions) implies the denial of (2) amounts to a kind of *Löwenheim-Skolem theorem* for first degree propositional relevance logic: i.e., if a quantifier-free set of fdfs is falsifiable, then it is so in an intensional lattice generated by a denumerable set. One can easily see that this is a "best possible" result by considering the task of simultaneously falsifying the set of all nontautological first degree entailments. Also note that it is the generators and not the lattice as a whole that we claim to be denumerable, but in the quantifier-free case that leads immediately—since the methods of generation of intensional lattices are finite—to the denumerability of the entire intensional lattice. This situation is to be contrasted with that for the case of (even) finite sets of fdfs with quantifiers, indeed even a single fdf with a single one-place predicate letter. There we can still make do with a denumerably *generated* complete intensional lattice, but now completeness tends to increase the size of the lattice itself. See §41 for further details.

We have as the principal corollary of Theorem 2 a decision procedure for validity and (equivalently) provability for finite S.

COROLLARY. If a sequence S of quantifier-free first degree formulas is finite, then there is a mechanical way of testing whether or not S is valid and (equivalently) provable in **E** or **R**.

Of the two procedures we provide, the first is more semantical, the second more proof-theoretical. In the first place, Theorem 2 implies that one can test S for validity in M^c, where c is the number of occurrences of the sign $\rightarrow$ in S. M^c will of course be finite, so the process of checking all possible models $\langle M^c, I, s\rangle$ (where I can be chosen arbitrarily since it plays no role) must terminate. Secondly, one can construct the full normal tree for S. The tree will be finite, by König's lemma, since, by §40.5.3(8), every branch of it will be finite; so one can simply check whether or not each branch is tautological, a procedure validated by Theorem 2. And of course S can consist of a single quantifier-free formula *A*; so we have a decision procedure for the first degree fragments of **E** and **R**.

§41. Undecidability of monadic first degree formulas. In §19, and then again in §40.7.2, we showed that the common first degree formula fragment (no nesting of arrows) of the various *propositional* relevance logics is decidable, and of course it goes without saying that the *full* common first degree fragment of these logics, quantifiers and all, is undecidable—effectively containing, as it does, the classical first-order functional calculus. In between stands the *monadic* first degree fragment: we show that the validity question for this fragment is undecidable.

We begin with a little history. Everyone knows that the full quantified classical calculus $\mathbf{TV}^{\forall\exists x}$ in even a single binary predicate letter is undecidable and that, in contrast, monadic $\mathbf{TV}^{\forall\exists x}$, with only one-place predicate letters, is fully decidable; this much is prehistory. We put in the same category the positive decidability results for **S5** and for **H** (intuitionism). What was new and interesting as history was the undecidability of monadic $\mathbf{S5}^{\forall\exists x}$ as established by Kripke 1962, in spite of the decidability of both **S5** and monadic $\mathbf{TV}^{\forall\exists x}$ (and the similar result for the monadic quantified intuitionist calculus $\mathbf{H}^{\forall\exists x}$ of Kripke 1965a).

Meyer 1968b showed in general a similar result for monadic $\mathbf{E}^{\forall\exists x}$ (which as he said followed from Kripke's result, basically because $\mathbf{E}^{\forall\exists x}$ is a subsystem of $\mathbf{S5}^{\forall\exists x}$) and monadic $\mathbf{R}^{\forall\exists x}$ (which required a new proof). At the time this was the best undecidability result going, but of course we now know that the *propositional* logics **E** and **R** are themselves undecidable (see §65 below), so the result as thus phrased is not so interesting.

But Meyer 1968b established not only that *all* of monadic $\mathbf{E}^{\forall\exists x}$ and $\mathbf{R}^{\forall\exists x}$ are undecidable, but that their monadic *first degree formula* fragments are; and since as in the classical case this fragment stands between the *decidable* propositional calculus of first degree formulas and the *undecidable* quantified calculus of first degree formulas admitting a single binary predicate letter, this is a result of enduring interest. We present it in a way that relies both on Meyer's insights and on the results of §40.

Before we get started, let us agree that "$\mathbf{TV}^{\forall\exists x}$-*valid*" shall mean validity in the usual classical sense (we take it as defined only for zero degree formulas, sans implications), and that "*fdf-valid*" shall mean (for quantified first degree formulas) "valid" in the sense of §40.4—i.e., valid in every propositional frame.

Let A be a zdf—that is, a zero degree formula, a formula hence of $\mathbf{TV}^{\forall\exists x}$—containing the two-place predicate letter S as its only nonlogical symbol. Define A^* as that monadic fdf (first degree formula) that results by replacing Sxy by the implication $Fx\rightarrow Gy$. We show that A is $\mathbf{TV}^{\forall\exists x}$-valid just in case the fdf A^* is fdf-valid. From this it immediately follows that fdf-validity is not decidable; for, if it were, we could decide classical $\mathbf{TV}^{\forall\exists x}$ validity by replacing the $\mathbf{TV}^{\forall\exists x}$-validity question for A by the fdf-validity question for A^*.

Suppose first that the zdf A is $\mathbf{TV}^{\forall\exists x}$-valid. Then A is fdf-valid, and since fdf-validity is closed under substitution of fdfs for predicate letters in zdfs, A^* is fdf-valid as required.

Now suppose that A is not $\mathbf{TV}^{\forall\exists x}$-valid and, hence (Löwenheim), can be classically falsified in a denumerable domain of individuals I by giving some relation S' on I × I as value to the predicate letter S and assigning each variable x a value in I. We need to show that A^* is not fdf-valid. Choose an intensional frame (§40.4) $\langle \mathrm{L}, \mathrm{I}\rangle$, where the new I is the same as the old I, and where L is a complete propositional lattice generated by a denumerable set G of independent generators, as promised by Assumption 3 of §40.3. There is, by denumerability, a one-one function from I onto G; for i in I, let $\mathrm{p_i}$ be the mate of i under this function (so $\mathrm{p_i} \in \mathrm{G}$). Now define an intensional model $\mathrm{Q} = \langle \mathrm{L}, \mathrm{I}, \mathrm{s}\rangle$, in the sense of §40.4, where s assigns variables x the same value as in the $\mathbf{TV}^{\forall\exists x}$ model given, and where s makes assignments to F and G dependent on S' as follows, for each i, j ∈ I:

$$\mathrm{s}(F)\mathrm{i} = \wedge_{\mathrm{j}\in\mathrm{I}}\{\mathrm{p_k}\colon S'\mathrm{ik}\}$$
$$\mathrm{s}(G)\mathrm{j} = \mathrm{p_j}$$

Because of the matching definitions with respect to the connectives and quantifiers (note same domain) for $\mathbf{TV}^{\forall\exists x}$-truth (i.e., Tarski) and fdf-truth (i.e., membership in the truth set $\mathrm{T_Q}$, §40.4), in order to show that A and A^* have the same "truth values" in their respective models (for A^* this means membership in T_Q), it suffices to show that this is so for atomic formulas: the classical value of Sxy must be the same as the fdf value of $Fx \rightarrow Gy$. Let the value assigned to x (in both models) be i and the value assigned to y be j (i, j ∈ I); then, by the classical truth conditions for Sxy on the one hand and the fdf truth conditions of §40.4 for $Fx \rightarrow Gy$ on the other, this amounts to showing that

$$S'\mathrm{ij} \text{ iff } \wedge_{k\in I}\{p_k\colon S'\mathrm{ik}\} \leq p_j.$$

From left to right is by the most elementary properties of meet, and from right to left is by the independence of the set G of generators (see §40.1 for help with both halves).

The Theorem of §40.7.1 may now be relied upon to transfer this undecidability result for monadic fdf-validity to the undecidability of provability of monadic first degree formulas in any of the relevance logics $\mathbf{E}^{\forall\exists x}$, $\mathbf{R}^{\forall\exists x}$, etc.

Meyer has pointed out (unpublished) that the result can be improved: even the fragment of the relevance logics consisting of first degree formulas based on a *single* monadic predicate letter is undecidable. The trick is to translate Sxy of the $\mathbf{TV}^{\forall\exists x}$ invalid formula as $Fx \rightarrow Fy$ (instead of as $Fx \rightarrow Gy$), with the same interpretation being given to F as indicated above. This translation will not work in general, but it *will* work whenever the interpretation S' of S is reflexive and transitive (and also antisymmetric if you like): under this condition one can show that

$$S'\mathrm{ij} \text{ iff } \wedge_{k\in I}\{p_k\colon S'\mathrm{ik}\} \leq \wedge_{k\in I}\{p_k\colon S'\mathrm{jk}\},$$

and one knows from Tarski 1949 and Tarski, Mostowski, and Robinson 1953 that the theory of a reflexive and transitive (and antisymmetric if you like) relation is undecidable.

§42. Extension of (γ) to $\mathbf{R}^{\forall\exists x}$ et al.

In §25.2 we answered affirmatively for the relevance logic **E** and its cousins the question, Is the rule (γ), from A and $\sim A \vee B$ to infer B, admissible? In fact in §25.3 we confirmed this welcome result with a second proof. In the present section we shall show that the answer to this question remains affirmative when we consider the first-order relevance logics. (See §25.1 for a setting-in-context of the matter.)

There are by now at least four variant proofs of the admissibility of (γ). The first three proofs (in chronological order): Meyer and Dunn 1969 (adapted for quantifiers in Meyer, Dunn, and Leblanc 1974), Routley and Meyer 1973, and Meyer 1976f are all basically due to Meyer (with some help from others), and all depend on the same first lemma. The fourth proof was obtained by Kripke in 1978 and is in Dunn and Meyer 1989.

All the Meyer proofs are what Smullyan 1968 would call "synthetic" in style and are inspired by Henkin-style methods. The Kripke proof is "analytic" in style and is inspired by Kanger-Beth-Hintikka tableau-style methods. In actual detail, Kripke's argument is modeled on completeness proofs for tableau systems, wherein a partial valuation for some open branch is extended to a total valuation. As Kripke has stressed, this avoids the apparatus of inconsistent theories which has hitherto been distinctive of the various proofs of (γ)'s admissibility.

The proof of §25.2—and §25.3—was the first of the Meyer proofs; and we shall leave a brief description of the second for §48.8, since it depends on semantic notions introduced there. What we do here is present the technique of the third proof in application to the question of (γ) for the first-order relevance logics.

The strategy of all the Meyer proofs can be divided into two segments: The Way Up and The Way Down. Of course we start with the hypotheses that $\vdash A$ and $\vdash \sim A \vee B$, yet assume not $\vdash B$ for the sake of *reductio*. We shall be more precise in a moment, but The Way Up involves constructing in a Henkin-like manner a maximal theory T′ (containing all the logical theorems) with $B \notin \mathrm{T}'$. The problem, though, is that T′ may be inconsistent in the sense of having both C, $\sim C \in \mathrm{T}'$ for some formula C. (Of course this could not happen in classical logic; for, by virtue of the paradox of implication $C \& \sim C \rightarrow B$, B would be a member of T′, contrary to construction.) The Way Down fixes this by finding in effect some subtheory $\mathrm{T}'_{\mathrm{m}} \subseteq \mathrm{T}$ that is both complete and consistent, and indeed is a "truth set" in the sense of Smullyan 1968 (Meyer has labeled The Way Down the "Converse Lindenbaum lemma"). Thus, for all formulas C and D, $\sim C \in \mathrm{T}'_{\mathrm{m}}$ iff $C \notin \mathrm{T}'_{\mathrm{m}}$, and $C \vee D \in \mathrm{T}'_{\mathrm{m}}$ iff at least one of C and D is in T'_{m}. So, since $\sim A \vee B \in \mathrm{T}'_{\mathrm{m}}$, at least one of $\sim A$ and B is in T'_{m}. But since $A \in \mathrm{T}'_{\mathrm{m}}$, $\sim A$ is not in T'_{m}. So B must be in T'_{m}(!?). But T'_{m} is a subset of T′, which was constructed to keep B out. So B cannot be in T'_{m}, and so by *reductio* we obtain $\vdash B$ as desired.

We remark that the proof as given here would appear to use the disjunctive syllogism in our use-language at just the point marked "(!?)"; but it can be restructured (indeed the original proof of Meyer and Dunn 1969 was so restructured) so as to avoid at least such an explicit use of the disjunctive syllogism. See §80.3 for a detailed discussion of this matter.

§42.1. Terminology for logics and theories. Enough of strategy. We now lay out a few notions needed for a more precise statement of The Way Up and The Way Down lemmas. We begin by collecting those properties of all the relevance logics **L** that we use over and over again when we are carrying out arguments like these; since the collection is dictated by The Way Up and The Way Down, we shall enter the following

Definition 1. A logic **L** is *Up-Down acceptable* provided it satisfies the following conditions, or at least such of them as pertain to the (primitive or defined) notation of **L**:

Modus ponens. If $\vdash_{\mathbf{L}} A \rightarrow B$, then if $\vdash_{\mathbf{L}} A$ then $\vdash_{\mathbf{L}} B$.

Reflexivity. $\vdash_{\mathbf{L}} A \rightarrow A$.

Transitivity. If $\vdash_{\mathbf{L}} A \rightarrow B$ and $\vdash_{\mathbf{L}} B \rightarrow C$ then $\vdash_{\mathbf{L}} A \rightarrow C$.

Conjunction. $\vdash_{\mathbf{L}} A \& B \rightarrow A$, $\vdash_{\mathbf{L}} A \& B \rightarrow B$, and if both $\vdash_{\mathbf{L}} A \rightarrow B$ and $\vdash_{\mathbf{L}} A \rightarrow C$ then $\vdash_{\mathbf{L}} A \rightarrow . B \& C$. If $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} B$ then $\vdash_{\mathbf{L}} A \& B$.

Disjunction. $\vdash_{\mathbf{L}} A \rightarrow . A \vee B$, $\vdash_{\mathbf{L}} B \rightarrow . A \vee B$, and if both $\vdash_{\mathbf{L}} A \rightarrow C$ and $\vdash_{\mathbf{L}} B \rightarrow C$ then $\vdash_{\mathbf{L}} (A \vee B) \rightarrow C$.

Distribution. $\vdash_{\mathbf{L}} A \& (B \vee C) \rightarrow . (A \& B) \vee (A \& C)$.

Universal quantifier. $\vdash_{\mathbf{L}} \forall x A x \rightarrow A a$ for each parameter a; if $\vdash_{\mathbf{L}} A \rightarrow B a$ and if a occurs in neither A nor $\forall x B x$, then $\vdash_{\mathbf{L}} A \rightarrow \forall x B x$; if $\vdash_{\mathbf{L}} A a$ then $\vdash_{\mathbf{L}} \forall x A x$, provided a does not occur in $\forall x A x$.

Existential quantifier. $\vdash_{\mathbf{L}} A a \rightarrow \exists x A x$ for each parameter a; if $\vdash_{\mathbf{L}} A a \rightarrow B$ and if a occurs in neither B nor $\exists x A x$, then $\vdash_{\mathbf{L}} \exists x A x \rightarrow B$.

Confinement. $\vdash_{\mathbf{L}} \forall x (A \vee B x) \rightarrow . A \vee \forall x B x$ and $\vdash_{\mathbf{L}} A \& \exists x B x \rightarrow \exists x (A \& B x)$, provided x is not free in A.

Negation. $\vdash_{\mathbf{L}} {\sim}{\sim} A \rightarrow A$, $\vdash_{\mathbf{L}} A \rightarrow {\sim}{\sim} A$, and if $\vdash_{\mathbf{L}} {\sim} A \rightarrow B$ then $\vdash_{\mathbf{L}} {\sim} B \rightarrow A$. Also, $\vdash_{\mathbf{L}} A \vee {\sim} A$.

$1\frac{1}{2}$ *degree properties.* $\vdash_{\mathbf{L}} A \& (A \rightarrow B) \rightarrow B$ and $\vdash_{\mathbf{L}} A \& {\sim} B \rightarrow . {\sim}(A \rightarrow B)$. (These relate implications to truth functions.)

Following §25.2 (and much other work in relevance logic inspired by Meyer), the concept of an "**L**-theory," that is, a collection of formulas that respects the logical standards of **L**, has assumed central importance:

DEFINITION 2. A set T of formulas is an **L**-*theory* provided it is closed under adjunction and *modus ponens* for implications holding in **L**:

If $A \in \mathrm{T}$ and $B \in \mathrm{T}$ then $A \& B \in \mathrm{T}$;
if $\vdash_{\mathbf{L}} A \rightarrow B$, then if $A \in \mathrm{T}$ then $B \in \mathrm{T}$.

Note very carefully that it is the implications of **L** (rather than of T itself) that must be respected via *modus ponens*. Also observe that there is no reason to expect that an **L**-theory T will contain **L**; *modus ponens* as above would lead to that feature only provided every theorem of **L** were implied by every formula whatsoever—not a provision typically satisfied by *relevance* logics. We shall, accordingly, need some special terminology covering the special case when T does in fact contain all theorems of **L**; from the beginning of the researches reported here, such **L**-theories have been called "regular"; however, we substitute the adjective "**L**-containing" for our immediate purposes as being more mnemonic.

Being "**L**-containing" is one way in which an **L**-theory might be "better" than its fellows; another is in being a truth set in the sense of Smullyan. The word "normal" has generally been used in something like that sense in the bulk of the research using the concept of **L**-theories, but again we substitute for the sake of memory, using the adjective form "truth-like" of Smullyan's "truth set." Thus the following definitions:

DEFINITION 3. T is **L**-*containing* if every theorem (member) of **L** is in T. T is *truth-like* if it satisfies the following equivalences for such notation as it contains (either primitive or defined):

$A\&B \in$ T iff $A \in$T and $B\in$T.
$A \vee B \in$ T iff $A\in$T or $B\in$T.
$\sim A \in$ T iff not $A \in$ T.
$\forall x Ax \in$ T iff for every parameter a, $Aa \in$ T.
$\exists x Ax \in$ T iff for some parameter a, $Aa \in$ T.

These special kinds of **L**-theories will prove important—and indeed not until §48 will there be cause to invoke **L**-theories that are not also **L**-containing.

It is clear that the Up-Down acceptability of **L** suffices to confer *some* of the properties of truth-likeness on each **L**-theory T, but by no means all, even if T is also **L**-containing. In particular, if **L** is Up-Down acceptable then **L**-theories are bound to satisfy both parts of the truth-likeness condition (Def. 3) on conjunction and half of the conditions on each of disjunction, the universal quantifier, and the existential quantifier. Nothing is known about negation for (even **L**-containing) **L**-theories, and only half of what we need for truth-likeness is known about disjunction and the two quantifiers. For this reason, it turns out to be convenient to give each of these "missing properties" a name.

DEFINITION 4. Let T be an **L**-theory.

T is $\vee$-*prime*: if $A \vee B \in$ T then either $A\in$T or $B\in$T.
T is $\sim$-*consistent*: not both $A\in$T and $\sim A \in$ T.
T is $\sim$-*complete*: either $A\in$T or $\sim A \in$ T.
T is $\forall$-*complete*: if $Aa \in$ T for all parameters a, then $\forall x Ax \in$ T.
T is $\exists$-*prime*: if $\exists x Ax \in$ T then $Aa \in$ T for some parameter a.

These definitions are intended to be in effect even if the mentioned notation is defined instead of primitive. And we sometimes use the English name instead of the symbol as a modifier, e.g., "negation-consistent."

One more definition. We are sometimes in a situation in which all the "positive" missing properties named in Def. 4 are available for an **L**-theory T; we shall then say that T is "prime."

DEFINITION 5. Let T be an **L**-theory. Then T is *prime* if it is $\vee$-prime, $\forall$-complete, and $\exists$-prime in the sense of Def. 4. If quantifiers are wholly missing from the language of **L**, then prime = $\vee$-prime. On the other hand, even if one of the quantifiers is primitive and the other is defined, primeness nevertheless requires both $\forall$-completeness and $\exists$-primeness (as well as $\vee$-primeness).

We observe that, for Up-Down acceptable **L**, an **L**-containing prime **L**-theory is bound to be negation-complete (via excluded middle) and, hence, to satisfy all the conditions on truth-likeness except negation-consistency; this is the situation we shall be in just after The Way Up and before The Way Down. Here are three more observations just to help with insight into the definitions. (1) If an **L**-theory T is both negation-consistent and complete, then it is also $\vee$-prime. In the absence of either negation property, however, $\vee$-primeness cannot be inferred. (2) From $\vee$-primeness one cannot in general infer negation-completeness—unless it should happen that an **L**-theory T also is known to contain each excluded middle. (Of course we know that Up-Down-acceptable **L** itself contains each excluded middle, but not in general that **L**-theories do.) (3) In the absence of either negation-consistency or negation-completeness, there is no connection between $\forall$-completeness and $\exists$-primeness; but if T is both negation-complete and negation-consistent, these quantifier properties are interdeducible via the De Morgan-like interdefinability of the quantifiers.

We are now in a position to state The Way Up and to hint at The Way Down:

THE WAY UP LEMMA. Let **L** be an Up-Down-acceptable logic. If A is not provable in **L**, then there is an **L**-containing prime **L**-theory T′ that excludes A. (The $1\frac{1}{2}$ degree properties of Def. 1 are not used on the Way Up.)

THE WAY DOWN LEMMA (MORE OR LESS). Let **L** be an Up-Down-acceptable logic that also satisfies certain other conditions to be specified later (§42.3). Let T′ be an **L**-containing prime **L**-theory such as that promised by The Way Up. Then there is an **L**-containing truth-like **L**-theory that is a subset of T′; that is, we can find a truth-like **L**-theory lying between **L** and T′.

Our plan is to treat these two lemmas successively in the next two sections, keeping in mind that truth-likeness is just what is wanted for our proof of the admissibility of the rule (γ).

§42.2. The Way Up. This lemma is Theorem 3 of Meyer, Dunn, and Leblanc 1974, and its proof is basically a Henkin-style proof with one novelty. In usual Henkin proofs one can assure $\forall$-completeness by building into the construction of T′ that, whenever $\sim\forall xBx$ is put in, then so is $\sim Ba$ for some *new* parameter a. This guarantees $\forall$-completeness, since if $Ba \in$ T for all a but $\forall xBx \notin$ T′ then, by $\sim$-completeness, $\sim\forall xBx \in$ T′, and so by the usual construction $\sim$Ba $\in$ T′ for some a, and so *by $\sim$-consistency* (??) Ba $\notin$ T′ for some a, contradicting the hypothesis for $\forall$-completeness. But for relevance logics we of course have no guarantee that T′ is $\sim$-consistent, as has been remarked above.

The novelty, then, of Meyer, Dunn, and Leblanc 1974 was to modify the construction so as to keep things out as well as put things in, though this last was emphasized. Full symmetry with respect to "good guys" and "bad guys" was finally obtained in what is called the Pair Extension lemma, due independently to Gabbay 1974; see Dunn 1986 for a little history. We need a bit of terminology for its statement.

DEFINITION 6. We call an ordered pair (Δ, Θ) of sets of formulas simply a *pair*. We shall say that one pair (Δ_1, Θ_1) *extends* another pair (Δ_0, Θ_0) if $\Delta_0 \subseteq \Delta_1$ and $\Theta_0 \subseteq \Theta_1$. A pair (Δ, Θ) is defined to be **L**-*exclusive* (for **L** a logic) if for no $A_1, \ldots, A_m \in \Delta$ and $B_1, \ldots, B_n \in \Theta$ do we have $\vdash_{\mathbf{L}} A_1 \& \ldots \& A_m \rightarrow . B_1 \vee \ldots \vee B_n$. And (Δ, Θ) is called *exhaustive* if, for every formula A, either $A \in \Delta$ or $A \in \Theta$. We say that a pair (Δ, Θ) is *quantifier-prime* if its left entry Δ is $\exists$-prime in the sense of Definition 4 of the last section and if its right entry Θ is "$\forall$-prime" in the matching sense: if $\forall x A x \in \Theta$, then $Aa \in \Theta$ for some parameter a.

What Definition 6 accomplishes is to replace the single theory of Henkin with *two* sets of formulas, the first to be interpreted as a theory or set of axioms, the second to be thought of as a set of "counteraxioms." These counteraxioms are to be considered as the rejectable (Łukasiewicz 1939, 1951) or refutable (Carnap 1942) items. (See Curry 1963, to whom the idea is also independently due, in Curry 1937, for a discussion of the matter and of its history.) It will be seen in this light that **L**-exclusivity is a kind of consistency property (you can consistently—by the standards of **L**—"assert" all the members of the left entry and "reject" or "deny" all the members of the right entry of an **L**-exclusive pair); however, **L**-exclusivity of a pair does not itself entail negation consistency of its left entry. Exhaustiveness is an obvious maximality property, as in quantifier-primeness.

We ought to remark in passing that it is typical of considerations arising out of relevance to generate *relational* analogues of concepts which in a classical setting can be one-place. Relevance just is an essentially relational idea, in that respect more in tune with the harmonies of modern logic since Frege and Russell than are the two-note cadenzas with which our ears have been so frequently addressed.

We can now state the

PAIR EXTENSION LEMMA. Let (Δ, Θ) be an **L**-exclusive pair, and let **L** be an Up-Down-acceptable logic. Further assume that Δ and Θ together omit infinitely many parameters. Then (Δ, Θ) can be extended to an **L**-exclusive, exhaustive, and quantifier-prime pair (T', F'). Furthermore, T' is a prime **L**-theory; T' is disjoint from F' and, accordingly disjoint from Θ; and if Δ contains the parameter-free theorems of **L**, then T' is **L**-containing as well.

Observe that the Pair Extension lemma does *not* claim the negation-consistency of T'.

Let us first see that the The Way Up lemma of the last section follows, trivially, from the Pair Extension lemma. For suppose A is not a theorem of an Up-Down-acceptable **L**. Then clearly ($\mathbf{L}_0$, $\{A\}$) is an **L**-exclusive pair omitting infinitely many parameters, where $\mathbf{L}_0$ is the set of theorems of **L** containing no parameters. The T′ delivered by the Pair Extension lemma will then be the required **L**-containing prime **L**-theory excluding A.

PROOF OF THE PAIR EXTENSION LEMMA. First observe that if **L** is Up-Down acceptable and (T, F) is **L**-exclusive, then (1) at least one of the pairs ($\mathrm{T}\cup\{A\}$, F) and (T, $\mathrm{F}\cup\{A\}$) is **L**-exclusive; (2) if $\exists xAx \in \mathrm{T}$ and a is a parameter not in any of T, F, $\exists xAx$, then ($\mathrm{T}\cup\{Aa\}$, F) is **L**-exclusive; and (3) if $\forall xAx \in \mathrm{F}$ and a is a parameter not in any of T, F, $\forall xAx$, then (T, $\mathrm{F}\cup\{Aa\}$) is **L**-exclusive. Now order the sentences $B_1, \ldots, B_n, \ldots$. Use this sequence to define inductively a sequence of pairs (T_n, F_n) as follows. First $(\mathrm{T}_0, \mathrm{F}_0) = (\Delta, \Theta)$. Then define ($\mathrm{T}_{n+1}$, F_{n+1}) from its predecessor according to a recipe designed to ensure a fast argument for both exhaustiveness and quantifier-primeness (in order to present the recipe perspicuously, we speak of "adding" elements even if they are already there).

(a) Suppose B_n can be added to T_n, maintaining **L**-exclusivity. First do so. Then check to see whether B_n has the form $\exists xAx$; if so, add Aa for the first parameter a that does not occur in any of T_n, F_n, $\exists xAx$. These additions define T_{n+1}; and just let $\mathrm{F}_{n+1} = \mathrm{F}_n$.

(b) Suppose B_n cannot be added to T_n, maintaining **L**-exclusivity. First add it to F_n. Then check to see whether B_n has the form $\forall xAx$; if so, add Aa for the first parameter a not occurring in any of T_n, F_n, $\forall xAx$. These additions define F_{n+1}; and just let $\mathrm{T}_{n+1} = \mathrm{T}_n$.

Now, as usual, define T′ {F′} as the union of all the T_n {F_n}. Finish the first part of the theorem by showing that (T′, F′) is (i) **L**-exclusive, (ii) exhaustive, and (iii) quantifier prime.

(i) The **L**-exclusivity of each (T_n, F_n) is a consequence of the observations (1)–(3) above, and that the instantiations called for by (a) and (b) can actually be done is a consequence of the availability of infinitely many parameters. (ii) and (iii) are built into the construction.

Turning now to the "furthermore" part of the theorem, disjointness follows from **L**-exclusivity and the **L**-theoremhood of $A \rightarrow A$ (reflexivity, Def. 1). It is tedious but straightforward to verify that the primeness and **L**-theoryhood of T′ follow from exhaustiveness and quantifier-primeness of (T′, F′), together with **L**-exclusivity. For example, suppose T′ contains $A \vee B$. Suppose now for *reductio* that neither A nor B is in T′; but then by exhaustiveness, both must be in F′; but then $\vdash_{\mathbf{L}} A \vee B \rightarrow . A \vee B$ violates **L**-exclusivity; so T′ must be $\vee$-prime. For one more example (part of what it is for T′ to be an **L**-theory), suppose $\vdash_{\mathbf{L}} A \rightarrow B$, and that $A \in \mathrm{T}'$. Now B cannot be in F′ on pain of violating **L**-exclusivity; so, by exhaustiveness, it must be in T′, as required.

The containment of **L** is trivial, though perhaps confusing. Suppose B is a theorem of **L**. Then, by Up-Down acceptability, so is some universal closure B' of B, which is, accordingly, in Δ and, hence, in T′. But $B' \rightarrow B$ is a theorem of **L** (by Up-Down acceptability), and we know that T′ is closed under *modus ponens* for implications of (not just Δ but) **L**; so B is in T′.

The Pair Extension theorem, then, and accordingly The Way Up lemma of the last section, have been established.

§42.3. The Way Down.

In the last section, given a nontheorem A of an Up-Down acceptable logic **L**, we ascended via The Way Up to an **L**-containing prime **L**-theory T′ excluding A. We have, however, no assurance that T′ is $\sim$-consistent; it is the task of The Way Down, then, to permit descent to a truth-like **L**-containing **L**-theory that is a subset of T′—and accordingly to a truth-like **L**-theory containing **L** that excludes A. We remind the reader that finding such a theory is the keystone of the proof that the rule (γ) is admissible in **L**.

What we require is an adaptation of the method of "metavaluation" as employed in §22.3.1 and §22.3.3. Here we are following Meyer 1976a; see also Meyer 1971a and 1976a for other applications of this fruitful technique. Because we have only a single goal in mind, we simplify the terminology.

DEFINITION 7. Given a set T′ of formulas, the set T'_m of *metatruths on* T′ is defined inductively as follows:

Atomic formulas: $A \in T'_m$ iff $A \in T'$.
$\sim A \in T'_m$ iff both $A \notin T'_m$ and $\sim A \in T'$.
$A \vee B \in T'_m$ iff either $A \in T'_m$ or $B \in T'_m$.
$A \& B \in T'_m$ iff both $A \in T'_m$ and $B \in T'_m$.
$A \rightarrow B \in T'_m$ iff both (if $A \in T'_m$ then $B \in T'_m$) and $A \rightarrow B \in T'$.
$\forall x A x \in T'_m$ iff $Aa \in T'_m$ for all parameters a.
$\exists x A x \in T'_m$ iff $Aa \in T'_m$ for some parameter a.

Further, we say that A *survives metavaluation* if, for every Up-Down acceptable logic **L** and every **L**-containing prime **L**-theory T′, if $A \in T'$ then $A \in T'_m$.

Observe the key reference to T′ in the clauses for negation and implication; these are the more "intensional" of our operators, and those over which we have too little control in the concept of an **L**-containing prime **L**-theory T′, even for Up-Down-acceptable **L**.

We are now ready to state The Way Down lemma. Our aim is to state it in such a way as to minimize what must be verified in applications; so, since our intended applications are to logics that can be axiomatized in the style of §38.2, we build that feature into the very lemma itself.

THE WAY DOWN LEMMA. Let **L** be an Up-Down-acceptable logic. Let T' be an **L**-containing prime **L**-theory.

Part 1: T'_m is then a truth-like subset of T'. Further suppose that, as in §38.2, **L** is axiomatizable with rules *modus ponens* and adjunction from a set of axioms, where that set of axioms is generated by the "axiom clause" *universal generalization* of §38.2 from a set of "base axioms." And suppose that these "base axioms" (i) are closed under substitution for parameters, and (ii) survive metavaluation in the sense of Def. 7.

Part 2: T'_m is then an **L**-containing **L**-theory. Accordingly, T'_m is a truth-like **L**-theory lying between **L** and T': $\mathbf{L} \subseteq T'_m \subseteq T'$.

(1) It is enough to show that if $A \notin T'_m$ then $\sim A \in T'$; for then Def. 7 guarantees negation-completeness. A straightforward induction on complexity of formulas suffices, using the negation-completeness of T', and—for the first time—that $\vdash_{\mathbf{L}} A\&\sim B\rightarrow.\sim(A\rightarrow B)$ (Def. 1).

(2) We know that **L** is axiomatizable relative to *modus ponens* and adjunction; since T'_m is trivially closed under these two rules by Def. 7, it suffices to establish that the axioms of **L** are in T'_m. We proceed by arguing inductively, on the use of universal generalization to generate axioms, that every substitution instance of every axiom is in T'_m. For the "base axioms," the hypotheses (i) and (ii) of The Way Down lemma are just right. Suppose then that every instance of Ba is in T'_m and that $\forall xBx$ comes therefrom by universal generalization, a not in $\forall xBx$. Take an arbitrary instance $(\forall xBx)'$ of $\forall xBx$ to be shown to be in T'_m, and define $B'x$ so that $(\forall xBx)' = \forall xB'x$. Choose c as arbitrary, and note that $B'c$ is an instance of Ba, hence in T'_m by inductive hypothesis. So $\forall xB'x = (\forall xBx)' \in T'_m$ by the clause for the universal quantifier in Def. 7.

§42.4. Admissibility of (γ) in $\mathbf{R}^{\forall\exists x}$ et al. For the sake of explicitness, we state our main

THEOREM. The rule (γ), from $\sim A\vee B$ and A to infer B, is admissible in $\mathbf{R}^{\forall\exists x}$, $\mathbf{E}^{\forall\exists x}$, etc., in the sense that, when its premisses are theorems, so is its conclusion.

PROOF. Given The Way Up and The Way Down of the previous sections, and the outline provided in the introduction to this section, it suffices to verify that $\mathbf{R}^{\forall\exists x}$, $\mathbf{E}^{\forall\exists x}$, etc., have the features required by The Ways Up and Down: they are (1) Up-Down-acceptable logics in the sense of Def. 3, and (2) axiomatizable via *modus ponens* and adjunction from some axioms that are generated by universal generalization from some "base axioms" that are closed under substitution for parameters and (key property) survive metavaluation in the sense of Def. 7. These logics were axiomatized in §38.2 (which refers to §R2) in the way required; it is a tedious but straightforward matter to verify that they have the needed properties. Skipping altogether the matter of Up-Down acceptability, which is clearly close to the surface, let us give an example or two of the capacity of our chosen "base axioms" to survive metavaluation per Def. 7. Verification of a number of these axioms relies on the fact that an **L**-theory T′ for Up-Down-acceptable **L** is closed not only under the implications of **L** (Def. 2), but under its own implications:

mp for T′: if $A \rightarrow B \in \mathrm{T}'$ and $A \in \mathrm{T}'$ then $B \in \mathrm{T}'$.

This follows from the first "$1\frac{1}{2}$ degree property" of Def. 1. We also write "mp for T′" when, in either *antecedent*, "T′" is instead "T'_{m}"—known to be a subset of T′.

As our first example we show that axiom R3 of §R2 survives metavaluation. We are assuming that **L** is Up-Down-acceptable and that T′ is an **L**-containing prime **L**-theory—and accordingly that T'_{m} is truth-like (by Part 1 of The Way Down). Suppose then that $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow .C \rightarrow B \in \mathrm{T}'$; to show that it belongs to T'_{m} and therefore survives metavaluation, it suffices to assume that (a) $A \rightarrow B \in \mathrm{T}'_{\mathrm{m}}$ and show that $C \rightarrow A \rightarrow .C \rightarrow B \in \mathrm{T}'_{\mathrm{m}}$. By mp for T′, $C \rightarrow A \rightarrow .C \rightarrow B \in \mathrm{T}'$; so it suffices to suppose that (b) $C \rightarrow A \in \mathrm{T}'_{\mathrm{m}}$ and show that $C \rightarrow B \in \mathrm{T}'_{\mathrm{m}}$. By mp for T′, $C \rightarrow B \in \mathrm{T}'$; so it suffices to suppose that (c) $C \in \mathrm{T}'_{\mathrm{m}}$ and show that $B \in \mathrm{T}'_{\mathrm{m}}$. But (a), (b), and (c) clearly are enough for this.

For our second and last example, we choose IQ6 of §38.2 (the case with just one quantifier). Suppose that $\forall x(Ax \rightarrow Ax) \rightarrow B \rightarrow B \in \mathrm{T}'$; to show that it belongs to T'_{m}, it suffices to assume that $\forall x(Ax \rightarrow Ax) \rightarrow B \in \mathrm{T}'_{\mathrm{m}}$ and show that $B \in \mathrm{T}'_{\mathrm{m}}$. For this it clearly suffices to show that $\forall x(Ax \rightarrow Ax) \in \mathrm{T}'_{\mathrm{m}}$; and, since T'_{m} is truth-like, for this we need $Aa \rightarrow Aa \in \mathrm{T}'_{\mathrm{m}}$, for each parameter *a*. But $Aa \rightarrow Aa \in \mathrm{T}'$, since T′ is **L**-containing; so obviously $Aa \rightarrow Aa \in \mathrm{T}'_{\mathrm{m}}$, as required.

Incidentally, to repeat what is essentially the same point as that made at the end of §25.3.1, we note that from the fact that all of some set of *axioms* for **L** survive metavaluation one cannot conclude that all its *theorems* will; for example, $A \rightarrow .A \rightarrow B \rightarrow B$ and $A \rightarrow B \rightarrow .{\sim}A \vee B$ survive, but ${\sim}A \vee (A \rightarrow B \rightarrow B)$ does not.

§43. Miscellany.

This section exists in order to maintain the integrity of the structure of Chapter VII as announced in the tentative table of contents presumptuously displayed in Volume I. Being reminded of computer manuals with their self-refuting messages "This page intentionally left blank," we solicit contributions to be included in the second edition of this volume.

CHAPTER VIII

ACKERMANN'S *STRENGE IMPLIKATION*

§44. Ackermann's Σ-systems. As we have already made clear several times, the philosophical views and mathematical results of this book were inspired almost entirely by Ackermann's remarkable 1956 paper, *Begründung einer strengen Implikation.* In this section we will discuss a formulation of **E** which reflects some intuitive ideas that we find (or perhaps apperceive) in that paper. The reader should be warned that the following account reads a good bit into his work and that Ackermann should not be held accountable for all the views expressed below.

Ackermann considers just two Σ-systems: the classical two-valued calculus, which we call **ΣTV**, and a system **Σ′**, which is equivalent to a Hilbert-style system called **Π′** by Ackermann, namely, **E** *together with the disjunctive syllogism* (γ) *as a primitive rule.* The Σ-formulations of various systems present in some respects the appearance of a consecution calculus, but the motivation and formulation are so vastly different from Gentzen's that it is doubtful whether they deserve to have this name in common. **Σ′** is not designed with an elimination theorem in view, nor has it the subformula property, nor any separation theorems, nor does it help in attacking the decision problem. But it does have the virtue of providing one more bit of evidence for a claim we have been making throughout this book, to wit, that the systems **R**, **E**, and **T** are *stable*, in the sense adumbrated in §7.1.

We refer to the Σ-formulations of these and other systems as **ΣR**, **ΣS4**, **ΣE**, **ΣT**, etc., selecting **ΣE** for detailed treatment and leaving the others to be dealt with by the reader.

§44.1 Motivation. The Σ approach to **E** and the "implicational" paradoxes is most easily understood against the background of §22.2.2–3. All the consecutions of **ΣE** have the form

$$A, B \vdash C,$$

where A, B, C are formulas of the system, any two of which may be void. If A is void, the consecution is to have its usual interpretation as $B \rightarrow C$. Ackermann's principal innovation is to eradicate the classical hovering between the two distinct interpretations that might be placed on such a consecution when all three are nonvoid (i.e., reading it as $A\&B \rightarrow C$ or as $A \rightarrow . B \rightarrow C$) by using an explicit notational device to make this important distinction:

$$A, B \vdash C \quad \text{means} \quad A\&B \rightarrow C,$$

and

$$A^*, B \vdash C \quad \text{means} \quad A \rightarrow . B \rightarrow C,$$

where the star distinguishes those formulas and consecutions to which we may apply the nested interpretation. The star has no meaning in isolation from a consecution, and in this respect it is like the subscripts in $\mathrm{FE}_{\rightarrow}$; it is a bookkeeper's mark, designed to tell us when and how the rules apply, and is a part of the analysis of proofs rather than of the object-language vocabulary.

Use of stars will also be of importance in connection with negation, in particular as it affects the antilogism. From

$$B \vdash C$$

we will expect (as usual) to be able to move to

$$\bar{C} \vdash \bar{B},$$

but, if a parameter is present, it must be on the left and distinguished (*ausgezeichnete*) by a star; i.e., from

$$A^*, B \vdash C$$

we will be able to go to

$$A^*, \bar{C} \vdash \bar{B}.$$

(If the star were not present, we would generate fallacies of the sort complained about in §22.2.3.)

It may be that both premisses in a consecution are starred, in which case permutation is allowed, consonantly with *restricted permutation* as in $\mathbf{E}_{\rightarrow}$ (see §4.2). (We leave it to the reader to see that, though we may have consecutions of the forms

$$A, B \vdash C,$$

$$A^*, B \vdash C,$$

and

$$A^*, B^* \vdash C,$$

consecutions of the form

$$A, B^* \vdash C$$

do not arise.)

Occasionally stars are dropped, as in rule VI(3) below: dropping stars amounts to application of the rule of importation. With this as background, we now proceed to state the axioms and rules of ΣE, reminding the reader again that this constitutes an adaptation of Ackermann's ideas to **E**, with a few simplifications thrown in. (In particular we have omitted some of Ackermann's rules, which are required for his Σ′, but not for ΣE.)

§44.2. ΣE.

Axioms

ΣE1 $\vdash A \to A$
ΣE2 $A \& B \vdash A$
ΣE3 $A \& B \vdash B$
ΣE4 $A \vdash A \vee B$
ΣE5 $B \vdash A \vee B$
ΣE6 $A, \mathrm{B} \vee C \vdash (A \& B) \vee C$

Rules

(In stating the rules, a star is put in parentheses to indicate that the rule holds in the presence or absence of the star. If stars are parenthesized in both premiss and conclusion of a rule, all of them must be present in both places or absent in both places.)

I *Permutation on the left*

$$\frac{A^*, B^* \vdash C}{B^*, A^* \vdash C}$$

II *Cut*

$$(1)\ \frac{\vdash A \qquad A \vdash B}{\vdash B} \qquad\qquad (2)\ \frac{\vdash A \qquad C^*, A^{(*)} \vdash B}{C \vdash B}$$

$$(3)\ \frac{\vdash A \qquad \vdash B \qquad A, B \vdash C}{\vdash C}$$

III *Entailment introduction*

$$(1)\ \frac{A \vdash B}{D \to A \vdash D \to B} \qquad\qquad (2)\ \frac{A^*, B^{(*)} \vdash C}{A^*, D \to B^* \vdash D \to C}$$

$$(3)\ \frac{A^{(*)}, B^{(*)} \vdash C}{D \to A^{(*)}, D \to B^{(*)} \vdash D \to C} \qquad\qquad (4)\ \frac{A \vdash B \vee C}{B \to D, C \to D \vdash A \to D}$$

IV *Entailment elimination*

$$(1)\ \frac{\vdash B\to C}{B\vdash C} \qquad\qquad (2)\ \frac{A\vdash B\to C}{A^{*}, B\vdash C}$$

V *Conjunction elimination*

$$\frac{A\&B\vdash C}{A, B\vdash C}$$

VI *Negation*

$$(1)\ \frac{A\vdash C}{\bar{C}\vdash \bar{A}} \qquad\qquad (2)\ \frac{A^{*}, C\vdash B}{A^{*}, \bar{B}\vdash \bar{C}}$$

$$(3)\ \frac{A^{*}, C\vdash B}{\bar{B}, C\vdash \bar{A}}$$

(4) A and $\bar{\bar{A}}$ are interreplaceable in any consecution

§44.3 ΣE contains E. We first establish that →E and &I hold in ΣE and then show that, if A is an axiom of E then $\vdash A$ is provable in ΣE.

→E. If $\vdash A$ and $\vdash A\to B$ then $\vdash B$.

Proof. By IV(1) we have $A\vdash B$, whence with $\vdash A$ we have $\vdash B$ by II(1).

&I. If $\vdash A$ and $\vdash B$ then $\vdash A\&B$.

Proof. By ΣE1 and IV(1) we have $A\&B\vdash A\&B$, whence, by V, A, $B\vdash A\&B$. Then $\vdash A\&B$ follows by II(3).

It will also be convenient to have a theorem and a couple of derived rules:

H1			$A\vdash A$	ΣE1 IV(1)
H2	(1)	1	$A\vdash B$	hypothesis
		2	$A\to A\vdash A\to B$	1 III(1)
		3	$\vdash A\to B$	2 ΣE1 II(1)
	(2)	1	$A^{*}, B^{(*)}\vdash C$	hypothesis
		2	$A^{*}, B\to B^{*}\vdash B\to C$	1 III(2)
		3	$A\vdash B\to C$	2 ΣE1 II(2)
		4	$\vdash A\to .B\to C$	3 H2(1)
H3		1	$A, B\vdash C$	hypothesis
		2	$A\&B\to A, A\&B\to B\vdash A\&B\to C$	1 III(3)
		3	$\vdash A\&B\to C$	ΣE2–3 H2(1) II(3)

It remains only to show that, if A is an axiom of **E** (see §R2), then $\vdash A$ is provable in ΣE. We examine each in turn.

E1		$A\to A\to B\to B$	
	1	$A\to A\to B\vdash A\to A\to B$	H1
	2	$A\to A\to B^*, A\to A\vdash B$	1 IV(2)
	3	$A\to A\to B\vdash B$	2 ΣE1 II(2)
	4	$\vdash A\to A\to B\to B$	3 H2(1)
E2		$A\to B\to.\ B\to C\to.A\to C$	
	1	$B\to C\vdash B\to C$	H1
	2	$B\to C^*, B\vdash C$	1 IV(2)
	3	$B\to C^*, A\to B^*\vdash A\to C$	2 III(2)
	4	$A\to B^*, B\to C^*\vdash A\to C$	3 I
	5	$\vdash A\to B\to.\ B\to C\to.A\to C$	4 H2(2)
E3		$(A\to.A\to B)\to.\ A\to B$	
	1	$A\to B\vdash A\to B$	H1
	2	$A\to B^*, A\vdash B$	1 IV(2)
	3	$A\to.\ A\to B^*, A\to A\vdash A\to B$	2 III(3)
	4	$A\to.A\to B\vdash A\to B$	3 ΣE1 II(2)
	5	$\vdash(A\to.A\to B)\to.\ A\to B$	4 H2(1)
E4–5		Use ΣE2–3 and H2(1)	
E6		$(A\to B)\&(A\to C)\to.\ A\to(B\&C)$	
	1	$B\&C\vdash B\&C$	H1
	2	$B, C\vdash B\&C$	1 V
	3	$A\to B, A\to C\vdash A\to(B\&C)$	2 III(3)
	4	$\vdash(A\to B)\&(A\to C)\to.\ A\to(B\&C)$	3 H3
E7		$\Box A\&\Box B\to\Box(A\&B)$ We prove the easier $((A\to A)\&(B\to B))\to C\to C$; see §26.1.	
	1	$((A\to A)\&(B\to B)\to C)^*, (A\to A)\&(B\to B)\vdash C$	H1 IV(2)
	2	$\vdash(A\to A)\&(B\to B)$	ΣE1 &I
	3	$(A\to A)\&(B\to B)\to C\vdash C$	1 2 II(2)
	4	$\vdash((A\to A)\&(B\to B)\to C)\to C$	3 H2(1)
E8–9		Use ΣE4–5 and H2(1).	
E10		$(A\to C)\&(B\to C)\to.\ (A\vee B)\to C$	
	1	$A\vee B\vdash A\vee B$	H1
	2	$A\to C, B\to C\vdash(A\vee B)\to C$	1 III(4)
	3	$\vdash(A\to C)\&(B\to C)\to.\ (A\vee B)\to C$	2 H3
E11		Use ΣE6 and H3.	

E12		$A\to\bar{A}\to\bar{A}$	
	1	$A\&A\vdash A\&A$	H1
	2	$A, A\vdash A\&A$	1 V
	3	$A\to A, A\to A\vdash A\to(A\&A)$	2 III(3)
	4	$\vdash A\to(A\&A)$	3 ΣE1 II(3)
	5	$A\to\bar{A}^*, A\vdash\bar{A}$	H1 IV(2)
	6	$\bar{\bar{A}}, A\vdash\overline{A\to\bar{A}}$	5 VI(3)
	7	$A, A\vdash\overline{A\to\bar{A}}$	6 VI(4)
	8	$\vdash A\&A\to\overline{A\to\bar{A}}$	7 H3
	9	$A\&A\vdash\overline{A\to\bar{A}}$	8 IV(1)
	10	$A\to(A\&A)\vdash A\to\overline{A\to\bar{A}}$	9 III(1)
	11	$\vdash A\to\overline{A\to\bar{A}}$	4 10 II(1)
	12	$A\vdash\overline{A\to\bar{A}}$	11 IV(1)
	13	$\overline{\overline{A\to\bar{A}}}\vdash\bar{A}$	12 VI(1)
	14	$A\to\bar{A}\vdash\bar{A}$	13 VI(4)
	15	$\vdash A\to\bar{A}\to\bar{A}$	14 H2(1)
E13		$A\to\bar{B}\to. B\to\bar{A}$	
	1	$A\to\bar{B}^*, A\vdash\bar{B}$	H1 IV(2)
	2	$A\to\bar{B}^*, \bar{\bar{B}}\vdash\bar{A}$	1 VI(2)
	3	$A\to\bar{B}^*, B\vdash\bar{A}$	2 VI(4)
	4	$\vdash A\to\bar{B}\to. B\to\bar{A}$	3 H2(2)
E14		Use H1, VI(4), and H2(1).	

So ΣE contains **E**, as advertised.

§44.4 E contains ΣE. To prove that whenever $\vdash A$ is provable in ΣE, A is a theorem of **E**, we need simply verify that the axioms and rules of ΣE hold in **E** under the following translation:

ΣE	E
$\vdash A$	A
$A\vdash B$	$A\to B$
$A, B\vdash C$	$A\&B\to C$
$A^*, B^{(*)}\vdash C$	$A\to. B\to C$

The proof is trivial, and will be left to the reader.

§45. Σ′, Π′, Π″, and E (historical). As may be expected of logicians trying to talk about History, we begin by setting down some axioms and rules to discuss. The axioms (1)–(15) and the rules (α)–(δ) below for Ackermann's calculus $\mathbf{\Pi}'$ are from his 1956 paper, verbatim (*sozusagen*):

Axioms

(1) $A\to A$,
(2) $(A\to B)\to((B\to C)\to(A\to C))$,
(3) $(A\to B)\to((C\to A)\to(C\to B))$,
(4) $(A\to(A\to B))\to(A\to B)$,
(5) $A\&B\to A$,
(6) $A\&B\to B$,
(7) $(A\to B)\&(A\to C)\to(A\to B\&C)$,
(8) $A\to A\vee B$,
(9) $B\to A\vee B$,
(10) $(A\to C)\&(B\to C)\to(A\vee B\to C)$,
(11) $A\&(B\vee C)\to B\vee(A\&C)$,
(12) $(A\to B)\to(\bar{B}\to\bar{A})$,
(13) $A\&\bar{B}\to\overline{A\to B}$,
(14) $A\to\bar{\bar{A}}$,
(15) $\bar{\bar{A}}\to A$.

Rules

(α) From A and $A\to B$ to infer B.
(β) From A and B to infer $A\&B$.
(γ) From A and $\bar{A}\vee B$ to infer B.
(δ) $A\to(B\to C)$ and B to infer $A\to C$.

Ackermann remarks that the definitions $\bar{A}\to A$ (of $\Box A$) and $A\to\bar{A}$ (of UA: "it is impossible that A") are not satisfactory for the system $\mathbf{\Pi}'$, since e.g. we would wish to regard A as impossible if $A\to .\,B\&\bar{B}$ were a theorem; but from this (he says) $A\to\bar{A}$ does not follow in $\mathbf{\Pi}'$, the required step $B\&\bar{B}\to\bar{A}$ being "paradoxical" and by Ackermann's matrix unprovable. There is a trivial mistake here: from $A\to .\,B\&\bar{B}$ we get $(A\to B)\&(A\to\bar{B})$, whence $(A\to B)\&(B\to\bar{A})$, and finally $A\to\bar{A}$. But this minor error does not obscure Ackermann's point: For we should wish equally to regard A as impossible if $A\to .\overline{B\to B}$ were a theorem; and from this, indeed, $A\to\bar{A}$ does not follow in $\mathbf{\Pi}'$, since neither the fallacy of relevance $\overline{B\to B}\to\bar{A}$ nor the fallacy of modality $\overline{B\to B}\to .B\to B$, nor anything more devious, is available to smooth the passage. So Ackermann is right: if $\Box A$ is defined as $\bar{A}\to A$, then it does not have the right properties (see §11).

He is therefore led, adapting an idea of Johansson 1936 (which, as Turquette has informed us in correspondence, is ultimately to be traced to Peirce, like so much else in modern symbolic logic), to formulate a system which we call $\mathbf{\Pi}''$, by adding to $\mathbf{\Pi}''$ a propositional constant $\boldsymbol{f}$ for *das Absurde*, and defining UA as $A\to\boldsymbol{f}$, $\Box A$ as U$\bar{A}$, and $\Diamond A$ as $\overline{\mathrm{U}A}$. (Ackermann used the label **Π′**, but he did not himself give any name to the system he defined on p. 124; so the name **Π″** is ours, but the system is not.) For **Π″** we then add the axioms

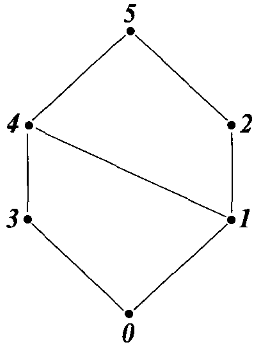
(16) $A \to f \to \bar{A}$, and

(17) $A \& \bar{A} \to f$,

together with the rule

(ε) From $A \to B$ and $(A \to B) \& C \to f$ to infer $C \to f$.

Ackermann demonstrates by means of a matrix that his system **Π″**, and *a fortiori* **Π′**, is free from what we call "fallacies of modality" (§5.2.1); that is, it has what we call "the Ackermann property" (§8.12): never $\vdash p \to .A \to B$, and more generally, if A is free of both f and →, then never $\vdash A \to .B \to C$. Ackermann's matrix is as follows: First, a Hasse diagram:



This is a redrawn and renumbered version ("the shield") of Six of §34.1; & is to be computed as greatest lower bound, ∨ as least upper bound, and f as *2*. Tables for negation and entailment:

→	0	1	2	3	4	5	~
0	3	3	3	3	3	3	5
1	0	3	3	0	3	3	4
2	0	0	3	0	0	3	3
*3	0	0	0	3	3	3	2
*4	0	0	0	0	3	3	1
*5	0	0	0	0	0	3	0

It is easy to see that $\{\mathit{1}, \mathit{4}\}$ is closed under the truth functions and that neither implies either of the entailment values *0* and *3*; so things are just as Ackermann said. As we noted in §22.1.1, the argument applies exactly to **E** and can be generalized along lines due to Maksimova.

Turning back now to the move from $\mathbf{\Pi}'$ to $\mathbf{\Pi}''$, we note that in the present context (16)–(17) and (ϵ) have no motivation other than permitting the introduction of modalities. It is therefore of interest to note that the additional axioms and rule are in this sense redundant, since UA (hence also the other modalities) may be defined in $\mathbf{\Pi}'$ in such a way as to obtain in $\mathbf{\Pi}'$ a theory of modalities equivalent to that of $\mathbf{\Pi}''$. We proceed to see that this is so.

§45.1. *f* goes. In $\mathbf{\Pi}'$ it is more natural to take $\Box$ as the fundamental modality, since the definition of $\Box$ takes the familiar §4.3 form:

$$\Box A =_{\mathrm{df}} (A \to A) \to A.$$

We refer to those formulas of $\mathbf{\Pi}''$ in which f occurs, if at all, only in contexts of the form $A \to f$ (so that f is always eliminable in favor of U) as "U-formulas" of $\mathbf{\Pi}''$. If A^{U} is any U-formula of $\mathbf{\Pi}''$, then the "$\Box$-transform" of A^{U} is the formula $A^{\Box}$ got by replacing every part of A^{U} of the form $B \to f$ by $(\bar{B} \to \bar{B}) \to \bar{B}$, i.e., by $\Box\bar{B}$. Evidently if A^{U} is a U-formula of $\mathbf{\Pi}''$, then $A^{\Box}$ will be a formula of $\mathbf{\Pi}'$ (as well as of $\mathbf{\Pi}''$), and we can now state the result in the following form.

THEOREM. If A^{U} is a U-formula of $\mathbf{\Pi}''$, and $A^{\Box}$ is its $\Box$-transform, then $\vdash A^{\mathrm{U}}$ in $\mathbf{\Pi}''$ if and only if $\vdash A^{\Box}$ in $\mathbf{\Pi}'$.

We first observe that in $\mathbf{\Pi}''$ we have

$$B \to f \rightleftarrows . \, \bar{B} \to \bar{B} \to \bar{B},$$

which is not hard to prove, and a derivable rule of substitution; so $\vdash A^{\mathrm{U}}$ in $\mathbf{\Pi}''$ iff $\vdash A^{\Box}$ in $\mathbf{\Pi}''$. This is half the battle; what remains is to see that $\mathbf{\Pi}''$ is a conservative extension of $\mathbf{\Pi}'$, i.e., that an f-free formula of $\mathbf{\Pi}''$ always has an f-free proof. Such a proof will also be a proof in $\mathbf{\Pi}'$, from which it will follow that if $\vdash A^{\Box}$ in $\mathbf{\Pi}''$ then $\vdash A^{\Box}$ in $\mathbf{\Pi}'$.

The leading idea is that, although f cannot be replaced by the same f-free formula in every proof, it is still possible to find for each proof of an f-free formula, a particular f-free formula that can replace f throughout that proof.

Let $A_1, \ldots, A_n$ ($A_n = A$) be a proof of A in $\mathbf{\Pi}''$, and let $p_1, \ldots, p_m$ be a list of all propositional variables occurring in the proof $A_1, \ldots, A_n$. Then, for this proof of A, we define f' as

$$\overline{(p_1 \to p_1) \,\&\, \ldots \,\&\, (p_m \to p_m)}.$$

Let A_i' be the result of replacing f throughout A_i by f'. (We notice for use below that $\overline{f'}$ is a theorem.) Since A is f-free, $A_n' = A$. We show inductively that each of $A'_1, \ldots, A'_n$ $(= A)$ has an $\boldsymbol{f}$-free proof in $\mathbf{\Pi}''$, which is to say a proof in $\mathbf{\Pi}'$, as required.

CASE 1. A_i is an axiom of $\mathbf{\Pi}''$.

(i) If A_i is one of the axioms (1)–(15) of $\mathbf{\Pi}''$, then $\vdash A'_i$ in $\mathbf{\Pi}'$ by the same axiom.

(ii) If A_i is an axiom (16) of $\mathbf{\Pi}''$, then A'_i has the form $C \rightarrow \boldsymbol{f}' \rightarrow \bar{C}$. By axiom (12) we have in $\mathbf{\Pi}'$ $\vdash C \rightarrow \boldsymbol{f}' \rightarrow . \overline{\boldsymbol{f}'} \rightarrow \bar{C}$, and $\overline{\boldsymbol{f}'}$ is provable, so by rule (δ), $\vdash A'_i$ in $\mathbf{\Pi}'$.

(iii) If A_i is axiom (17) of $\mathbf{\Pi}''$, then A_i' has the form $C \& \bar{C} \rightarrow \boldsymbol{f}'$; we need to show that A_i' is provable in $\mathbf{\Pi}'$.

Let $q_1, \ldots, q_k$ be all the variables occurring in C. Then an easy induction on the length of C shows that $\&_1^k\ (q_j \rightarrow q_j) \rightarrow . C \rightarrow C$. Evidently, $\&_1^m\ (p_j \rightarrow p_j) \rightarrow \&_1^k (q_j \rightarrow q_j)$, since the q_j are all among the p_i, so by transitivity, double negation, and the definition of $\boldsymbol{f}'$ we have $\overline{\boldsymbol{f}'} \rightarrow .\ C \rightarrow C$, hence $\overline{C \rightarrow C} \rightarrow \boldsymbol{f}'$. But then axiom (13) and transitivity give us $C \& \bar{C} \rightarrow \boldsymbol{f}'$, as required.

CASE 2. A_i is a conclusion of a rule.

(i) If A_i is a conclusion from premisses A_j and A_k by (α), (β), (γ), or (δ), then $\vdash A'_j$ and $\vdash A'_k$ in $\mathbf{\Pi}'$ by the inductive hypothesis, and $\vdash A'_i$ in $\mathbf{\Pi}'$ by the same rule.

(ii) If A_i is the conclusion from A_j and A_k by rule (ϵ) in $\mathbf{\Pi}''$, then A'_i has the form $C \rightarrow \boldsymbol{f}'$, and A'_k (say) has the form $A'_j \& C \rightarrow \boldsymbol{f}'$. By the hypothesis of the induction, (13), etc., we have in $\mathbf{\Pi}' \vdash \overline{A'_j \& C} \vee \boldsymbol{f}'$, and then, by De Morgan's laws and commutation, $\vdash \boldsymbol{f}' \vee \overline{A'_j} \vee \bar{C}$. By (γ) twice, we have $\vdash \bar{C}$ in $\mathbf{\Pi}'$. Then, by axiom (12) and rule (δ), we have $\vdash C \rightarrow C \rightarrow \bar{C}$, and, contrapositively, $\vdash C \rightarrow \overline{C \rightarrow C}$. But we saw under Case 1 that $\overline{C \rightarrow C} \rightarrow \boldsymbol{f}'$; so, by transitivity, we have, in $\mathbf{\Pi}'$, $C \rightarrow \boldsymbol{f}'$, i.e., A'_i, as required.

This completes the proof of the theorem, which shows essentially that the addition of $\boldsymbol{f}$, axioms (16)–(17), and the rule (ϵ) is otiose, since $\mathbf{\Pi}'$ already contains an equivalent theory of modality. This was the first step in the course of arriving at **E** from $\mathbf{\Pi}''$, and we are now left with $\mathbf{\Pi}'$: axioms (1)–(15) and rules (α)–(δ).

§45.2. (δ) goes. As we pointed out in §8.2, (δ) can be dispensed with in favor of one of several axioms in the pure arrow theory. And in §26.1 we saw that the same effect can be obtained in **E**, by adding one of several axioms to $\mathbf{\Pi}'$, involving (in effect) necessity and conjunction. The only difference between the resulting system and **E** is, then, that the former takes (γ) as primitive.

§45.3. (γ) goes. As the reader can verify by trying to construct proofs, addition of (γ) as primitive to **E** destroys practically every nice property **E** has.

The Fitch-style natural deduction formulation **FE** and the entailment theorem of §23.6 both go to pieces, as do the other results that depend on these. And it is not hard to see why. Most of the metalogical proofs are by induction on the length of formal proofs in **E**, and, in the absence of $A\&(\bar{A}\vee B) \rightarrow B$, there is apparently no way of getting over the inductive step where (γ) is used.

This fact led to the observation that, if $A\&(\bar{A}\vee B) \rightarrow B$ does not belong in a theory of entailment, which it *obviously* (by this time) does not, then the primitive rule (γ) does not belong there either. So attention was turned (in 1958) to the problem of showing (γ) to be an admissible rule.

(γ) might fail in either of two ways. We might find formulas A and B such that $\vdash A$ and $\vdash \bar{A}\vee B$, but $\vdash \bar{B}$. This would of course be disastrous, since &I and De Morgan would lead to $\vdash \overline{\bar{A}\vee B}$, in outright contradiction to $\vdash \bar{A}\vee B$; but **E** is easily seen to be consistent, so this possibility need occasion no alarm.

But a less comfortable alternative is also available, in principle at least. For it might be that $\vdash A$ and $\vdash \bar{A}\vee B$, but still $\nvdash B$, which would indicate that the system was askew with respect to the intended interpretation of the wedge and the overbar. Counterexamples of this sort do exist for the system **E** without distribution, as was pointed out at the end of §25.2.3. Had (γ) failed in this second way, we might have felt that **E** was incomplete, and that further (or fewer?) axioms were needed.

These considerations led to the belief that, if **E** was a *nice* system, it would have the completeness conferred by having (γ) admissible. In 1958 we thought a proof of this was just around the corner. It proved to be a long corner; the Meyer-Dunn proof of (γ), reported in §25.2, was not even envisaged in 1958, and did not arrive until ten years later.

Lest the upshot have been lost in the forest, we note that it is a consequence of the eliminability of *f* as in §45.1, of (δ) as in §45.2, and of (γ) as in this section, that our calculus **E** and Ackermann's calculus $\mathbf{\Pi}'$ are equivalent as to theorems, and that not only is $\mathbf{\Pi}''$ their conservative extension, but also all the modal work of $\mathbf{\Pi}''$ can already be done in $\mathbf{\Pi}'$ or in **E**.

§46. Miscellany.

This section is like §8 of Volume I.

§46.1. Ackermann on strict "implication".

Both C. I. Lewis and Ackermann were motivated in part by the hope of finding a formal treatment of "if . . . then—" that obviated the absurdities of material "implication." Many natural and obvious questions arise concerning relations between the two solutions, and among the first papers to discuss the topic was Ackermann 1958.

Ackermann opined (correctly, as shown in §29.12) that *strenge Implikation* was not definable in terms of strict implication, and, though he did not say so, the reason seems obvious: relevance cannot be articulated in terms of truth functions and modalities alone. But we are left wondering whether the

Lewis notions can be embodied in a system that, like Ackermann's, recognizes truth functions, modality, *and* relevance.

In his 1958 paper, Ackermann, in response to inquiries by Bernays, attacked just this question, and solved it to the following extent: if we translate $A \strictif B$ as $A\&\bar{B}\&\bar{f} \to f$ in $\mathbf{\Pi}''$, then we can prove analogues of all the theorems and rules of Lewis's system **S2**: $\mathbf{\Pi}''$ contains at least **S2** under translation.

In 1958 Ackermann did not know that the modal structure of $\mathbf{\Pi}''$ itself is like that of **S4**; with this additional information it is easy to show that, under the usual Lewis definition of $\Box A$ as $\bar{A} \strictif A$, one has, under the translation, the standard **S4** axiom $\Box A \strictif \Box\Box A$; so that, given the proposed translation, $\mathbf{\Pi}''$ contains not only **S2** but all of **S4**.

Also in 1958 information was meager about what $\mathbf{\Pi}''$ fails to contain, so much so that Ackermann guessed that a solution to the question whether his translation was exact (provable if and only if) would have to wait on a decision procedure for the parent system $\mathbf{\Pi}''$. But although, by §65 below, no decision procedure is possible, the translation is indeed an exact translation of (not **S2** but) **S4** into $\mathbf{\Pi}''$. This follows from Meyer 1970a, which contains other results and details; here we present an argument based on the foregoing sections.

To save fussing about hooks and arrows, we begin by thinking of **S4** as formulated with the arrow—for strict implication—as primitive, and we rely on Hacking 1963 to secure us a formulation, which we might call $\mathbf{S4}_{\to\&\vee\sim}$ but do call just **S4**, of the strict-implication-conjunction-disjunction-negation fragment of Lewis's **S4**. Let A be an **S4**-formula, and let A' be its translation into $\mathbf{\Pi}''$ by means of the replacement of $B \to C$ with the Ackermann translation $B\&{\sim}C\&{\sim}f \to f$. We know by verification that, for every theorem A of **S4** in the Hacking 1963 formulation, its translation A' is provable in $\mathbf{\Pi}''$, since the availability of (γ) in $\mathbf{\Pi}''$ reduces this claim to proof in $\mathbf{\Pi}''$ of the translation of axioms of **S4**.

Suppose for the converse that, for a certain **S4**-formula A, its translation A' is provable in $\mathbf{\Pi}''$. Then, as in §45.1, the result A'^* of replacing f by suitable $f^* = {\sim}((p_1 \to p_1)\& \ldots \&(p_n \to p_n))$ is provable in Ackermann's $\mathbf{\Pi}''$. Now $\mathbf{\Pi}'$ is easily seen to be a subsystem of **S4**, so A'^* is a theorem of **S4**. The desired provability of A itself in **S4** is a consequence of the fact that **S4** warrants the interprovability of A'^* and A because it holds equivalent the formulas $B \to C$ and $B\&{\sim}C\&{\sim}f^* \to f^*$. This completes the argument.

One further note: because we know from §25 that **E** admits (γ) and is thus equivalent to $\mathbf{\Pi}'$ in point of theorems and also because f can be added to **E** conservatively, the Ackermann translation can be seen as a way to embed **S4** in **E** as well as in $\mathbf{\Pi}''$.

Let us add what we think is still an open question: using $\Box$ for the necessity of **E**, what about the system in $\&$, $\vee$, $\sim$, and $\strictif$, where $A \strictif B$ is defined as

$\Box(\bar{A}\vee B)$? (Before the reader assigns this problem to his friends, we note that it is probably not very interesting, inasmuch as the cockle-warming principle $(A ⥽ B)\&\Box A ⥽ \Box B$ fails to be forthcoming—an unpublished result shown semantically by Meyer.)

§46.2. An interesting matrix. In §8.10 we set as a problem the finding of some interesting matrix to stand as witness to the unprovability in $\mathbf{E}_{\to}$ of the formula $(A\to B\to B\to A)\to .\ A\to A$. R. Z. Parks obtains the following solution (correspondence, 1988):

→	*0*	*1*	2	*3*
0	*3*	*3*	*3*	*3*
1	*0*	2	*0*	*3*
*2	*0*	*0*	2	*3*
**3*	*0*	*0*	*0*	*3*

The values *2* and *3* are designated. It is straightforward to check that the axioms E1–E3 of §R2, which, according to §8.3.3, are sufficient for $\mathbf{E}_{\to}$, all uniformly take designated values, and that the rule *modus ponens* of $\mathbf{E}_{\to}$ preserves this property. And the formula to be excluded takes the value *0* when *A* and *B* each take the value *1*, thus settling a small problem, but one that had been open for at least thirteen years.

CHAPTER IX

SEMANTICS

§47. Semilattice semantics for relevance logics (by Alasdair Urquhart).

The present section records an attempt to develop a natural semantical analysis of relevance logics which would reflect more or less directly the preformal intuitions underlying these systems. (The model theory was conceived independently by Richard Routley, the author of the present section, and Kit Fine. See §48.3 for some related historical details.) This attempt has met with only partial success. Natural and elegant semantics are provided for a wide family of pure implicational logics including many of those treated in earlier sections, and especially $\mathbf{R}_{\rightarrow}$, $\mathbf{E}_{\rightarrow}$, and $\mathbf{T}_{\rightarrow}$, respectively, of §3, 4, and 6. These results extend to conjunction, but break down when disjunction and negation are added. Nevertheless the semantical theory developed here is of interest in spite of this failure: it leads to new relevance logics which are worth considering in their own right, and it provides the starting point for the more general semantics and completeness results to be explained in §48. To emphasize the intuitive basis of the program we shall begin with some considerations which are philosophical rather than mathematical.

§47.1. Semantics for $\mathbf{R}_{\rightarrow}$.

The concept of a "piece of information," to be explained below, will be basic throughout the semantical analysis. Let us suppose that we have a particular topic or subject under consideration and a language in which to formulate discussions about this subject. It is to be supposed that from the sentences of this language we can isolate the basic or atomic sentences from which logically complex sentences are formed by operations such as conjunction and implication. Thus if the subject under consideration were number theory the basic sentences would be numerical equations, if physics, simple statements of experimental results, and so forth. A *piece of information* is to be thought of as an arbitrary set of basic sentences. Such a set could be given as a finite list or, if infinite, listed in some mechanical way, possibly even given in some nonmechanical manner—for instance in physics we might think of the set of all experimental results to be established in the future.

The concept of a "piece of information" is to be contrasted with two less general concepts, those of an "evidential situation" and of a "possible world." The former is a concept suitable for an analysis of intuitionistic logic (Kripke

1965a); the latter is familiar from work on modal logic (Kripke 1959, 1963). An *evidential situation* is to be thought of as a set of propositions that have been established as true during the course of some investigation into a subject. It must, hence, satisfy the requirement of *consistency*. The concept of a *possible world* is still narrower: since it is intended to be a *total* description of some possible situation, it must satisfy not only the requirement of consistency but also that of *completeness*. The distinctions between the three concepts emerge in two ways; first, in the mathematical structures that it is natural to consider by abstracting from these ideas, secondly, in the truth conditions that obtain for complex statements relative to these structures, given the truth conditions for basic statements. We now consider these questions for the case of pieces of information.

Let us suppose that we have a given subject under consideration, and, further, a set S of possible pieces of information x which would be relevant to argumentation or communication about the subject. What can we say about the structure of S? At the least, it would seem, we would wish to include the empty piece of information $\mathit{0}$ in S; further, it seems clear that if x and y are in S so is $x \cup y$, that is, the piece of information that is the union of the pieces of information x and y. Thus S has the structure of a join (or upper) *semilattice with zero*; that is, it is closed under a binary operation $\cup$ for which the equations:

1. (Identity) $\mathit{0} \cup x = x$
2. (Commutativity) $x \cup y = y \cup x$
3. (Associativity) $(x \cup y) \cup z = x \cup (y \cup z)$
4. (Idempotence) $x \cup x = x$

hold for all x, y, z in S.

It is to be noted that the semilattice structure can also reasonably be imposed on a set of evidential situations. For if such a set S is taken to represent a set of statements established during investigation into some fixed subject matter, we must suppose that any x and y in S are jointly consistent, so that $x \cup y$ would again be an evidential situation in S. On the other hand, closure under the semilattice operation makes no sense when considered as applying to a set of possible worlds. There is no reason to expect two possible worlds x, y to be jointly consistent so that $x \cup y$ would again be a possible world; in fact the metalogical usefulness of the concept of a "possible world" lies precisely in the idea that a statement might be true in one world, but false in another, so that the two worlds are jointly inconsistent.

Before we can say anything useful about a semilattice S of pieces of information, we need one further concept, namely, a primitive notion of *consequence* or *entailment*. A piece of information x will in general entail certain basic statements *p*; let us write x $\Vdash p$ if this relationship holds. For instance, we might have:

$\{1+1=2, 2=2+0\} \Vdash 1+1=2+0$,
{John is a bachelor} $\Vdash$ John is unmarried,
{Harry is taller than Fred, Jim is taller than Harry}
$\Vdash$ Jim is taller than Fred,

and so forth. This consequence relation is essentially *logic free*; that is, it holds not by virtue of the logical complexities of the statements involved, but by virtue of (a) the meanings of the predicates occurring in the basic sentences and (b) certain background facts presupposed in the context of discourse. Hence there is no circularity involved in defining the notion of consequence for complex statements in terms of a postulated consequence relation for basic statements. Finally, note that we do not postulate the condition: if $x \Vdash p$, then $x \cup y \Vdash p$. The reason is of course that we require the consequence relation to be one of *relevant* entailment.

Given a semilattice S and a consequence relation for basic statements relative to the elements of S, the consequence relation for complex statements can be defined recursively. Let us suppose for the moment that the language has implication as its sole logical connective. If the consequence relation has been extended to the statements *A* and *B*, what are the truth conditions for $A \to B$? Well, since → represents the notion of logical consequence, we wish it to be the case that $A \to B$ is a consequence of x whenever *B* is a consequence of x and *A*. We could then write: $x \Vdash A \to B$ if $x \cup \{A\} \Vdash B$. However, this will not do as a definition; *A* may be logically complex so that $x \cup \{A\}$ would not be a piece of information. The intention can be reproduced, nevertheless, in a more general form: $x \Vdash A \to B$ if and only if, whenever $y \Vdash A$, $x \cup y \Vdash B$, for any y in S. Since this statement expresses exactly the sense in which → represents deducibility, we take it as the recursive consequence definition for that connective.

The definition we have just arrived at has, when properly viewed, a familiar look: when $x \Vdash A \to B$ is written as $(A \to B)_x$, with x a class of numerals, and if $x \cup y$ means set union, we find we have simply an abstraction from the subscripting requirements of §3: write $(A \to B)_x$ iff from A_y you can pass to $B_{x \cup y}$, y arbitrary ("new"). We have, so to speak, given the subscripts a life of their own beyond the role of bookkeeping tags which they perform in the subproof formulations. What we have been emphasizing is the naturalness and philosophical plausibility of the requirement.

Let us restate the semantics in a more formal manner. Given a semilattice S with *0* the lattice zero, a *valuation on* S is a function v which assigns to each propositional variable *p* a subset of S, v(*p*). A pair $Q = \langle S, v \rangle$ we shall refer to as a *consequence model*, or *c-model*. Given a c-model Q, the consequence relation relative to Q, $\Vdash_{\mathrm{Q}}$, is defined recursively as follows:

1. $\mathrm{x} \Vdash_{\mathrm{Q}} p$ iff x is in $\mathrm{v}(p)$.
2. $\mathrm{x} \Vdash_{\mathrm{Q}} A \rightarrow B$ iff for all y in S either not $\mathrm{y} \Vdash_{\mathrm{Q}} A$ or $\mathrm{x} \cup \mathrm{y} \Vdash_{\mathrm{Q}} B$.

A formula A is *true in* Q if $\mathit{0} \Vdash_{\mathrm{Q}} A$; *c-valid* if it is true in all c-models.

The set of c-valid formulas coincides with the theorems of $\mathbf{R}_{\rightarrow}$. It is left to the reader to check that all theorems of $\mathbf{R}_{\rightarrow}$ are c-valid; in fact we already proved this when we showed that $\mathbf{R}_{\rightarrow}$ is contained in $\mathbf{FR}_{\rightarrow}$. To show the converse, let $\langle A_1, \ldots, A_n\rangle$ be a finite sequence of formulas of $\mathbf{R}_{\rightarrow}$. Then $A_1, \ldots, A_n \vdash B$ is defined to hold if $A_1 \rightarrow . \ldots \rightarrow . A_n \rightarrow B$ is provable in $\mathbf{R}_{\rightarrow}$. (Note: this is a local use of "$\vdash$".) We list some derived rules in terms of this definition; they are easily checked by the subproof formulation.

DR1. If $\alpha \vdash A$, then $\alpha' \vdash A$, where α' is a permutation of α,
DR2. If $\alpha \vdash A \rightarrow B$ and $\beta \vdash A$, then $\alpha, \beta \vdash B$,
DR3. If $\alpha, A, A, \beta \vdash B$, then $\alpha, A, \beta \vdash B$,
DR4. If $\alpha, A \vdash B$, then $\alpha, A \rightarrow A \rightarrow A \vdash B$.

THEOREM. A formula of $\mathbf{R}_{\rightarrow}$ is provable in $\mathbf{R}_{\rightarrow}$ iff it is c-valid.

PROOF. Let S be the set of all finite sets of formulas of $\mathbf{R}_{\rightarrow}$. S is a semilattice under the operation of set union, with the empty set $\mathit{0}$ the lattice zero. For x in S, let x be in $\mathrm{v}(p)$ if $\bar{\mathrm{x}} \vdash p$ for some sequence $\bar{\mathrm{x}}$ consisting of the elements of x (without repetitions). Thus $\mathrm{Q} = \langle \mathrm{S}, \mathrm{v}\rangle$ is a c-model. We show first the following

FACT. For any x in S, any formula A, $\mathrm{x} \Vdash_{\mathrm{Q}} A$ iff $\bar{\mathrm{x}} \vdash A$ for some ordering $\bar{\mathrm{x}}$.

This holds by definition for propositional variables; let it be assumed to hold for A and B. Now if $\bar{\mathrm{x}} \vdash A \rightarrow B$, then, if $\mathrm{y} \Vdash_{\mathrm{Q}} A$, $\bar{\mathrm{y}} \vdash A$, by induction hypothesis; so $\bar{\mathrm{x}}, \bar{\mathrm{y}} \vdash B$ by DR2. By DR1 and DR3, repetitions in $\langle \bar{\mathrm{x}}, \bar{\mathrm{y}}\rangle$ may be eliminated; so $\overline{\mathrm{x} \cup \mathrm{y}} \vdash B$, and hence $\mathrm{x} \cup \mathrm{y} \Vdash_{\mathrm{Q}} B$. Thus $\mathrm{x} \Vdash_{\mathrm{Q}} A \rightarrow B$. Now assume conversely that $\mathrm{x} \Vdash_{\mathrm{Q}} A \rightarrow B$. Define $\Box^0 A = A$; $\Box^{k+1} A = \Box^k A \rightarrow \Box^k A \rightarrow \Box^k A$. Let m be the least k such that $\Box^k A$ is not in x. Now $\{\Box^m A\} \Vdash_{\mathrm{Q}} A$ by DR4 and induction hypothesis; hence, by assumption, $\mathrm{x} \cup \{\Box^m A\} \Vdash_{\mathrm{Q}} B$. By induction hypothesis, $\overline{\mathrm{x} \cup \{\Box^m A\}} \vdash B$ for some ordering; since $\Box^m A$ is not in x, $\overline{\mathrm{x} \cup \{\Box^m A\}}$ is $\langle \bar{\mathrm{y}}, \Box^m A, \bar{\mathrm{z}}\rangle$, $\mathrm{x} = \mathrm{y} \cup \mathrm{z}$. Hence $\bar{\mathrm{x}}, \Box^m A \vdash B$ by DR1, so $\bar{\mathrm{x}} \vdash A \rightarrow B$, since $A \rightarrow \Box^m A$ is a theorem of $\mathbf{R}_{\rightarrow}$.

So much for the Fact. Now, if A is c-valid, then $\mathit{0} \Vdash_{\mathrm{Q}} A$; hence, by the Fact, A is provable in $\mathbf{R}_{\rightarrow}$.

§47.2. Semantics for $\mathbf{E}_{\rightarrow}$. In the foregoing account of logical consequence a factor has been omitted which may be held to be an essential part of the theory of entailment. A primitive, logic-free consequence relation holding between pieces of information and atomic statements was postulated; this relation was assumed to hold by virtue of (a) the meanings of words in the basic statements and (b) certain presupposed background facts. Now in the above account the set of background facts is ignored, or rather is considered as fixed or invariable. However, if we take into account the idea that there may be alternative backgrounds of fact, the picture changes. For instance, "{I saw Herman Wouk} ⊩ (I saw the author of *Youngblood Hawke*)" is true given the present background facts, but would be false against a background in which, say, Youngblood Hawke was the author of *Herman Wouk*. In other words, the fundamental notion for entailment is not simply logical consequence, but logical consequence *relative* to a set of background facts; we write "$x, w_i \Vdash p$" for "the piece of information x entails p if the facts are as in possible world w_i." Given a class W of possible worlds and a semilattice of pieces of information, one further notion is required to determine the truth conditions of complex statements, namely a relation $\leq$ of relative possibility or accessibility (Kripke 1963) defined on W. With this, we are ready to state a formal semantics for $\mathbf{E}_{\rightarrow}$ (§4).

Let S be a semilattice with zero, W a nonempty set, $\leq$ a reflexive, transitive relation defined on W. A quadruple $Q = \langle S, W, \leq, v\rangle$ is an *entailment model*, or *e-model*, if v is a function that assigns to each propositional variable p a subset of $S \times W$. For purely implicational formulas the consequence relation relative to Q is defined recursively as follows, for x in S, w_i in W.

1. $x, w_i \Vdash_Q p$ iff $\langle x, w_i\rangle$ is in $v(p)$;
2. $x, w_i \Vdash_Q A \rightarrow B$ iff, for all y in S and all w_j such that $w_i \leq w_j$, either not $y, w_j \Vdash_Q A$ or $x \cup y, w_j \Vdash_Q B$.

A formula A is *true in* Q if $\mathit{0}, w_i \Vdash_Q A$ is true for all w_i in W, and *e*-valid if true in all e-models. The truth conditions for arrow statements, it may be noted, correspond to the idea that entailment is necessary relevant implication (see §4); $A \rightarrow B$ is a consequence of x relative to w_i if, relative to all accessible w_j, it is a consequence of x that A relevantly implies B.

THEOREM. A formula of $\mathbf{E}_{\rightarrow}$ is provable in $\mathbf{E}_{\rightarrow}$ if and only if it is e-valid.

PROOF. An ordered pair of which the first member is a formula of $\mathbf{E}_{\rightarrow}$ and the second a finite set of positive integers, we shall refer to as a *term*. A term may be written as a formula with a subscript, for instance, as $A \rightarrow A_{\{1,2\}}$ instead of $\langle A \rightarrow A, \{1, 2\}\rangle$. Now let W be the set of w_i satisfying the conditions:

(i) w_i is a set of terms;
(ii) The union of all subscripts occurring in any term in w_i is finite;

(iii) If A is a theorem of $\mathbf{E}_{\rightarrow}$, then $A_{\boldsymbol{0}}$ is in w_i;
(iv) If $A \rightarrow B_{\mathrm{x}}$ and A_{y} are in w_i, then $B_{\mathrm{x} \cup \mathrm{y}}$ is in w_i.

For w_i, w_j in W, let $\mathrm{w}_i \leq \mathrm{w}_j$ hold if, for all A, B, x, if $A \rightarrow B_{\mathrm{x}}$ is in w_i, then $A \rightarrow B_{\mathrm{x}}$ is in w_j. For X a finite set of terms, let a *proof of a term* A_{y} *from* X be defined as a sequence of terms such that each term in the sequence either is in X or is $C_{\boldsymbol{0}}$, where C is a theorem of $\mathrm{E}_{\rightarrow}$, or is derived from preceding terms by $\rightarrow$E (with union of subscripts as in §3) and such that the last term is A_{y}. Now, for w_i in W, let w'_i be the set of terms $A \rightarrow B_{\mathrm{x}}$ in w_i; let k be a number greater than any occurring in any subscripts in w_i. Define $\mathrm{P}(\mathrm{w}_i, A_{\{k\}})$ to be the set of all B_{y} such that there is a proof of B_{y} from $\mathrm{w}'_i \cup \{A_{\{k\}}\}$.

LEMMA. If $B_{\mathrm{x} \cup \{k\}}$ is in $\mathrm{P}(\mathrm{w}_i, A_{\{k\}})$, then $A \rightarrow B_{\mathrm{x}}$ is in w_i.

The proof of this lemma follows exactly the proof of the deduction theorem for $\mathbf{E}_{\rightarrow}$ in §4. Now, noting that $\mathrm{P}(\mathrm{w}_i, A_{\{k\}})$ is in W, we have as a corollary that, for any w_i, $A \rightarrow B_{\mathrm{x}}$ is in w_i if and only if, for all w_j such that $\mathrm{w}_i \leq \mathrm{w}_j$, if A_{y} is in w_j then $B_{\mathrm{x} \cup \mathrm{y}}$ is in w_j.

Let S be the set of all finite sets of positive integers, including the empty set; this is a semilattice under set union. For x in S, w_i in W, let $\langle \mathrm{x}, \mathrm{w}_i \rangle$ be in $\mathrm{v}(p)$ if p_{x} is in w_i. The quadruple $\mathrm{Q} = \langle \mathrm{S}, \mathrm{W}, \leq, \mathrm{v} \rangle$ is easily seen to be an e-model. What remains to be shown is that, for all x in S, w_i in W, we have $\mathrm{x}, \mathrm{w}_i \Vdash_{\mathrm{Q}} A$ iff A_{x} is in w_i, for all A; this may be proved by induction, using the corollary to the lemma. Now the set $\mathrm{w}_{\boldsymbol{0}}$ of terms $A_{\boldsymbol{0}}$ such that A is a theorem of $\mathbf{E}_{\rightarrow}$ is in W. Hence if a formula A is e-valid, $\boldsymbol{0}, \mathrm{w}_{\boldsymbol{0}} \Vdash_{\mathrm{Q}} A$, so $A_{\boldsymbol{0}}$ is in $\mathrm{w}_{\boldsymbol{0}}$; hence A is a theorem of $\mathbf{E}_{\rightarrow}$. This concludes the completeness proof.

§47.3. Semantics for $\mathbf{T}_{\rightarrow}$. The semantic analysis of $\mathbf{T}_{\rightarrow}$ (see §6) proceeds along slightly different lines. In the case of $\mathbf{E}_{\rightarrow}$, a set of "possible worlds" is added to the semilattice of pieces of information, together with a binary relation. In the case of $\mathbf{T}_{\rightarrow}$, however, we add a binary relation to the semilattice itself. In a c-model, the consequences of a given piece of information are dependent on all pieces in the model; $A \rightarrow B$ is a consequence of x if *all* pieces of information y in the model satisfy the appropriate condition. However, adhering to the ticket/fact distinction of §6, we might postulate that $\mathrm{x} \Vdash A \rightarrow B$ is to hold when the appropriate condition is satisfied by all pieces of information y *which give at least as much information as* x. We write $\mathrm{x} \leq \mathrm{y}$ for "y contains the same or more information than x." Note that this relation is not necessarily to be identified with set-theoretical containment; that is, we do *not* postulate that $\mathrm{x} \leq \mathrm{y}$ holds if and only if $\mathrm{x} \cup \mathrm{y} = \mathrm{y}$. As a counterexample, consider x = {John has a wife} and y = {John's wife is sick}. It seems evident that $\mathrm{x} \leq \mathrm{y}$ holds, while on the other hand x is not contained in y. The relation $\leq$ must nevertheless satisfy several postulates characteristic of containment; for instance, $\mathrm{x} \leq \mathrm{x} \cup \mathrm{y}$ must be true for all x and y.

A triple $Q = \langle S, \leq, v\rangle$ is defined to be a *t-model* if: S is a semilattice with zero; $\leq$ is a transitive binary relation defined on S such that, for any x, y in S, $\mathit{0} \leq x$, and if $x \leq y$ then $x \cup z \leq y \cup z$ for any z; and v is a valuation on S which assigns to each propositional variable *p* a subset of S, v(*p*). The consequence relation relative to Q is defined as follows:

1. $x \Vdash_Q p$ iff x is in v(*p*),
2. $x \Vdash_Q A \rightarrow B$ iff for all y such that $x \leq y$ either not $y \Vdash_Q A$ or $x \cup y \Vdash_Q B$.

A formula *A* is *true* in Q if $\mathit{0} \Vdash_Q A$, and *t-valid* if true in all t-models.

The completeness proof for $\mathbf{T}_{\rightarrow}$, like the completeness proof for $\mathbf{E}_{\rightarrow}$, is essentially an adaptation of the methods used in proving the equivalence of the axiomatic and subproof formulations. Let S be the set of all finite sets of terms. For x in S, let s(x) be the union of all subscripts occurring in x. A *proof of A from* x is defined to be a sequence of terms with last term $A_{s(x)}$, such that each term B_y in the sequence either is in x or is B_{0}, where *B* is a theorem of $\mathbf{T}_{\rightarrow}$, or is inferred from preceding terms by →E, the ticket restriction being satisfied. The relation $x \vdash A$ is to hold if there is a proof of *A* from x. For x, y in S, let $x \leq y$ hold if max(s(x)) is less than or equal to max(s(y)), where max(x) is the greatest member of x if $x \neq \mathit{0}$, and max($\mathit{0}$) is zero.

LEMMA 1. Suppose $x \vdash A \rightarrow B$, $y \vdash A$, and $x \leq y$. Then $x \cup y \vdash B$.

By hypothesis, there are sequences α and β which are, respectively, proofs of $A \rightarrow B$ from x and of *A* from y. It is easily seen that the sequence $\langle \alpha, \beta, B_{s(x) \cup s(y)}\rangle$ is a proof of *B* from $x \cup y$; so $x \cup y \vdash B$.

LEMMA 2. If $x \cup \{A_{\{k\}}\} \vdash B$, where *k* is greater than any number in s(x), then $x \vdash A \rightarrow B$.

The proof of this lemma is a straightforward adaptation of the method used in §6 to eliminate the innermost subproof of a proof in $\mathrm{FT}_{\rightarrow}$.

Now consider $Q = \langle S, \leq, v\rangle$ where S is the semilattice of all finite sets of terms, $\leq$ is as defined above, and x is in v(*p*) if and only if $x \vdash p$. Q is easily seen to be a t-model. It remains to be shown that $x \Vdash_Q A$ iff $x \vdash A$ for every formula *A*; it is left to the reader to show this by an induction on the complexity of *A*, using Lemmas 1 and 2. Now if a formula *A* is t-valid, then $\mathit{0} \Vdash_Q A$; hence $\mathit{0} \vdash A$. So *A* is a theorem of $\mathbf{T}_{\rightarrow}$. Since every theorem of $\mathbf{T}_{\rightarrow}$ is t-valid, we have as a

THEOREM. A formula of $\mathbf{T}_{\rightarrow}$ is provable in $\mathbf{T}_{\rightarrow}$ if and only if it is t-valid.

§47.4. Variations on a theme. Before proceeding to the problems raised by the addition of connectives other than the arrow, we shall discuss a few of the many possible variations and extensions of the semantic analyses.

The concept of a c-model may be modified by either strengthening or weakening the requirements. A family of sublogics of $\mathbf{R}_{\rightarrow}$ is generated by considering models $\langle \mathrm{S}, \mathrm{v}\rangle$ in which S is closed under an operation $\cup$ which may satisfy some but not all of the semilattice conditions, retaining the definition of consequence in a model for implicational formulas. Thus we may define a *(c – w)-model* as a pair $\langle \mathrm{S}, \mathrm{v}\rangle$ in which S is closed under an associative, commutative operation $\cup$, with $\mathrm{x}\cup \mathit{0} = \mathrm{x} = \mathit{0}\cup\mathrm{x}$ (a commutative monoid—see §28.2.1), and v is a valuation on S. The set of formulas true in all (c – w)-models is axiomatized by simply omitting the contraction schema $(A\rightarrow.A\rightarrow B)\rightarrow.A\rightarrow B$ from $\mathbf{R}_{\rightarrow 1}$ (§8.3.4); this system is discussed in Meredith and Prior 1963 with the name **BCI**.

Still weaker is the logic that arises from dropping the requirement of commutativity of $\cup$ from the definition of a (c – w)-model; let us call a structure $\langle \mathrm{S}, \mathrm{v}\rangle$ in which $(\mathrm{x}\cup\mathrm{y})\cup\mathrm{z} = \mathrm{x}\cup(\mathrm{y}\cup\mathrm{z})$, $\mathrm{x}\cup\mathit{0} = \mathit{0}\cup\mathrm{x} = \mathrm{x}$ for x, y in S (a monoid; §28.2.1), an *m-model*. The set of formulas true in all m-models is axiomatized by the schemata:

$\mathbf{M}_{\rightarrow}1$ $A\rightarrow A$,
$\mathbf{M}_{\rightarrow}2$ $A\rightarrow B\rightarrow.C\rightarrow A\rightarrow.C\rightarrow B$,

together with *modus ponens* and the additional rule of inference: *from A to infer $A\rightarrow B\rightarrow B$.* This system, $\mathbf{M}_{\rightarrow}$, appears more natural and interesting if reformulated as a consecution calculus (§7). In this formulation, the axioms all have the form $A\vdash A$; there are two rules:

$$(\rightarrow\vdash)\quad \frac{\beta\vdash A \qquad \alpha, B, \gamma\vdash C}{\alpha, A\rightarrow B, \beta, \gamma\vdash C}$$

$$(\vdash\rightarrow)\quad \frac{\alpha, A\vdash B}{\alpha\vdash A\rightarrow B}$$

Note that there are no structural rules whatever. A proof of the Elimination theorem for the consecution formulation, which is easily given, allows us to prove equivalence of the two formulations. $\mathbf{M}_{\rightarrow}$ seems another natural candidate for the role of minimal logic in Church's sense (see §8.11)—though the concept of minimality does not appear to be definite enough to allow of a decision.

It is possible to go still further in weakening the semilattice requirements. The weakest possible requirement we can make is to demand only that $\mathit{0}\cup\mathrm{x} = \mathrm{x}$ (even $\mathrm{x}\cup\mathit{0} = \mathrm{x}$ may not necessarily hold). A model $\langle \mathrm{S}, \mathrm{v}\rangle$ in which S satisfies this condition we shall call an *i-model*. The set of formulas true in all i-models is axiomatized by the schema $A\rightarrow A$ with no rules of inference.

As the completeness argument is short and neat, we sketch it here. Let S be the set of all sets of formulas, and, for x and y in S, let $x \cup y$ be defined as the set of all formulas B such that, for some A, $A \rightarrow B$ is in x and A is in y; let $\mathbf{0}$ be the set of formulas having the form $A \rightarrow A$. Now if a formula A is in $\mathbf{0} \cup x$ then, for some B, $B \rightarrow A$ is in $\mathbf{0}$, B is in x; but B must be identical with A, so A is in x. Conversely, if A is in x, $A \rightarrow A$ is in $\mathbf{0}$; hence A is in $\mathbf{0} \cup x$. It follows that $\mathbf{0} \cup x = x$; so S together with the defined operation satisfies the required condition. Let x be in $v(p)$ if and only if p is in x. It is left to the reader to supply the simple proof that $x \Vdash_Q A$ iff A is in x, where $Q = \langle S, v \rangle$. From this it follows that, if A is i-valid, A is in $\mathbf{0}$ and so has the form $B \rightarrow B$.

We may strengthen the requirements on a c-model by adding conditions to the valuation function v. If we require that $x \cup y$ be in $v(p)$ if x is in $v(p)$, then $A \rightarrow .B \rightarrow A$ is valid. The set of valid formulas then coincides with the set of theorems of $\mathbf{H}_{\rightarrow}$ (§1). The weaker requirement that $x \cup y$ be in $v(p)$ if both x and y are in $v(p)$ validates $A \rightarrow .A \rightarrow A$; $\mathbf{RM0}_{\rightarrow}$ (§8.15) is complete with respect to this class of models.

The ideas introduced in the semantic analysis of $\mathbf{E}_{\rightarrow}$ allow of even greater variation. The two components of an e-model, the semilattice and the possible-world structure, can be tinkered with independently. Thus if we reduce the set W to a single world we obtain the class of c-models, or, rather, structures semantically interchangeable with c-models; if we reduce S to a single piece of information, there results a semantics with respect to which $\mathbf{S4}_{\rightarrow}$ (§2) is complete (Kripke 1963, Hacking 1963). More interestingly, we can vary the requirements on the accessibility relation. If we require $\leq$ merely to be reflexive, then a pure implicational logic results which is akin to the system **G** of §29.1.1. We may equally make the requirements more stringent; let us call an e-model in which the accessibility relation is symmetric an *e5-model*, to remind us of **S5**. The schema $A \rightarrow .(A \rightarrow .B \rightarrow C) \rightarrow .B \rightarrow C$ is valid in all e5-models, though refutable in an e-model. Fine 1976a shows that the addition of this schema to $\mathbf{E}_{\rightarrow}$ axiomatizes the e5-valid formulas.

The semantics of $\mathbf{T}_{\rightarrow}$ allows of similar variations to those sketched for the case of c-models. For example, a *(t−w)-model* may be defined to be exactly like a t-model save that $x \cup x = x$ is not postulated. The set of (t−w)-valid formulas coincides with the set of theorems of $\mathbf{T}_{\rightarrow}$−W, $\mathbf{T}_{\rightarrow}$ minus contraction (§8.11).

To sum up, well-motivated, natural, and elegant analyses of many purely implicational intensional logics fit into the present semantical framework. We have still to consider the problem of adding other connectives. Conjunction presents no problems; in a c-model or a t-model we extend the consequence relation by defining

$$x \Vdash_Q A \& B \text{ iff } x \Vdash_Q A \text{ and } x \Vdash_Q B,$$

and, in an e-model,

$$\mathrm{x}, \mathrm{w}_i \Vdash_Q A\&B \text{ iff } \mathrm{x}, \mathrm{w}_i \Vdash_Q A \text{ and } \mathrm{x}, \mathrm{w}_i \Vdash_Q B.$$

With this extension, the completeness proofs already given go through easily with respect to the implication and conjunction fragments of the appropriate logics (§27.1.1). Before we leave the topic of conjunction it might be mentioned that "relevant consistency" or "intensional conjunction" (§27.1.4) or "fusion" (§30.4) allows of a natural treatment in the present framework. We define, relative to a c-model:

$$\mathrm{x} \Vdash_Q A \circ B \text{ iff, for some y, z, x} = \mathrm{y} \cup \mathrm{z} \text{ and both y} \Vdash_Q A \text{ and } z \Vdash_Q B.$$

This definition appears to give the right properties to the connective. It is interesting to notice the meaning of the definition: $A \circ B$ follows from x just in case A and B follow, not necessarily each from the whole of x but from jointly exhaustive parts of x.

Conjunction, as we noted, poses no problems, which makes it appear at first sight as if disjunction is equally unproblematic. We need simply to define in a c-model or t-model

$$\mathrm{x} \Vdash_Q A \vee B \text{ iff x} \Vdash_Q A \text{ or x} \Vdash_Q B$$

and, in an e-model,

$$\mathrm{x}, \mathrm{w}_i \Vdash_Q A \vee B \text{ iff } \mathrm{x}, \mathrm{w}_i \Vdash_Q A \text{ or } \mathrm{x}, \mathrm{w}_i \Vdash_Q B.$$

These definitions, along with those preceding, indeed validate all theorems following from the negation-free schemata of **R**, **E**, and **T**. The negation-free fragments of these logics, however, are incomplete with respect to the appropriate account of validity, as is shown by a counterexample due to the joint effort of Dunn and Meyer. The schema

$$(A \rightarrow A)\&(A\&B \rightarrow C)\&(A \rightarrow .B \vee C) \rightarrow .\ A \rightarrow C$$

is valid under all three accounts, but it is not provable in **R**, as can be seen from the matrices of §22.1.3, which satisfy all the axioms and rules of inference of **R**. If we give A the value $+3$, B the value $+0$, and C the value -0, then the schema takes the value -3. A slightly simpler schema,

(D) $(A \rightarrow .B \vee C)\&(B \rightarrow D) \rightarrow .\ A \rightarrow .D \vee C$,

is also valid under all three accounts, and may be falsified in the same matrices by giving A, B, and C the same values as before, and D the value -0. The first counterexample is deducible from the scheme (D) in the context of **T**, with the help of the distribution axiom. (See §27.1.1 for mention of this formula. It answers to the rule $\vee E^s$ of §27.2.)

The semilattice semantics with disjunction as above has been investigated by Fine 1976 and Charlwood 1978, 1981. The latter offers two natural deduction systems, one with subscripts and one without; this last is in fact the (positive) system of Prawitz 1965, which Prawitz wrongly conjectured to be the same as $\mathbf{R}_+$. Charlwood proves normalization (as did Prawitz for his system—incidentally the problem of normalization for $\mathbf{R}_+$ itself seems still open). Charlwood also carries out in detail the engineering needed to implement the Fine 1976 axiomatization of these semantics; that is, it is shown that what is wanted is to add the following rule to $\mathbf{R}_+$ as formulated in §R2:

First premiss:

$$(A\&(B_1\&p_1\rightarrow(B_2\&p_2\rightarrow(\ldots\rightarrow(B_n\&p_n\rightarrow D_1)\ldots))))\rightarrow.\;(C_1\&p_1\rightarrow(C_2\&p_2\rightarrow(\ldots\rightarrow(C_n\&p_n\rightarrow E)\ldots)))$$

Second premiss: the above with D_2 for D_1

Conclusion:

$$(A\&(B_1\rightarrow(B_2\rightarrow(\ldots\rightarrow(B_n\rightarrow(D_1\vee D_2))\ldots))))\rightarrow.\;(C_1\rightarrow(C_2\rightarrow(\ldots\rightarrow(C_n\rightarrow E)\ldots)))$$

Proviso: the p_i are all distinct propositional variables, and they occur only where indicated.

The rule is of course not pretty, but it does solve the problem of providing an axiomatization of the semilattice semantics for implication, conjunction, and disjunction.

Still, we must recall that the semilattice semantics had as its original target the system $\mathbf{R}_+$, not some other system. We must therefore record that this particular semantical analysis breaks down in a surprising fashion in the presence of disjunction, and this failure seems irreparable. There appears to be no plausible substitute for the obvious evaluation rule for disjunction. It follows that the evaluation rules for implication, though completely successful where implication alone is concerned, must be altered if the full systems of intensional logic are to be treated.

The failure becomes still more evident if we consider what semantic rules can be introduced to deal with negation. The "obvious" rule

$$\mathrm{x} \Vdash_{\mathrm{Q}} {\sim}A \text{ iff it is not the case that } \mathrm{x} \Vdash_{\mathrm{Q}} A$$

is of course no good here, for it validates $(A\&{\sim}A)\rightarrow B$ and other implicational paradoxes. The reason for this is that the "classical" negation rule given above automatically excludes inconsistent pieces of information, i.e., pieces of information x such that $\mathrm{x} \Vdash A$ and $\mathrm{x} \Vdash {\sim}A$. However, as we argued informally above, a piece of information, in contrast to a "possible world" or an "evidential situation," cannot in general be expected to be consistent. On such grounds it is *natural* to expect $(A\&{\sim}A)\rightarrow B$ and disjunctive syllogism to be invalid in semantics founded on the idea of a "piece of information." Philosophers and logicians who have argued that these principles must be included in any coherent system of entailment have evidently had one of the less general concepts in mind—hence their mistaken conclusions.

Negation in relevance logics, however, clearly has many of the features of classical negation. The problem is to preserve these features while invalidating the paradoxes. One plausible way to do this (see §48.2 and §48.5) is to add to the semilattice S a function * under which S is closed, such that $\boldsymbol{0}^* = \boldsymbol{0}$ and such that, for all x in S, $x^{**} = x$; and then define

$x \Vdash_Q {\sim}A$ iff it is not the case that $x^* \Vdash_Q A$,
$x, w_i \Vdash_Q A$ iff it is not the case that $x^*, w_i \Vdash_Q A$.

This definition has many of the right features. Exactly the right zero and first degree entailments are validated; none of the paradoxes are valid. This last may be shown by exhibiting an *extended c-model*, or *c*-model*.

Let $S = \{\boldsymbol{0}, \boldsymbol{a}, \boldsymbol{b}\}$, and let $\cup$ and * be defined by the tables:

$\cup$	$\boldsymbol{0}$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{0}$	$\boldsymbol{0}$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{a}$	$\boldsymbol{a}$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{b}$	$\boldsymbol{b}$	$\boldsymbol{b}$	$\boldsymbol{b}$

*	
$\boldsymbol{0}$	$\boldsymbol{0}$
$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{b}$	$\boldsymbol{a}$

Let $v(p) = \{\boldsymbol{a}\}$, $v(q) = \{\boldsymbol{b}\}$. Then $(p \& \bar{p}) \rightarrow q$, $p \rightarrow (q \vee \bar{q})$, $(\bar{p} \& (p \vee q)) \rightarrow q$ are all falsified in the model; note that $\boldsymbol{a}$ is an inconsistent piece of information.

When we go beyond the first degree fragment, however, the picture changes. Both contraposition and $A \rightarrow \bar{A} \rightarrow \bar{A}$ are invalid in the semantics. Let $S = \{\boldsymbol{0}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, let $\cup$ and * be defined by the tables below:

$\cup$	$\boldsymbol{0}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{0}$	$\boldsymbol{0}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{a}$	$\boldsymbol{a}$	$\boldsymbol{a}$	$\boldsymbol{c}$	$\boldsymbol{c}$
$\boldsymbol{b}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{c}$	$\boldsymbol{c}$	$\boldsymbol{c}$	$\boldsymbol{c}$	$\boldsymbol{c}$

*	
$\boldsymbol{0}$	$\boldsymbol{0}$
$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{b}$	$\boldsymbol{a}$
$\boldsymbol{c}$	$\boldsymbol{c}$

and let $v(p) = \{\boldsymbol{b}\}$, $v(q) = \{\boldsymbol{c}\}$. This c*-model falsifies both $q \rightarrow \bar{p} \rightarrow . \; p \rightarrow \bar{q}$ and $p \rightarrow \bar{p} \rightarrow \bar{p}$.

It is possible of course to introduce a variety of negation for which these last two principles are assured. If we add a constant f to each of the systems and define ${\sim}A$ as $A \rightarrow f$, then they are automatically valid. However, in this case the zero and first degree entailments do not fit the desired pattern. $A \vee \bar{A}$,

$\bar{\bar{A}}\to A$, and $(\overline{A\&B})\to(\bar{A}\vee\bar{B})$ are invalid, as are all schemata that are not intuitionistically valid.

The lack of an appropriate construction for negation in the present semantical framework can be stated in a rather strong form. We shall deal only with the case of c-models, but the argument given below applies to all three categories of model. A model ⟨S, U, v⟩ is an *expansion* of a c-model if ⟨S, v⟩ is a c-model and U is a set of relations and functions defined on S. Now let us suppose that the concept of c-validity has been extended in the following sense: a class of models—call them *cn-models*—has been defined, each cn-model being an expansion of a c-model; and the concept of consequence has been extended to include negation, the consequence definition remaining unchanged for the positive connectives. Further, let it be the case that each c-model has an expansion that is a cn-model. Then not all theorems of **R** are cn-valid. The reason is that if the concept of c-validity is extended as above, then every cn-valid negation-free formula is also c-valid. However, if all theorems of **R** are cn-valid, then, since the schema (D) must be cn-valid, it follows by some simple manipulations that the schema (see §27.1.1)

$$(A\&B\to C)\&(D\to B)\to.\ A\&D\to C$$

must be cn-valid. This last schema, though, is not c-valid, which contradicts the supposition that all theorems of **R** are cn-valid.

A related result is that, if we define Rabc as $a\cup b = c$ and subject * to the same constraints (Period two, Inversion) of §48.5 below, it is easy to prove (and left to the reader) that all pieces of information in the model are identical, and so we get classical logic.

Returning to the informal motivation underlying the semantics, it can be seen that failures with respect to negation are only to be expected. For instance, is it plausible to suppose that the schema $A\vee\bar{A}$ should be valid? To suppose so is to posit that for every statement A either $\mathbf{0}\Vdash A$ or $\mathbf{0}\Vdash\bar{A}$—but this seems quite implausible on the informal interpretation. With no information about, say, Milton Zysman, I can neither assert that Zysman is fat nor that he is thin. In other words, it seems obvious that both

$\mathbf{0}\Vdash$ Milton Zysman is fat

and

$\mathbf{0}\Vdash$ $\sim$(Milton Zysman is fat)

are false; so we would expect the law of excluded middle to be invalid. The same type of remark applies to the law of double negation and to other intuitionistically invalid formulas. The style of negation that seems consonant with the ideas underlying the semantics is *constructive*, resembling the second type of negation discussed. That is, it appears that, to follow through the philosophical ideas concerning "pieces of information," we should introduce

$\bar{A}$ as $A \rightarrow f$, where f is a propositional constant about which no further assumptions are made.

Of course, in following this line of thought we have strayed far from the systems of relevance logic which were the original objects of investigation. The systems defined by the model theory appear, however, well motivated and worthy of investigation.

PROBLEM. Axiomatize these systems.

This problem and a wide variety of related questions provide an intriguing and challenging field of research.

§48. Relational semantics for relevance logics.

The principal aim of this section is to present a brief view of the Routley-Meyer three-termed relational semantics for the chief relevance logics—semantics set out in detail in various Routley-Meyer publications as listed in the Bibliography and especially in their book: Routley, with Plumwood, Meyer, and Brady 1982. We begin by setting the matter in context, reaching the relational semantics itself only in §48.3.

§48.1. Algebraic vs. set-theoretical semantics.

In the "open problems" paper, Anderson 1963, the last major question listed, almost as if an afterthought, was the question of the semantics of **E** and $\mathbf{E}^{\forall\exists x}$. Despite this appearance, on page 16 we find that "the writer does not regard this question as 'minor'; it is rather the principal large question remaining open." Cited approvingly was earlier work (described here in §§18, 19, and 40) on providing an algebraic semantics for first degree entailments, but it was noted that the general problem of finding a semantics for the whole of **E**, with an appropriate completeness theorem, remained unsolved.

It is interesting to observe that Anderson 1963 appeared in the same *Acta filosophica fennica* volume as the now classic paper of Kripke 1963, which provided what is now simply called "Kripke-style" semantics for a variety of modal logics (Kripke 1959 of course provided a semantics for **S5**, but it lacked the accessibility relation R which is so versatile in providing variations).

Of course ARA knew of this work long before 1963, since he was one of those who corresponded with Kripke in the mid-fifties while the latter was working out his ideas. When ARA was writing his "open problems" paper, however, the dominant paradigm of a semantical analysis of a nonclassical logic was probably still something like the work of McKinsey and Tarski 1948, which provided interpretations for modal logic and intuitionistic logic by way of certain algebraic structures analogous to the Boolean algebras that are the appropriate structures for classical logic. But since then the Kripke-style semantics (sometimes referred to as "possible-worlds semantics" or "set-theoretical semantics") seems to have become the paradigm. We rightly call the paradigm "Kripke-style" since it was his elegant work, first published in Kripke 1959, that had the effect of creating a surge of interest in modal logic, although it was not without precursors in Meredith 1958 (or 1956), Kanger 1957, and Bayart 1958; and certainly the independent project first reported in Hintikka 1961 has had a heavy influence on subsequent research. Words apart, however, what we are indicating is that **E** and **R** now have both an algebraic semantics and a Kripke-style semantics. We shall first distinguish in a kind of general way the differences between these two main approaches to semantics, before going on to explain the particular details of the semantics for relevance logics (again **R** will be our paradigm).

It is convenient to think of a logical system as having two distinct aspects: syntax (well-formed strings of symbols, e.g., sentences) and semantics (what, e.g., these sentences mean, i.e., propositions). These two aspects compete with each other, as can be seen in the competing usages "sentential calculus" and "propositional calculus," but we should keep both aspects firmly in mind.

Since sentences can be combined by way of connectives, say the conjunction sign &, to form further sentences, typically there is for each logical system at least one natural algebra arising at the level of syntax, the algebra of sentences. (If one has a natural logical equivalence relation, there is yet another that one obtains by identifying logically equivalent sentences together into equivalence classes—the so-called "Lindenbaum algebra.") And since propositions can be combined by the corresponding logical operations, say conjunction, to form propositions, there is an analogous algebra of propositions.

Now undoubtedly some readers, who were taught to "Quine" propositions from an early age, will have troubles with the above story. The same readers would most likely find uncompelling any particular metaphysical account we might give of numbers. We ask those readers then at least to suspend *dis*belief in propositions so that we can get on with the mathematics.

We shall not pause to survey algebraic semantics for relevance logics, since we have devoted other parts of this book to just those topics, especially §18, which provides extended motivation for algebraic considerations in general as well as some details for first degree entailments; §19, which extends the same sort of treatment to first degree formulas of relevance logic; §40, which pursues the same sort of goals for first degree formulas with quantifiers, and §28.2, which treats the algebra of all of **R**. (There are numerous other places where algebraic considerations are invoked; §25.2 is one example among many.) Many of the structures we use are summarized in §28.2.1. We wish only to call to mind the following. (1) Intensional lattices as defined in §18.2 turned out to be the right algebraic family for *first degree formulas* (formulas without nesting of arrows), as demonstrated in §19. We showed in §18.8 that these lattices also correspond to *first degree entailments* (entailments between truth functions), but we also mentioned in passing that we could instead have relied upon *De Morgan lattices* for first degree entailments (but not for first degree formulas). We shall be thinking of De Morgan lattices as structures $\langle L, \vee, \wedge, \sim \rangle$, with $\vee$ and $\wedge$ as (distributive) lattice operations on L, and $\sim$ satisfying De Morgan properties, including double negation (§28.2.1). (2) Among De Morgan lattices, the four-element one of §15.3 (picture in §24.4.1, called "SL" in §34.1 and "**L4**" in §81.1.1) plays the same role that the two-element Boolean algebra plays among Boolean algebras generally; here we call it "**L4**" and use the same labeling as in §81.1.1. And (3) we know from §28.2 that De Morgan monoids are the right algebraic structure for **R**.

There is an alternative approach to semantics which can be described by saying that, rather than taking propositions as primitive, it "constructs" them out of certain other semantical primitives. Thus there is, as a paradigm of this approach, the so-called "U.C.L.A. proposition" as a set of "possible worlds." (Actually the germ of this idea was already in Boole—see Dipert 1978—although apparently Boole thought of it as an analogy rather than as a reduction.)

We here want to stress the general structural idea, not placing any emphasis on the particular choice of "possible world" as the semantic primitive. One reason is the following. In the more sophisticated applications of the apparatus, one wants to quantify over whatever-it-is that the semantic primitive refers to. There are three related points that we wish to make about such quantifications. The first is that philosophers can be taught to understand the idea of a "possible world" as a value of a variable and to understand the explication of necessity as truth in all possible worlds and to understand possibility as truth in some possible worlds (or relatively possible worlds). The second point is that there is nothing *idiomatic* about such an explication—the teaching is required just because the phrase "possible world" is not an everyday idiom. The third point is that, even though there is nothing idiomatic about the connection between the necessity-possibility modalities and the "possible worlds" quantifications, there are *other* modality-quantification pairs whose connection *is* firmly rooted in idiom. For example, most of us would have little choice between members of the following lists:

> It is inevitable that X will win tomorrow; in any *case* X will win tomorrow; in any *circumstances* X will win tomorrow.
>
> It is possible that X will win tomorrow; there is at least one *case* (or *circumstance* or *situation*) in which X will win: he cheats.

What we are claiming is that at least occasionally we are idiomatically comfortable with a trade-off between modals and quantifications; and there is a natural and easy passage from these idiomatic modality-quantification connections to the nonidiomatic connection between the necessity-possibility modalities and quantification over possible worlds. This indirect ground in idiom is, we think, a substantial part of the persuasiveness of the Kripke explications of necessity and possibility in terms of possible worlds, and a seldom-mentioned reason that the Kripke paradigm has "caught on."

A second sort of reason is expressed at the beginning of §47: "possible world" suggests both consistency and completeness (how could a possible world be impossible, or be less than a fully determined world?), whereas whatever-it-is that we shall be thinking about will not in general have these features. It would be acceptable for us on these grounds to invoke one of the idiomatic words "case," "circumstance," or "situation," since one can imagine "the case when it rains" without deciding whether in "that case" the wind blows from the east or not and since one can imagine what are at bottom impossible situations (the circle has just been squared!); but since in the proceedings of this section our aim is to concentrate on the mathematics rather than on the philosophical underpinnings, we prefer to join the Routleys (1972) in using a more neutral term: *set-up*.

Such "set-theoretical" semantical accounts do not always explicitly exhibit such a construction of propositions. Indeed perhaps the more common approach is to provide an interpretation that says whether a formula A is true or false at a given set-up S, writing $\varphi(A, \mathrm{S}) = T$ or $\mathrm{S} \vDash A$ or some such thing. Think of Kripke's 1963 presentation of his semantics for modal logic. But (unless one has severe ontological scruples about sets) one might just as well interpret A by assigning it a class of set-ups, writing $\Phi(A)$ or $|A|$ or some such thing. One can go from one framework to the other by way of the equivalence

$$\mathrm{S} \in |A| \text{ iff } \mathrm{S} \vDash A.$$

§48.2. Set-theoretical semantics for first degree relevant implications. Dunn 1966 (see also Dunn 1967) considered a variety of (effectively equivalent) representations of De Morgan lattices as structures of sets. We shall here discuss the two of these which have been the most influential in the development of set-theoretical semantics for relevance logic.

The earliest one of these is due to Białynicki-Birula and Rasiowa 1957 and goes as follows. Let U be a nonempty set, and let g be a function on U of period two, i.e.,

$$\mathrm{g}(\mathrm{g}(\mathrm{x})) = \mathrm{x}, \text{ for all } \mathrm{x} \in \mathrm{U}.$$

(We shall call the pair (U, g) an *involuted set*—g is the *involution*, and is clearly one-one.) Let Q(U) be a "ring" of subsets of U (closed under $\cap$ and $\cup$) closed as well as under the operation of "quasi-complement":

$$\sim\mathrm{X} = \mathrm{U} - \mathrm{g}[\mathrm{X}] \qquad (\mathrm{X} \subseteq \mathrm{U}).$$

$(\mathrm{Q}(\mathrm{U}), \cup, \cap, \sim)$ is called a *quasi-field of sets* and is a De Morgan lattice.

QUASI-FIELDS OF SETS THEOREM (Bialynicki-Birula and Rasiowa 1957). Every De Morgan lattice is isomorphic to a quasi-field of sets.

PROOF. Let U be the set of all prime filters (§18.1) of a De Morgan lattice $\langle L, \vee, \wedge, \sim\rangle$, and let P range over U. Let $\sim P = \{\sim a\colon a \in P\}$, and define $g(P) = U-(\sim P)$. We leave it to the reader to verify that U is closed under g. For each element $a \in L$ set

$$f(a) = \{P\colon a \in P\}.$$

Clearly f is one-one because of Stone's prime filter theorem (§18.1, or Fact 2 of §25.3.3), so we need only check that f preserves the operations.

ad $\wedge$: $P \in f(a \wedge b)$ iff $a \wedge b \in P$ iff (filterhood) $a \in P$ and $b \in P$ iff $P \in f(a)$ and $P \in f(b)$ iff $P \in f(a) \cap f(b)$. So $f(a \wedge b) = f(a) \cap f(b)$ as desired.

ad $\vee$: The argument that $f(a \vee b) = f(a) \cup f(b)$ is exactly parallel, using primeness (or, alternatively, this can be skipped using the fact that $a \vee b = \sim(\sim a \wedge \sim b)$).

ad $\sim$: $P \in f(\sim a)$ iff $\sim a \in P$ iff $a \in \sim P$ iff $a \notin g(P)$ iff $g(P) \notin f(a)$ iff $P \notin g[f(a)]$ iff $P \in U - g[f(a)]$.

We shall now discuss a second representation. Let U be a nonempty set, and let R be a ring of subsets of U (closed under intersection and union, but not necessarily under complement, quasi-complement, etc.). By a *polarity* in R we mean an ordered pair $X = (X_1, X_2)$ such that $X_1, X_2 \in R$. We define a relation and operations as follows, given polarities $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$:

$$X \leq Y \text{ iff } Y_1 \subseteq X_1 \text{ and } X_2 \subseteq Y_2$$
$$X \wedge Y = (X_1 \cap Y_1, X_2 \cup Y_2)$$
$$X \vee Y = (X_1 \cup Y_1, X_2 \cap Y_2)$$
$$\sim X = (X_2, X_1)$$

By a *field of polarities* we mean a structure $(P(R), \leq, \wedge, \vee, \sim)$, where P(R) is the set of all polarities in some ring of sets R, and the other components are defined as above. We leave to the reader the easy verification that every field of polarities is a De Morgan lattice.

We shall prove the following

POLARITIES THEOREM (Dunn 1966). Every De Morgan lattice is isomorphic to a field of polarities.

PROOF. Given the previous representation, it clearly suffices to show that every quasi-field of sets is isomorphic to a field of polarities.

The idea is to set $f(X) = (X, U-g[X])$. Clearly f is one-one. We check that it preserves operations.

$$\begin{array}{ll}
\textit{ad}\ \wedge: & f(X\cap Y) = (X\cap Y, U-g[X\cap Y]) \\
 & \quad = (X\cap Y, (U-g[X])\cup(U-g[Y])) \\
 & \quad = (X, U-g[X])\wedge(Y, U-g[Y]) \\
 & \quad = f(X)\wedge f(Y). \\
\textit{ad}\ \vee: & \text{Similar.} \\
\textit{ad}\ {\sim}: & f({\sim}X) = ({\sim}X, U-g({\sim}X)) = (U-g[X], U-g(U-g[X])) \\
 & \quad = (U-g[X], X) = {\sim}f(X).
\end{array}$$

We now discuss informal interpretations of the representation theorems that relate to semantic treatments of relevant first degree implications ($\mathbf{R}_{\mathrm{fde}} = \mathbf{E}_{\mathrm{fde}}$ = the tautological entailments of §15).

Routley and Routley 1972 presented a semantics for $\mathbf{R}_{\mathrm{fde}}$, the main ingredients of which were a set K of "atomic set-ups" (to be explained) and an involution * defined on K. An "atomic set-up" is just a set of propositional variables, and it is used to determine inductively when complex formulas are also "in" a given set-up. A set-up is explained informally as being like a possible world except that it is not required to be either consistent or complete. The Routleys' 1972 paper seems to conceive of set-ups very syntactically as literally being sets of formulas, and in §16.2.1 we reified them as certain conjunctions; but Routley and Meyer 1973 conceives of them more abstractly. We shall think of them this latter way here so as to simplify exposition. The Routleys' models can then be considered a structure $(K, *, \vDash)$, where K is a nonempty set, * is an involution on K, and $\vDash$ is a relation from K to zero degree formulas. We read "$a \vDash A$" as: the formula A holds at the set-up a:

1 $(\&\vDash)$ $a \vDash A\&B$ iff $a \vDash A$ and $a \vDash B$;
2 $(\vee\vDash)$ $a \vDash A\vee B$ iff $a \vDash A$ or $a \vDash B$;
3 $({\sim}\vDash)$ $a \vDash {\sim}A$ iff $a^* \nvDash A$.

The important thing to observe about the clause for negation is that the value for A at a set-up a is made to depend on the value of A at some *different* set-up a*, in this respect contrasting with the clauses for conjunction and disjunction.

The connection of the Routleys' semantics with quasi-fields of sets will become clear if we let (K, *) induce a quasi-field of sets Q with quasi-complement ~, and let $|\ |$ interpret sentences in Q subject to the following conditions:

1′ $|\&|$ $|A\&B| = |A|\cap|B|$;
2′ $|\vee|$ $|A\vee B| = |A|\cup|B|$;
3′ $|{\sim}|$ $|{\sim}A| = {\sim}|A|$.

Clause (&⊨) results from clause |&| by translating $a \in |X|$ as $a \models X$ (see end of §48.1). Thus clause |&| says

$$a \in |A\&B| \text{ iff } a \in |A| \text{ and } a \in |B|;$$

i.e., clause |&| translates as clause (&⊨). The case of disjunction is obviously the same. The case of negation is clearly of special interest; so we write it out. Thus clause |∼| says

$$a \in |{\sim}A| \text{ iff } a \in {\sim}|A|,$$
$$a \in |{\sim}A| \text{ iff } a \in K-(|A|^*),$$
$$a \in |{\sim}A| \text{ iff } a \notin |A|^*,$$
$$a \in |{\sim}A| \text{ iff } a^* \notin |A|.$$

But the translation of this last is just clause (∼⊨).

There are at least two philosophical interpretations to be put on fields of polarities; we defer discussion of "proposition surrogates" and "situations" to §50.6 below.

§48.3. Three-termed relational (Routley-Meyer) semantics for $\mathbf{R}_+$. As indicated at the beginning of §47, Routley had the basic idea of the operational semantics at about the same time as Urquhart. Priority would be hard to assess. At any rate we first received some details concerning both their work in early 1971, although J. Garson told us of Urquhart's work in December of 1970, and we have seen references made to a typescript of Routley's with a 1970 date on it (in Charlwood 1978).

The operational semantics and the relational semantics as well were also conceived a little later by Fine in complete independence. Fine heard NDB lecture on relevance logics at the presemantic level in Oxford in early 1970, and he obtained essentially the whole semantics by the middle of 1971. In April of 1972 Segerberg passed along to NDB a prepublication copy of Fine's completed paper, which because of publication vagaries did not appear until 1974. (This is the paper that constitutes §51.)

Returning to the thread leading to the work reported in this section, Meyer and JMD were colleagues at the time in early 1971 when Routley sent a somewhat incomplete draft of his ideas to each of Meyer and NDB. This was a courageous and open communication in response to our keen interest in the topic (instead he might have sat on it until it was perfected). The draft favored the operational semantics, indeed the semilattice semantics of §47, and it was not clear that this was not the way to go for the calculus **R**. But the draft started with a more general point of view, suggesting the use of a three-placed accessibility relation R (of course a two-placed operation like ∪ of §47 is a three-placed relation, but not always conversely), with the following valuation clause for →:

(→) $a \models A \to B$ iff, for all $b, c \in K$, if $Rabc$ and $b \models A$ then $c \models B$.

Forgetting negation for a while, the clauses for & and $\vee$ are "truth-functional," just as for the operational semantics.

Meyer, having observed with JMD the lack of fit between the semilattice semantics and **R** (described in §47.4), was all primed to make important contributions to Routley's suggestion. In particular, he saw that the more general three-placed relation approach could be made to work for all of **R**. In interpreting Rxyz in the context of **R**, perhaps the best reading is to say that the combination of the pieces of information x and y (not necessarily the union) is a piece of information included in z (in bastard symbols, $x \circ y \leq z$). Routley himself called the x, y, etc. "set-ups" and conceived of them as being something like possible worlds except that they were allowed to be inconsistent and incomplete (but always prime). On this reading Rxyz can be regarded as saying that x and y are compatible according to z, or some such thing. (We postpone for a bit our remarks on how to read "R" in a more general setting.)

Before going on, we want merely to refer to some other work from the early seventies, namely, Gabbay 1976, especially Chapter 15.

We now set out in more formal detail a version of the Routley-Meyer semantics for $\mathbf{R}_+$ (see §R2 for $\mathbf{R}_+$; negation will be reserved for the next section). The techniques are novel and the completeness proof quite complicated, so we shall be reasonably explicit about details. The presentation here is very much indebted to work (some unpublished) of Routley and Meyer.

By an ($\mathbf{R}_+$) *frame* (or *model structure*) is meant a structure (K, R, 0), where K is a nonempty set (the elements of which are called *set-ups*), R is a three-placed relation on K, $0 \in K$, all subject to five conditions we shall state after a few definitions. We define for $a, b \in K$, $a \leq b$ (Routley and Meyer used $<$) iff R0ab, and R^2abcd iff $\exists x$(Rabx and Rxcd). We also write this last as $R^2(ab)cd$ and distinguish it from: $R^2a(bc)d$ iff $\exists x(Raxd \wedge Rbcx)$. The variables "a," "b," etc. will be understood as ranging over the elements of some K fixed by the context of discussion.

Transcribing the condition of §47.1 on the semilattice semantics as closely as we can into this framework, we get the following as the first four conditions on an $\mathbf{R}_+$-frame:

1. (Identity)	R0aa, i.e., $a \leq a$;
2. (Commutativity)	if Rabc then Rbac;
3. (Associativity)	if $R^2(ab)cd$ then $R^2a(bc)d$; and
4. (Idempotence)	Raaa.

It should be remarked that these conditions fail to pick up the whole strength of the corresponding §47.1 semilattice conditions. Thus, e.g., Identity here picks up only $0 \cup a \leq a$ and not conversely, and similarly for Idempotence (also of course Commutativity and Associativity do not require any identity, but this is a slightly different point). We also note that Routley and Meyer 1973 employed

3′. (Pasch's law) if R^2abcd then R^2acbd,

in place of 3 (also, 5 below was misprinted there).

We need (for technical reasons) one more condition:

5. (Monotony) if Rabc and $a' \leq a$ then Ra′bc.

By an $\mathbf{R}_+$-*model* we mean a structure (K, R, 0, ⊨), where (K, R, 0) is an $\mathbf{R}_+$-frame and ⊨ is a relation from K to sentences of $\mathbf{R}_+$ satisfying the following conditions:

ATOMIC HEREDITARY CONDITION. For a propositional variable *p*, if a ⊨ *p* and a ≤ b, then b ⊨ *p*.

VALUATIONAL CLAUSES. For formulas *A*, *B*,

(→) $a \vDash A \to B$ iff ∀b,c∈K (if Rabc and $b \vDash A$, then $c \vDash B$);
(&) $a \vDash A \& B$ iff $a \vDash A$ and $a \vDash B$;
(∨) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$.

Were we to admit the connectives ∘ and ***t***, they would be awarded the following clauses: (∘) $a \vDash A \circ B$ iff for some b, c, Rbca and $b \vDash A$ and $c \vDash B$; (***t***) $a \vDash \boldsymbol{t}$ iff $0 \leq a$.

At this point we interrupt the stream of definitions to consider the intuitive significance of the three-termed relation R. In §48.1 we endorsed the view that an automatic intuitive punch accrues to the explanation of modal speech in terms of quantificational locutions (e.g., necessity in terms of quantification over "cases" or "situations") from the fact that these interchanges are licensed in our naive speech. In the same spirit, however, it must be recognized that there is no similar naive foundation for the two-termed accessibility relation of "relative possibility"; perhaps this can most clearly be seen by contrasting the intuitively *strained* deployment of the "relative possibility" relation between possible worlds in modal logic, with the naively *evident* use of the "earlier than" relation between times in tense logic. That is to say, in modal logic we think that the quantificational idea and the domain of quantification are intuitive whereas the ordering relation is not; but in tense logic we think that not only times, but *also* an ordering relation among them, is for us commonsensical, so the tense logicians' explanations ("*will* means *some time later*") fall easily on the naive ear. Not so for the modal logicians: their "situations" or such have a familiar sound, but they have to be thoroughly patient in explaining their outré relational structure of relative possibility.

Lest there be any misunderstanding, we emphasize that we mean here only to be pointing out easy facts; we certainly believe neither that naive familiarity automatically legitimates an explication nor that employment of non-intuitive ideas is somehow illegitimate. We present the matter as we have only to make the following point: the three-termed relation of the Routley-Meyer semantics seems to us more like the two-termed relation of modal logic than like the two-termed relation of tense logic: it can be explained, but it does not in itself have close ties to naive intuition.

Here is what the two parties (Routley-Meyer and Fine) who have done the most work with the three-termed relation in relevance logics each have to say.

> We might put it this way—relative to the *laws* in a, Rabc means that c is accessible from b; i.e., if the *antecedent* of an a-law is realized in b, then its *consequent* is realized in c. Peter Woodruff suggests that we might account for this situation not by thinking of R as a single ternary relation but as a *set* of binary relations R_a, R_b, etc., indexed by set-ups. (Routley and Meyer 1972b, p. 195.)
>
> ... a relativized inclusion relation Rabc (also written b $\leq_a$ c) with the sense that c is as strong as b relative to a. (Fine 1974, p. 362.)

Clearly both of these accounts derive such power as they have from the role that R plays in the semantic clause ($\rightarrow$) for the arrow—another example of the "circularity" of semantics mentioned above in §48.2 in our discussion of the * operation. And again we want to suggest that this sort of circularity is not necessarily a matter of evil; since, however, we ourselves have nothing further to offer in explication of R, we return to formal matters with just this final word: we are convinced of the high probability that a mathematical apparatus of such proven formal power will eventually find its concrete applications and its resting place in intuition (think of tensors).

We shall say that A is *verified* on a model if $0 \vDash A$ and that A *entails* B on a model if $\forall a \in K$ (if $a \vDash A$, then $a \vDash B$). We say that A is *valid* if A is verified on all models.

It is easy to prove, by an induction on A, the following (note how Monotony enters in):

HEREDITARY CONDITION. For an arbitrary formula A, if $a \vDash A$ and $a \leq b$, then $b \vDash A$.

VERIFICATION LEMMA. If, in a given model $(K, R, 0, \vDash)$, A entails B in the sense just defined then $A \rightarrow B$ is verified in the model, i.e., $0 \vDash A \rightarrow B$.

PROOF. Suppose that R0ab and $a \vDash A$. By the hypothesis of the lemma, $a \vDash B$, and, by the Hereditary condition, $b \vDash B$, as is required for $0 \vDash A \rightarrow B$.

We are now in a position to prove the

SOUNDNESS THEOREM. If $\vdash_{\mathbf{R}_+} A$ then A is valid.

PROOF. Much of this will be left to the reader, but we take the trouble to verify the axioms R1–R4 of §R2, and the rules.

ad R1. That $0 \vDash A\to A$ is immediate from the Verification lemma (which relies, as we saw, on the Hereditary condition).

ad R2. By the Verification lemma, we need to suppose (1) $a \vDash A\to B$ and show (*) $a \vDash B\to C\to.A\to C$. To obtain (*), suppose (2) Rabc and (3) $b \vDash B\to C$ and show (**) $c \vDash A\to C$. To obtain (**), suppose (4) Rcde and (5) $d \vDash A$ and show (***) $e \vDash C$. Putting (2) and (4) together by definition gives R^2abde, whence R^2b(ad)e by Associativity, that is, (6) Rbxe and (7) Radx, for some x. Now (7), (1), and (5) yield $x \vDash B$, which together with (6) and (3) finally gives (***).

ad R3. To show that $A\to.A\to B\to B$ is valid, it suffices by the Verification lemma to assume $a \vDash A$ and show $a \vDash A\to B\to B$. For this last we assume Rabc and $b \vDash A\to B$, and show $c \vDash B$. By Commutativity, Rbac. By ($\to$), since we have $b \vDash A\to B$ and $a \vDash A$, we get $c \vDash B$ as desired.

ad R4. Direct verification would be at best contrived, but it is an easy consequence of condition

4′. if Rabc then R^2abbc,

and 4′ comes as follows. Start with Rabc, and obtain Rbac by Commutation. Because also Rbbb, we have R^2(bb)ac by definition, and so R^2b(ba)c by Associativity, i.e., Rbxc and Rbax, some x. Two uses of commutation give Rabx and Rxbc, i.e., R^2abbc, as required. Now, with the Verification lemma in mind, suppose (1) $a \vDash A\to.A\to B$; to show the needed $a \vDash A\to B$, suppose (2) Rabc and (3) $b \vDash A$, and show $c \vDash B$. Use condition 4′ on (2) to get R^2abbc, i.e., (4) Rabx and (5) Rxbc, for some x. Now (1) with (4) and (3) gives $x \vDash A\to B$, which with (5) and (3)—again—gives $c \vDash B$ as wanted.

Verification of the conjunction and disjunction axioms is routine and is safely left to the reader.

It remains to be shown only that the rules *modus ponens* and adjunction preserve validity. Actually something stronger holds. It is easy to see that, for any $a\in K$ (not just 0), if $a \vDash A\to B$ and $a \vDash A$ then $a \vDash B$ (by virtue of Raaa), and of course it follows immediately from (&) that if $a \vDash A$ and $a \vDash B$ then $a \vDash A\&B$.

We next go about the business of establishing the

COMPLETENESS THEOREM. If E is valid, then $\vdash_{\mathbf{R}_+} E$.

The main idea of the proof is similar to that of the by now well-known Henkin-style completeness proofs for modal logic. We suppose that not $\vdash_{\mathbf{R}_+} E$

and construct a so-called "canonical model," the set-ups of which are certain prime $\mathbf{R}_+$-theories (playing the role of the maximal theories of modal logic). The base set-up 0 is constructed as a prime $\mathbf{R}_+$-containing $\mathbf{R}_+$-theory (for the terminology "prime," etc. consult the numbered definitions of §42.1) that excludes *E*.

Remark. The presentation of these matters would have been made in some respects more attractive had we availed ourselves of the optional fusion connective $\circ$ and the propositional constant ***t***. And, with or without these connectives, we could have rendered our presentation more elegant by developing the properties of what Routley and Meyer 1973 call the calculus of "intensional **R**-theories" (these are also independently important to Fine 1974). We forego these niceties, however, in consonance with our aim of giving the reader the flavor of the enterprise by establishing the main results as cheaply as possible. For a vastly more complete presentation, we recommend the later work of Routley and Meyer, especially Routley with Plumwood, Meyer, and Brady 1982.

Let us now look at the details. Since $\mathbf{R}_+$ is Up-Down acceptable in the sense of Def. 1 of §42.1, we may use the Way Up lemma (stated at the end of §42.1 and proved in §42.2) to yield a prime $\mathbf{R}_+$-containing $\mathbf{R}_+$-theory (Defs. 2, 3, and 5 of §42.1) not containing *E*; here we call that theory "0". Henceforth we shall let 0 (not $\mathbf{R}_+$) take the place of "L" in the numbered definitions of §42.1; and, in particular, we define K = the set of prime (but *not* necessarily 0-containing) 0-theories (Defs. 2 and 5 of §42.1). Then we define the three-termed accessibility relation R "canonically" as follows:

Rabc iff for all A, B, if $A \rightarrow B \in \mathrm{a}$ and $A \in \mathrm{b}$, then $B \in \mathrm{c}$.

Theorem 1. The canonically defined structure (K, 0, R) is indeed an $\mathbf{R}_+$-frame; that is, $0 \in \mathrm{K}$, and the conditions 1–5 of Identity, Commutativity, Associativity, Idempotence, and Monotony, laid above on 0 and the three-termed relation R, are all satisfied.

Theorem 1 will not finish our work, for it yields only a "frame" and not a "model"; but it gives us a start. Before commencing its proof, it is convenient to enter a couple of facts.

Fact 1 We first observe

1. 0 (not just $\mathbf{R}_+$) is Up-Down acceptable in the sense of Def. 1 of §42.1—the positive propositional parts only, of course. 0 is a 0-theory.

Furthermore, the following are theorems of $\mathbf{R}_+$, *and* they are also theorems of 0 (since 0 is an $\mathbf{R}_+$-containing $\mathbf{R}_+$-theory), *and* all members of K, including

0 itself, are closed under the passage from the antecedents to the consequents of these implications (since the members of K are 0-theories).

2. $A \rightarrow . A \rightarrow B \rightarrow B$
3. $((A \rightarrow B) \& A) \rightarrow B$
4. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$
5. $((A \rightarrow B) \& (C \rightarrow D)) \rightarrow . A \& C \rightarrow . B \& D$
6. $((A \rightarrow B) \& (C \rightarrow D)) \rightarrow . A \vee C \rightarrow . B \vee D$
7. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$
8. $((A \rightarrow B) \& (A \rightarrow C)) \rightarrow . A \rightarrow . B \& C$

PROOF. By the natural deduction techniques of §R3.

FACT 2. If R0ab then a $\subseteq$ b.

PROOF. $A \rightarrow A \in 0$, by Fact 1-1; so, given R0ab, if A is in a, it must be in b, by the canonical definition of R.

PROOF OF THEOREM 1. *ad* 0$\in$K. We know from Fact 1-1 that 0 is a 0-theory, hence in K, since it was constructed by the Way Up to be prime.

ad Identity, R0aa. Given the canonical definition of R, this is just the requirement that a be a 0-theory.

ad Commutativity. Suppose (1) Rabc. To show Rbac, suppose (2) $A \rightarrow B \in$ b and (3) $A \in$a. Fact 1-2 and (3) yield that $A \rightarrow B \rightarrow B \in$ a, from which $B \in$c follows with (1) and (2).

ad Idempotence, Raaa. Suppose $A \rightarrow B \in$ a and $A \in$a; but then $B \in$a via Fact 1-3.

ad Monotony. Suppose a′ $\leq$ a, that is, R0a′a, and Rabc; but then Ra′bc is immediate from Fact 2.

ad Associativity. This is the final piece of Theorem 1, and by far the least trivial. Let us then assume that R^2(ab)cd, i.e., (1) Rabx and (2) Rxcd, for some x. We need to find a prime 0-theory y such that Rayd and Rbcy, i.e., R^2a(bc)d. Set $y_0 = \{B\colon \exists A(A \rightarrow B \in \text{b and } A \in \text{c})\}$. (This is a way of defining "b$\circ$c" in the "calculus of intensional theories.") Clearly (3) Rbcy_0.

Observe first that y_0 is a 0-theory; for suppose that $B \in y_0$ and that $\vdash_0 B \rightarrow B'$. Then $\vdash_0 A \rightarrow B \rightarrow . A \rightarrow B'$, by Fact 1-4; so $A \rightarrow B' \in$ b, which puts B' in y_0. But also y_0 is closed under adjunction, for suppose $B, B' \in y_0$ in virtue of $A \rightarrow B$, $A' \rightarrow B' \in$ b and $A, A' \in$ c. Use Fact 1-5 to calculate that $A \& A' \rightarrow . B \& B' \in$ b, whence, since $A \& A' \in$ c, $B \& B' \in y_0$ as required.

We next verify that Ray_0d. Suppose (4) $A \rightarrow B \in$ a and $A \in y_0$, i.e., (5) $C \rightarrow A \in$ b and (6) $C \in$c, for some C. Now (4) and Fact 1-4 yield (4′)

$C\to A\to .C\to B \in$ a, and (1), (4′), and (5) imply that $C\to B\in$ x, which with (2) and (6) yields $B\in$d, as required.

The reader is excused a bit if the thread has been lost and it is thought that we have now finished verifying the associativity of R. We wanted some prime 0-theory y that fills in the blanks of

Ra__d and Rbc__,

and we have just finished verifying that y_0 is a 0-theory that does fill in the blanks. The kicker is that y_0 need not be prime. So we work next at pumping up y_0 to make it prime while continuing to fill in the blanks. It clearly suffices to prove

THE SQUEEZE LEMMA. Let a and y_0 be 0-theories (they need not be prime) for 0 an $\mathbf{R}_+$-containing prime $\mathbf{R}_+$-theory, and let d be a prime 0-theory. If Ray$_0$d then there exists a prime 0-theory y such that (i) $y_0\subseteq$ y and (ii) Rayd.

PROOF. This could be accomplished by a Lindenbaum-like inductive construction or via Zorn's lemma as in Routley and Meyer 1973; for us it is easiest to rely on the already established Pair Extension lemma of §42.2, having noted in Fact 1-1 that 0 is Up-Down acceptable.

Thus set $\Delta = y_0$ and $\Theta = \{A: \exists B(A\to B)\in$ a and $B\notin$d$\}$. We need to check that (Δ, Θ) is 0-exclusive (Def. 6 of §42.2), and to this end we observe first that Θ is closed under disjunction. Suppose A_1, $A_2\in\Theta$; then $A_1\to B_1$, $A_2\to B_2\in$ a, and yet $B_1, B_2\notin$ d, for some B_1, B_2. The first part yields $A_1\vee A_2\to .B_1\vee B_2\in$ a by Fact 1-6, and the second part yields $B_1\vee B_2\notin$ d by the primeness of d; so $A_1\vee A_2\in\Theta$ as promised. Since Δ, being a 0-theory, is closed under adjunction, this means that if the pair (Δ, Θ) were to fail to be 0-exclusive then we would have (7) $\vdash_0 A\to B$ for some (8) $A\in\Delta$ and $B\in\Theta$, i.e., (9) $B\to C\in$ a and (10) $C\in$d, for some C. But (7) with Fact 1-7 yields $\vdash_0 B\to C\to .A\to C$, which with (9) gives $A\to C\in$ a. But now this, with (8) and (10), conflicts with the hypothesis Ray$_0$d of the lemma.

Now that we know that (Δ, Θ) is an 0-exclusive pair, we apply the Pair Extension lemma to get a disjoint pair (y, y′) with $y_0 = \Delta\subseteq$ y, $\Theta\subseteq$ y′, and y a prime 0-theory. The disjointness of y from Θ immediately implies that Rayd, completing the proof of the Squeeze lemma, which actually does finish the verification that the relation R is associative and, thus, completes the proof of Theorem 1.

Having finally verified that the canonical (K, 0, R) has all the properties of an $\mathbf{R}_+$-frame, we need now to define an appropriate relation $\vDash$ on it. The natural definition is: a $\vDash A$ iff $A\in$a, and we adopt it; but we need to verify that this has the properties (1) and (2) required of $\vDash$ above in our definition of an $\mathbf{R}_+$-model.

THEOREM 2. The canonically defined (K, 0, R, $\models$) is indeed an $\mathbf{R}_+$-model.

PROOF. *ad* (1): the Atomic Hereditary condition. Suppose that $a \leq b$, i.e., R0ab, and that $p \in a$. Then $p \in b$, by Fact 2.

ad (2): the Valuation clauses. The clauses (&) and ($\vee$) are more or less immediate (primeness is of course needed for half of ($\vee$)). The clause of interest is ($\rightarrow$). Applying the canonical definition of $\models$, this amounts to

$(\rightarrow_c)$ $A \rightarrow B \in a$ iff $\forall$b,c (if Rabc and $A \in b$, then $B \in c$).

Left-to-right is just the canonical definition of R. Right-to-left, to which we turn, involves another application of the Pair Extension lemma. Thus suppose contrapositively that $A \rightarrow B \notin a$. We need prime 0-theories b and c, with $A \in b$ and $B \notin c$. Let $\Delta_b = \{A' \colon \vdash_0 A \rightarrow A'\}$; this contains A, and is clearly a 0-theory, by Fact 1-1. Also let $\Delta_c = \{C \colon A \rightarrow C \in a\}$; this is a 0-theory, by Facts 1-7 and 1-8. Hence, given our supposition, $(\Delta_c, \{B\})$ is a 0-exclusive pair; so, by the Pair Extension lemma, there is a prime 0-theory c that includes Δ_c and excludes B. We may use Fact 1-7 to verify that $Ra\Delta_b\Delta_c$, hence $Ra\Delta_b c$. We are now in a position to apply the Squeeze lemma, obtaining a prime theory b including Δ_b such that Rabc, and clearly also such that $A \in b$ and $B \notin c$.

So (K, 0, R, $\models$) is an $\mathbf{R}_+$-model. If we now recall that, by construction, 0 excludes our chosen nontheorem E and that the canonical definition of $\models$ agrees with membership, we shall have finished our proof of completeness: for each nontheorem E of $\mathbf{R}_+$, there is an $\mathbf{R}_+$-model in which E is not verified (not $0 \models E$).

§48.4. Strong completeness for $\mathbf{R}_+$. It is fashionable these days always to prove *strong* completeness. This could have been done. Thus define A to be an *Official* $\mathbf{R}_+$*-consequence* of a set of formulas Γ iff A is verified (holds at 0) in every $\mathbf{R}_+$-model that verifies every member of Γ. This is a kind of classical or official notion and should not be confused with any kind of relevant consequence; thus, for example, where B is a theorem of $\mathbf{R}_+$, since always $0 \models B$, B will be an Official $\mathbf{R}_+$-consequence of any set Γ. Define A to be *Officially* $\mathbf{R}_+$*-deducible* (or *derivable*) *from* Γ iff A in every $\mathbf{R}_+$-containing $\mathbf{R}_+$-theory containing Γ (this is a restatement in the present context of §22.2.1.). Appropriate modifications of the work above will show that Official $\mathbf{R}_+$-consequence is equivalent to Official $\mathbf{R}_+$-deducibility. Or we may ring the changes in another way. Say that A is a *relevant* $\mathbf{R}_+$*-consequence* of Γ iff, for every $\mathbf{R}_+$-model, $a \models A$ whenever $a \models B$ for every member B of Γ, for (not just 0 but) *all* $a \in K$. And say that A is *relevantly* $\mathbf{R}_+$*-deducible* (or *derivable*) from Γ iff A is in *every* $\mathbf{R}_+$-theory (whether $\mathbf{R}_+$-containing or not) containing Γ (this is a restatement in the present context of §23.6). Then relevant $\mathbf{R}_+$-consequence agrees with relevant $\mathbf{R}_+$-deducibility. (For the "completeness" side of this, obtain 0 by applying the Pair Extension lemma of §42.2 to (Δ, Θ), where $\Delta = \mathbf{R}_+$ and Θ is the set of all implications $(B_1\& \ldots \& B_n) \rightarrow A$, with all the B_i in Γ. It is easy to see that this pair must be $\mathbf{R}_+$-exclusive. Then obtain the required prime 0-theory a by applying the Pair Extension lemma to the 0-exclusive pair $(\Gamma, \{A\})$.)

§48.5. Relational semantics for all of R. We now discuss the Routley-Meyer semantics for the whole system **R**. The idea is simply to add the Routleys' treatment of negation using the * operator (discussed in §48.2). (This is not difficult, and there is very little reason to segregate it off into this separate section, except that we thought that the treatment of $\mathbf{R}_+$ was complicated enough.)

Thus an **R**-*frame* is a structure, (K, R, 0, *) where (K, R, 0) is an $\mathbf{R}_+$-frame and K is closed under the unary operation * satisfying:

Period two. a** = a;
Inversion. If Rabc then Rac*b*.

For an **R**-model the valuational clauses for the positive connectives are as for an $\mathbf{R}_+$-model, and we of course add clause 3 of §48.2 (slightly renamed):

$(\sim)$ $\quad$ $\mathrm{a} \vDash {\sim} A$ iff $\mathrm{a}^* \nvDash A$.

The soundness and completeness results are relatively easy modifications of those for $\mathbf{R}_+$. That * is of period two naturally is used in the verification of Double Negation (R13 of §R2), and Inversion is central to the verification of Contraposition (R12 of §R2). For completeness, a* is defined canonically as $\{A\colon {\sim}A \notin \mathrm{a}\}$ (see the definition of the analogue g(P) in the proof of Białynicki-Birula and Rasiowa's representation of De Morgan lattices in §48.2), and one of course has to show that a* is a prime 0-theory when a is. One also has to show that canonical * is of period two and satisfies Inversion, and that canonical $\vDash$ satisfies $(\sim)$ above, i.e., $A \in \mathrm{a}$ iff ${\sim}A \in \mathrm{a}^*$, i.e., ${\sim}A \notin \{B\colon {\sim}B \in \mathrm{a}\}$, i.e., ${\sim}{\sim}A \in \mathrm{a}$, which of course just uses Double Negation.

It is worth remarking that since the canonical 0 is a prime **R**-containing **R**-theory, then since $\vdash_{\mathbf{R}} A \vee {\sim}A$, then 0 is negation complete (but not necessarily negation consistent—this is relevant to the proof of the admissibility of (γ) in §48.8 below). For your garden variety Routley-Meyer **R**-model (not necessarily canonical) notice also that either $0 \vDash A$ or $0 \vDash {\sim}A$. This follows ultimately from $0^* \leq 0$, i.e., R00*0, which is proved below:

1. R0*0*0* Idempotence (§48.3)
2. R0*00 1 Inversion, Period two
3. R00*0 2 Commutation (§48.3)

Now $0^* \leq 0$ means, by the Hereditary condition of §48.3, that if $0 \nvDash \mathrm{A}$ then $0^* \nvDash \mathrm{A}$, i.e., $0 \vDash {\sim}\mathrm{A}$ as desired.

It should be said that, although §48.2 shows that either the four-valued treatment or the *-operator treatment of negation works equally well for *first degree* relevant implications (at least from a technical point of view), the *-operator treatment seems to win hands down in the context of all of **R**. Meyer 1979b has succeeded in giving a four-valued treatment of all of **R**, but at the price of great technical complexity (e.g., the accessibility relation has to be made four-valued as well, and that is just for starters). Further, as Meyer points out, one's models still have to be closed under *, so it still can be said to sneak in the back door. Routley 1984a has given a more orthodox four-valued semantics for **R**, but at the complication of providing two two-valued accessibility relations. (The reader can see that two two-valued relations R_1 and R_2 can simulate one four-valued relation, since either of the two can hold by itself, or both or neither can hold.)

§48.6. Relational semantics for E. We know that **E**, as axiomatized by E1–E14, *modus ponens*, and adjunction in §R2, is motivated by a combination of considerations involving *both* relevance and necessity; it is therefore interesting (because surprising) that a semantics can be given for **E** which is based on no elements other than those required for the nonmodal calculus **R**. The following is due in essence to Meyer (unpublished), though we have played with details.

Our immediate aim is to define what it is for a structure (K, 0, R, *, ⊧) to be an "E model." We take over the definitions of "$a \leq b$", "R^2abcd" (or "$R^2(ab)cd$"), and "$R^2a(bc)d$" from §48.3, and we add a key definition, answering to the modal character of **E**, describing a set-up a as verifying all those entailments verified at the base set-up 0: Za iff, for every x, y, if Raxy then R0xy.

The first conditions are basic structural requirements on (K, 0, R, *, ⊧).

1. $0 \in K$; R is a three-placed relation on K; * is an operation on K; ⊧ is a relation from members of K to formulas.
2. $a \leq a$; if $a \leq a'$ and $Ra'bc$ then Rabc.

The next conditions correspond to the postulates E1, E2, E3, E12, E13, and E14 of **E** (§R2), and are numbered accordingly. (Other postulates except for E7—discussed below—are already verified by 1, 2, and 4.)

3.1. For each a there is an x such that Zx and Raxa.

(This is a kind of partial or dependent right identity, in contrast to R0aa (i.e., $a \leq a$), which is a full or independent left identity.)

3.2. If R^2abcd then $R^2b(ac)d$.
3.3. If Rabc then R^2abbc.
3.12. Raa^*a.
3.13. If Rabc then Rac^*b^*.
3.14. $a^{**} = a$.

So much gives us an "**E**-*frame*," to use the language of §48.3. We complete the definition of an "**E**-*model*" with the following:

4. The Atomic Hereditary condition and Valuation clauses (&), ($\vee$), and ($\rightarrow$) of §48.3 and the Valuation clause ($\sim$) of §48.5, all hold.

It is now straightforward to verify a Hereditary condition (for all formulas), a Verification lemma, and a Soundness theorem, each in analogy with §48.3. The only ugly bit comes in regard to postulate E7 (of §R2), which uses both 3.1 and 3.2 in its verification. But it is perhaps better to note that 3.1 suffices not only for E1, but also for its strengthening $((A\rightarrow A)\&(B\rightarrow B))\rightarrow C\rightarrow C$, which is shown in §26.1 to make both E1 and E7 redundant.

Completeness goes just as in §48.3 and §48.5; in particular, the canonical definition of (K, 0, R, *, $\vDash$) relative to a chosen nontheorem E is "the same." The only new mildly interesting feature is the verification of the modal condition 3.1. (Meyer points out, incidentally, that 3.1′ Ra0a would suffice on the side of soundness, but does not hold in all canonical models, and in fact would verify the nontheorem $\sim A\vee(A\rightarrow A\rightarrow A)$.) The idea is to start with $x_0 = \{A\colon \vdash_0 A\rightarrow A\rightarrow A\}$. This is a 0-theory; and also, canonically, Rax_0a; and, finally, if Rx_0bc then R0bc, for all b, c—that is, Zx_0. The Squeeze lemma of §48.3 now allows us to replace x_0 by a member x of K (i.e., a prime 0-theory) satisfying the same conditions. (Note how, although we begin with x_0 independent of a, the result x of the Squeeze lemma unavoidably depends on a.)

§48.7. Relational semantics for T, RM, etc. The research of Routley and Meyer has established that the three-placed relational style of semantics is applicable to a large variety of logical systems indeed. We confine ourselves to the briefest mention of two systems, **T** and **RM**, which we have frequently discussed in this book, and to equally brief mention of the systems $\mathbf{B}_+$ and **B**, to which we have not given equal time.

One obtains a semantics for **T** by modifying that of §48.6 (for **E**) as follows: *subtract* the modal requirement 3.1, and *add* a condition answering to A2 of §R2: if R^2abcd then $R^2a(bc)d$.

There are two ways to obtain a semantics for **RM**. To the definition of an **R** model in §48.5 (which refers to §48.3), add *either* (1) $0 \leq a$ or $0 \leq a^*$, as in Routley and Meyer 1973, *or* (2) if Rabc then either $a \leq c$ or $b \leq c$, as in Dunn 1979a. In §49.1 we discuss an alternative *binary* relational semantics for **RM**.

The "basic" positive calculus $\mathbf{B}_+$ corresponds to the postulation of *only* Monotony (5 of §48.3; this defines a "$\mathbf{B}_+$ model structure"). It can be defined by the following axioms and rules. *Axioms* of $\mathbf{B}_+$: from §27.1.1 or §R2 take R1, R5–11. *Rules* of $\mathbf{B}_+$: *modus ponens*, adjunction, "rule prefixing" (from $A\rightarrow B$ to infer $C\rightarrow A\rightarrow .C\rightarrow B$) and "rule suffixing" (from $A\rightarrow B$ to infer $B\rightarrow C\rightarrow .A\rightarrow C$).

The more or less "basic" calculus **B** with negation, due to Meyer and referred to in §29.12, is more complicated to describe. A **B** model structure is obtained from the notion of a $\mathbf{B}_+$ model structure as defined above by keeping Period two (a** = a) of §48.5, but replacing full Inversion as given there by (1) if R0ab then R0b*a* and (2) R00*0. For *axioms* of **B**, add double negation R13 from §R2 *and* its converse $A \rightarrow {\sim}{\sim}A$ to the axioms of $\mathbf{B}_+$. Also add excluded middle, $A \vee {\sim}A$. For *rules* of **B**, (1) add "rule contraposition," from $A \rightarrow B$ to infer ${\sim}B \rightarrow {\sim}A$, and (2) allow all five rules to "operate in disjunctive contexts"; e.g., "disjunctive rule prefixing" is the rule, from $D \vee (A \rightarrow B)$ to infer $D \vee (C \rightarrow A \rightarrow . C \rightarrow B)$.

§48.8. Spinoffs from relational semantics. One may if one likes make the question of the *philosophical* importance of the relational semantics a matter for debate, but one cannot question its *mathematical* power. This lesson is best learned from Routley's and Meyer's own works; here we only indicate with the utmost brevity how their techniques can be used to prove a variety of results concerning the systems **R**, **E**, etc., which were either more complicated using other methods or even impossible.

Meyer and Routley 1974, for example, offer a virtually complete list of *conservative extension results* for the calculuses **E**, **R**, and $\mathbf{R}^{\Box}$, using the semantic methods described above. Part of what they show is that the notational resources of the implication-negation fragment $\mathbf{R}_{\overset{\sim}{\rightarrow}}$ of **R** are adequate for the construction of a canonical **R** model ($A \circ B = {\sim}(A \rightarrow {\sim}B)$ takes over the central role of &, and $A + B = {\sim}A \rightarrow B$ takes over from $\vee$); this suffices to show that **R** is a conservative extension of $\mathbf{R}_{\overset{\sim}{\rightarrow}}$. (We permit this reference to Meyer and Routley 1974 ("**E** is a conservative extension of $\mathbf{E}_{\mathrm{I}}$") to do duty for a tentative section (60.4) on this topic to which we referred, for example, in §24.5.)

Also, it is possible to give a proof of the admissibility of (γ) (see Routley and Meyer 1973 for **R** and 1982 for **E**) which is easier than the original algebraic proof (though not as easy as Meyer's latest proof using metavaluations—as in §42). Admissibility of (γ) amounts to showing that if A is refutable in a given **R**-model (K, R, 0, *, ⊨) then A is refutable in a *normal* **R**-model (K′, R′, 0′, *′, ⊨′) (one where 0′*′ = 0′) obtained by adding 0′ as a new "zero" and redefining R′ and *′ and ⊨′ in a certain way from R and * and ⊨.

Perhaps the most interesting new property to emerge this way is "Halldén completeness," i.e., if $\vdash_{\mathbf{R}} A \vee B$ and A and B share no propositional variables in common, then $\vdash_{\mathbf{R}} A$ or $\vDash_{\mathbf{R}} B$ (Routley and Meyer 1973).

Another direction that the Routley-Meyer semantics has taken quickly ends up in heresy: classical (Boolean) negation $\neg$ can be added to **R** with horrible theorems resulting, like $(A \wedge \neg A) \rightarrow B$, and yet (surprisingly) **R** does not collapse to classical logic. Indeed, no new theorems emerge in the original vocabulary of **R**. The idea is to take a normal **R**-model (K, R, 0, *, ⊨) and turn it in for a new **R**-model (K′, R′, 0′, *′, ⊨′), whose 0′ is a new element, K′ = K∪{0′}; *′ is like * but with 0′*′ = 0′, and R′ is like R with the additional features:

R′0′ab iff R′a0′b iff a = b,
R′ab0′ iff a = b*.

Also, ⊨′ is just like ⊨ but with 0′ ⊨ *A* if 0 ⊨ *A*. The whole point of this exercise is to provide refuting **R**-models that have the property: if a ≤ b (i.e., R0′ab) then a = b. These are called "classical **R**-models" (first studied in Meyer and Routley 1973, 1974a), and upon them one can define

(¬) a ⊨ ¬*A* iff not a ⊨ *A*.

One could not do this on ordinary **R**-models without things' coming apart at the seams, because in order to have the theorem ¬p→¬p valid, one would need the Hereditary condition of §48.3 to hold for ¬*p*, i.e., if a ≤ b, then if a ⊨ ¬*p* then b ⊨ ¬*p*, i.e., if a ⊭ *p* then b ⊭ *p*. But one has no reason to think that is the case, since all one has is the converse coming from the fact that the Hereditary condition holds for *p*. The inductive proof of the Hereditary condition breaks down in the presence of Boolean negation, but of course with classical **R**-models the Hereditary condition becomes vacuous and there is no need for a proof.

This leads to certain technical simplicities; e.g., it is possible to give Gödel-Lemmon-style axiomatizations of relevance logics like the familiar ones for modal logics (Lemmon 1957), where one takes among one's axioms all classical tautologies (using ¬)—see Meyer 1974a.

But it also leads to certain philosophical perplexities. For example, what was all the fuss about "fallacies of relevance" (see Volume I, *passim*)? What were the complaints lodged against contradictions' implying everything (§16) and against the disjunctive syllogism (§25)? Boolean negation trivially satisfies these principles; so what can be the interest of De Morgan negation's failing to satisfy them? Will the real negation please stand up?

It is possible to disagree over the propriety of thinking of Boolean negation as part of relevance logic; see §80.2 below for the "con" side and Meyer 1978 for the "pro." Here we want only to enter the following: the mathematical fact that "Boolean" negation can be conservatively added to relevance logics is established—and interesting. But, as in the case for instance of the Löwenheim-Skolem theorem, the nature of that interest remains an open topic for philosophical argle-bargle. Perhaps the following will make our point clear. The Kripke 1965a semantics for intuitionism establishes beyond a shadow of a doubt that one can conservatively add a "Boolean" negation to intuitionism—just use the clause (¬) above, and do not expect to save substitution for propositional variables. But it is now quite another question what philosophical sense such an addition makes.

We leave the matter for further discussion in §80, with just this final perspective: the question at issue may well be taken to be, Should relevance logic be considered an *addition* to classical Boolean logic (like adding modalities) or should it be considered an *alternative* to this logic (as intuitionism is usually construed)? (We are thinking of Haack 1974 and more especially of Wolf 1978.)

§48.9. Relational semantics for quantifiers. As indicated in Routley and Meyer 1973, the relational semantics can be extended in an easy way to quantifiers, at least to the extent of providing a so-called "constant domain" variety. We carry out the task for $\mathbf{R}^{\forall\exists x}$, but it will be seen that there is nothing special about this choice.

Thus, a $\mathbf{R}^{\forall\exists x}$ *constant-domain model* is a structure $(\mathrm{K}, 0, \mathrm{R}, {*}, \mathrm{D}, \varphi, \vDash)$ such that:

1. $(\mathrm{K}, 0, \mathrm{R}, {*})$ is an **R**-frame, as before (§48.3 and §48.5).
2. D is a nonempty set (of "individuals").
3. φ maps each individual parameter into D, and φ also maps each set-up *cum* n-ary predicate letter into a subset of D^n, subject to the following analogue to the Atomic hereditary condition of §48.3:

> Predicate Hereditary condition: if $\mathrm{a} \leq \mathrm{b}$ then $\varphi(\mathrm{a}, F) \subseteq \varphi(\mathrm{b}, F)$, for all $\mathrm{a}, \mathrm{b} \in \mathrm{K}$ and n-ary predicates F. ("$\mathrm{a} \leq \mathrm{b}$" is defined in §48.3.)

4. For each assignment α of members of D to the individual variables, $\vDash_\alpha$ relates set-ups to formulas much as before, with the clauses for $(\rightarrow)$, $(\&)$, $(\vee)$, and $(\sim)$ being the same as in §48.3 and §48.5, and with three new clauses covering "atomic" formulas and quantifications (for $\mathrm{d} \in \mathrm{D}$ and x a variable, $\alpha[\mathrm{d}/x]$ is defined as like α, except $\alpha[\mathrm{d}/x](x) = \mathrm{d}$):

> $\mathrm{a} \vDash_\alpha Ft_1 \ldots t_n$ iff $\langle \delta(t_1), \ldots, \delta(t_n) \rangle \in \varphi(\mathrm{a}, F)$ (where $\delta(t_i)$ is $\alpha(t_i)$ or $\varphi(t_i)$ according as t_i is a variable or parameter).
>
> $(\forall)$ $\mathrm{a} \vDash_\alpha \forall x A$ iff for every $\mathrm{d} \in \mathrm{D}$, $\mathrm{a} \vDash_{\alpha[\mathrm{d}/x]} A$.
>
> $(\exists)$ $\mathrm{a} \vDash_\alpha \exists x A$ iff for some $\mathrm{d} \in \mathrm{D}$, $\mathrm{a} \vDash_{\alpha[\mathrm{d}/x]} A$.

It is easy to check that the calculus $\mathbf{R}^{\forall\exists x}$ defined in §38 is sound in this semantics.

It is also pretty close to the surface that the set of $\mathbf{R}^{\forall\exists x}$ of valid formulas is recursively enumerable: for each formula A of $\mathbf{R}^{\forall\exists x}$ one can see how to write down a first-order formula A' (in K, R, 0, etc.) that is first-order valid just in case A is $\mathbf{R}^{\forall\exists x}$-valid.

The question of whether or not the particular calculus $\mathbf{R}^{\forall\exists x}$ is complete in these semantics was open for many years, only to be closed in the negative by the research of Fine reported in §52; and similarly for the analogous calculuses based on E, etc. But even though we shall need to look to §52 to settle the question, we can say here something about why it resists the techniques we have been describing in this section. If one attempts to solve the problem in the fashion of §48.3, one soon finds out that the Squeeze lemma (stated for $\mathbf{R}^{\forall\exists x}$) is not just hard to prove, but false. Suppose, for example, that a is the theory (in an appropriate sense) generated by the set of all implications $Fa\rightarrow q$, where a is any parameter; that b is the set of consequences of $\exists xFx$; and that c is some prime theory (same sense) not containing q. So Rabc, plausibly enough; but there is now no way to "pump up" b to an $\exists$-prime theory (§42.1, Def. 4) b′ since one cannot add any Fa without violating Rab′c.

It is possible to avoid this problem for weaker systems—those without existential conditions on R, such as Associativity or if R^2abcd then R^2a(bc)d (notation as in §48.3).

Given not only the local problem just described, however, but also the definite negative results of §52, an apt semantics for $\mathbf{R}^{\forall\exists x}$ must be something quite different; which indeed is provided by Fine in §53.

§49. Binary relational semantics for the mingle systems RM and $\mathbf{RM}^{\forall\exists x}$.

We introduced the mingle idea in §8.15 in a pure implicational setting; we axiomatized it with truth functions as **RM** in §27.1.1 (and again in §R2); we gave it an algebraic model theory in §29.3–4 in terms of Sugihara matrices (§26.9); we complained about it a bit in §29.5; and we noted in §48.7 that it can be given a Routley-Meyer three-termed relational semantics. In this section we show that the mingle idea can be treated from the point of view of a *binary* relational semantics. In §49.1 we treat the propositional calculus **RM**, and in §49.2 we extend the treatment to the calculus $\mathbf{RM}^{\forall\exists x}$ with individual quantifiers (axiomatized in §38).

§49.1. Binary relational semantics for RM.

We offer a model theory using Kripke's device of model structures with a binary accessibility relation. The principal point of departure from Kripke is the consideration of models that allow sentences to be simultaneously both "true" and "false," as in §48.2 and §81.

When the basic results here were first obtained back in 1969 (in Dunn 196+, 197+a, 1976b) there was no need to mention explicitly that the semantics presented used a *binary* accessibility relation. That was surely a part of the ordinary meaning of the phrase "Kripke-style semantics." But, since then, the work of Routley and Meyer (§48) and others has shown how to extend Kripke-style semantics to allow for ternary accessibility relations. This has been particularly fruitful in the case of the relevance logics, and §48 outlines completeness results for various of these logics, including **R** and **RM**.

It must be frankly confessed that the ingenuity and the power of the ternary semantics were so overwhelming that they caused delay until 1975 of the full exposition of the binary semantics for **RM**. There had been at first the vain hope that the binary semantics could be extended to the other relevance logics, and then, with the success of the ternary approach, the binary semantics began to appear old-fashioned and special.

Although all of that still remains true, nonetheless there are reasons why the binary semantics for **RM** remains interesting. After all, a binary accessibility relation is less complex than a ternary one. On the philosophical level, as it turns out, this leads to a rather familiar glossing of the semantics for **RM** in terms of the flow of time and increasing information, whereas it seems that there has been as yet no completely satisfactory glossing of the ternary semantics (see §48.3). On the mathematical level, it happens that one can use the method of filtration straightforwardly on the binary semantics to show that **RM** is decidable, whereas the method of filtration seems to have at least no straightforward application to the ternary semantics. (Of course, Meyer previously showed **RM** decidable using algebraic methods; see §29.3.2.)

§49.1.1. The binary semantics. We define an **RM** *model structure* to be an ordered triple (G, K, R), where K is a set, G∈K, and R is a (weak) linear ordering of K, i.e., a reflexive, antisymmetric, transitive, connected relation. Further, it does not hurt to require that G be the least element of K under R, as the reader can check for himself as he works through the proofs. An **RM**-*model* on such a model structure is a function $\varphi(p, \mathrm{H})$, where p ranges over sentential variables and H ranges over members of K. We specify that the possible values of φ are $\{T\}$, $\{F\}$, and $\{T, F\}$, and also require the

HEREDITARY CONDITION: For all H, H′ ∈ K, if HRH′ then $\varphi(p, \mathrm{H}) \subseteq \varphi(p, \mathrm{H}')$.

We then extend φ to complex sentences inductively as follows:

$(\sim T)$	$T \in \varphi(\sim A, \mathrm{H})$ iff $F \in \varphi(A, \mathrm{H})$;
$(\sim F)$	$F \in \varphi(\sim A, \mathrm{H})$ iff $T \in \varphi(A, \mathrm{H})$;
$(\&T)$	$T \in \varphi(A\&B), \mathrm{H})$ iff $T \in \varphi(A, \mathrm{H})$ and $T \in \varphi(B, \mathrm{H})$;
$(\&F)$	$F \in \varphi(A\&B, \mathrm{H})$ iff $F \in \varphi(A, \mathrm{H})$ or $F \in \varphi(B, \mathrm{H})$;
$(\vee T)$	$T \in \varphi(A\vee B, \mathrm{H})$ iff $T \in \varphi(A, \mathrm{H})$ or $T \in \varphi(B, \mathrm{H})$;
$(\vee F)$	$F \in \varphi(A\vee B, \mathrm{H})$ iff $F \in \varphi(A, \mathrm{H})$ and $F \in \varphi(B, \mathrm{H})$;
$(\rightarrow T)$	$T \in \varphi(A\rightarrow B, \mathrm{H})$ iff, for all H′∈K such that HRH′, (i) $T \in \varphi(A, \mathrm{H}')$ only if $T \in \varphi(B, \mathrm{H}')$, and (ii) $F \in \varphi(B, \mathrm{H}')$ only if $F \in \varphi(A, H')$;
$(\rightarrow F)$	$F \in \varphi(A\rightarrow B, \mathrm{H})$ iff either (i) $T \notin \varphi(A\rightarrow B, \mathrm{H})$ or (ii) $T \in \varphi(A, \mathrm{H})$ and $F \in \varphi(B, \mathrm{H})$.

Note that an easy induction shows that $\varphi(A, \mathrm{H})$ is always nonempty.

We next define a sentence A to be an *Official* **RM**-*consequence* of a set of sentences S (in symbols $\mathrm{S} \vDash_{\mathbf{RM}} A$) iff, for all models φ on all **RM** model structures (G, K, R), if, for all B∈S, $T \in \varphi(B, \mathrm{G})$ then $T \in \varphi(A, \mathrm{G})$. The chief result of this section is that the semantic notion of Official **RM**-consequence is coincident with the syntactic notion of Official **RM**-derivability.

§49.1.2. Informal interpretation. In §48.2 a semantics was presented for the first degree entailments (no nesting) of Chapter III, using the idea that an adequate modeling of a system of beliefs would permit the assignment to a given sentence of both or neither of the truth values *T* and *F* (as well as, of course, the usual assignments of exactly one) (see also §§50 and 81). Pains were taken to stress that such modelings were regarded as epistemologically rather than ontologically based. One is sometimes told (whether by informants, nature, theory, intuition, whatever) that *A* is both true and false, and at other times one has no information at all regarding *A*'s truth or falsity. And yet in fact presumably *A* is precisely true or precisely false. It is remarked in §50.3 that one was able to capture the first degree implications of **RM** by basically considering only those "ambivalent" models in which sentences were always assigned at least one truth value. We here extend this observation, arguing in effect that **RM** is the logic appropriate to reasoning in a situation of complete but not necessarily consistent information.

In §50 below these ambivalent models are essentially "static." No account has been taken of change in information concerning *A* over time. Now Kripke 1965a and Grzegorczyk 1964, independently, developed a semantics for intuitionistic logic which can be thought of as "dynamic." Since we are using Kripke's model structures, it is natural to talk in terms of them, but sometimes we borrow a particularly vivid image from Grzegorczyk's motivations.

The rough idea is that the members of K are evidential situations (G being the actual situation) and that the accessibility relation R is to be understood as the relation of possible extension of one evidential situation so as to obtain another. The Kripke model structures for intuitionism require R to be reflexive and transitive, but not necessarily connected or antisymmetric. The last requirement could have been made with no harm; however, connectedness would give rise to a semantics for Dummett's **LC**, an extension of the intuitionist logic (see Segerberg 1968). This is pleasant, since **LC** can be translated into **RM** (see Dunn and Meyer 1971), and there must be some connection there. Once R is connected, there seems no reason not to think of it simply as the relation of temporal priority.

There is in the Kripke-Grzegorczyk semantics an asymmetrical treatment of truth and falsity. Thus Kripke 1965a, p. 98, says: "But $\varphi(A, \mathbf{H}) = \mathrm{F}$ does not mean that *A* has been proved *false* at **H**. It simply is not (yet) proved at **H**, but may be established later." Grzegorczyk 1964, who seems to have at the base of his motivations the idea that the atomic sentences are something like observation sentences, rings a more philosophical note when he says (p. 596): "The compound sentences are not a product of experiment, they arise from reasoning. This concerns also negations: we see that the lemon is yellow, we do not see that it is not blue."

Now there need ultimately be nothing wrong with such a preferred treatment of truth, and indeed it seems consonant with the original motivations of intuitionism. But the semantics we are presenting here is more even-handed in its treatment of truth and falsity. It takes a more "positive" stance toward falsity. (Perhaps, contra Grzegorczyk, we do after all see that the lemon is not blue.) In this it is quite similar to the Thomason 1969 study of constructible falsity. The Kripke-Grzegorczyk semantics makes its prejudice in favor of truth formally explicit in that it requires that, once a sentence is true in an evidential situation, it remain true in all later evidential situations, but the corresponding requirement is not made for falsity. Thomason does make the same requirement for falsity as for truth, and we do so also. It is obvious that this cannot be done while working with models that give each sentence precisely one truth value (as do Kripke's models) without the models' degenerating into what are in effect static models; for all the evidential situations would be indistinguishable in terms of which sentences they established. Thomason works with models in which some sentences have no truth value, whereas we are working the other side of the street.

The idea that sentences can be valued as simultaneously both true and false is admittedly rather odd. The reader wanting motivation should consult §50.2. Incidentally, K. Pledger has suggested privately that our motivation is unduly pessimistic, since the Hereditary condition has things getting more and more contradictory as time goes on *if* one regards HRH′ as indicating that the evidential situation H temporally *precedes* the situation H′. But Pledger suggests that the temporal order of the accessibility relation should be thought of optimistically in the reverse order. Thus one starts with a situation in which many sentences (for all one knows) are just as much true as false, and then one improves on this situation as time goes on by accumulating evidence that occasionally decides things one way or the other.

§49.1.3. Semantical soundness. We shall draw much of our terminology from the numbered definitions of §42; since almost all those definitions will be used, it might be worth while for the reader to review them. For now, we recall that an **RM**-*theory* (Def. 2) is closed under adjunction and *modus ponens* for implications in **RM** and that such a theory is **RM**-*containing* (Def. 3) if it contains every theorem of **RM**. And, following §48.5, we say that A is *Officially* **RM**-*derivable* from a set S of sentences if A belongs to every **RM**-containing **RM**-theory that contains S; i.e., if A can be obtained from S and the theorems of **RM** by adjunction and *modus ponens* for **RM**-implications. We write

$$\mathrm{S} \vdash_{\mathbf{RM}} A$$

for **RM**-derivability. Further, noting that we shall be dealing almost entirely with **RM**-containing **RM**-theories, when T is such it is convenient—and not misleading—to write

$$\vdash_{\mathrm{T}} A \text{ iff } \mathrm{T} \vdash_{\mathbf{RM}} A \text{ iff } A \in \mathrm{T}, \text{ and}$$
$$\mathrm{S} \vdash_{\mathrm{T}} A \text{ iff } \mathrm{T} \cup \mathrm{S} \vdash_{\mathbf{RM}} A.$$

We can now state the

SEMANTICAL SOUNDNESS THEOREM. If $S \vdash_{\mathbf{RM}} A$ then $S \vDash_{\mathbf{RM}} A$.

PROOF. The only rules of **RM** are modus ponens and adjunction. Both of these rather obviously preserve truth. (For the former, look at $(\to \boldsymbol{T})$ and recall that R is reflexive; for the latter, just look at $(\& \boldsymbol{T})$.) Thus the proof reduces to verification of axioms (given in §R2). This is even more tedious than usual because of the "double-entry bookkeeping" needed because of clause (ii) in $(\to \boldsymbol{T})$.

We verify first the characteristic **RM** axiom RM0: $A \to (A \to A)$. In checking for $\boldsymbol{T} \in \varphi(A \to (A \to A), \mathrm{G})$ it suffices to show that if GRH then

(i) $\boldsymbol{T} \in \varphi(A, \mathrm{H})$ only if $\boldsymbol{T} \in \varphi(A \to A, \mathrm{H})$, and
(ii) $\boldsymbol{F} \in \varphi(A \to A, \mathrm{H})$ only if $\boldsymbol{F} \in \varphi(A, \mathrm{H})$.

Now $\boldsymbol{T} \in \varphi(A \to A, \mathrm{H})$ can easily be seen to hold, since it boils down to the tautology that $\boldsymbol{T} \in \varphi(A, \mathrm{H}')$ only if $\boldsymbol{T} \in \varphi(A, \mathrm{H}')$, and the same thing for $\boldsymbol{F}$. So (i) is trivially true by virtue of a true consequent. As for (ii), it is easy to see that, since always $\boldsymbol{T} \in \varphi(A \to A, \mathrm{H})$, if $\boldsymbol{F} \in \varphi(A \to A, \mathrm{H})$ this must be because of $\to \boldsymbol{F}$ (ii). So we have $\boldsymbol{F} \in \varphi(A, \mathrm{H})$.

REMARK. Note that neither the linearity of the accessibility relation nor the assumption that each sentence is either true or false was used in the verification of the characteristic **RM** axiom. And yet adding that axiom to **R** produces $\vdash_{\mathbf{RM}} (A \to B) \vee (B \to A)$, verification of which seems to require both assumptions.

The verification of the other axioms is left to the reader. The following will be extremely useful for that purpose:

HEREDITARY LEMMA. For any sentence A, if HRH' then $\varphi(A, \mathrm{H}) \subseteq \varphi(A, \mathrm{H}')$.

PROOF is by straightforward induction on the length of A. The only case to give any pause is when $A = B \to C$ and $\boldsymbol{F} \in \varphi(A, \mathrm{H})$. There are three subcases (note that in 1 and 2 we use linearity of **R**—only there does linearity enter on the side of soundness):

Subcase 1. $\exists \mathrm{H}_1$; so HRH_1 and $\boldsymbol{T} \in \varphi(B, \mathrm{H}_1)$ and $\boldsymbol{T} \notin \varphi(C, \mathrm{H}_1)$. Either $\mathrm{H}'\mathrm{RH}_1$ or $\mathrm{H}_1\mathrm{RH}'$. If the first, then, by $\to \boldsymbol{T}$, clearly $\boldsymbol{T} \in \varphi(B \to C, \mathrm{H}')$ and so, by $\to \boldsymbol{F}$(i), $\boldsymbol{F} \in \varphi(B \to C, \mathrm{H}')$. If the second, then $\boldsymbol{T} \in \varphi(B, \mathrm{H}')$ by inductive hypothesis, since $T \in \varphi(B, \mathrm{H}_1)$. Also, since $T \notin \varphi(C, \mathrm{H}_1)$, $F \in \varphi(C, \mathrm{H}_1)$, and, again by inductive hypothesis, $F \in \varphi(C, \mathrm{H}')$. But, since $T \in \varphi(B, \mathrm{H}')$ and $F \in \varphi(C, \mathrm{H}')$, by $\rightarrow F$(ii), $F \in \varphi(B \rightarrow C, \mathrm{H}')$.

Subcase 2. $\exists \mathrm{H}_1$; so HRH_1 and $F \in \varphi(C, \mathrm{H}_1)$ and $F \notin \varphi(\mathrm{B}, \mathrm{H}_1)$. Argued symmetrically to subcase 1.

Subcase 3. $T \in \varphi(B, \mathrm{H})$ and $F \in \varphi(C, \mathrm{H})$. Then, by inductive hypothesis, $T \in \varphi(B, \mathrm{H}')$ and $F \in \varphi(C, \mathrm{H}')$, and so by $\rightarrow F$(ii), $F \in \varphi(B \rightarrow C, \mathrm{H}')$.

§49.1.4. Semantical completeness. We first define the requisite notions, recalling from Defs. 4 and 5 of §42.1 that an **RM**-theory is *prime* if, whenever it contains $A \vee B$, it also contains either A or B. Let T_0 be a prime **RM**-containing **RM**-theory. We define the *canonical model structure* determined by T_0 to be $(\mathrm{G}_{\mathrm{T}_0}, \mathrm{K}_{\mathrm{T}_0}, \mathrm{R}_{\mathrm{T}_0})$, where $\mathrm{G}_{\mathrm{T}_0} = \mathrm{T}_0$, $\mathrm{K}_{\mathrm{T}_0}$ is the set of all prime T_0-containing **RM**-theories, and $\mathrm{R}_{\mathrm{T}_0}$ is the subset relation on $\mathrm{K}_{\mathrm{T}_0}$. We remark that members of $\mathrm{K}_{\mathrm{T}_0}$ are also **RM**-containing T_0-theories. The *canonical model* determined by T_0, φ_{T_0}, is then defined on this model structure so that (i) $T \in \varphi_{\mathrm{T}_0}(p, \mathrm{T})$ iff $p \in \mathrm{T}$, and (ii) $F \in \varphi_{\mathrm{T}_0}(p, \mathrm{T})$ iff $\sim p \in \mathrm{T}$.

We next prove a series of lemmas.

LEMMA 1. Let T be an **RM**-containing **RM**-theory not containing A. Then there is a prime T-containing T-theory that also excludes A.

PROOF. Not only is **RM** Up-Down acceptable (Def. 1 of §42.1), but so is every **RM**-containing **RM**-theory. We may therefore apply the Way Up lemma as stated at the end of §42.1 (and proved in §42.2).

LEMMA 2. Let T_0 be a prime **RM**-containing **RM**-theory and let T_1 and T_2 be T_0-theories. Then either $\mathrm{T}_1 \subseteq \mathrm{T}_2$ or $\mathrm{T}_2 \subseteq \mathrm{T}_1$.

PROOF. Suppose for *reductio* that $\vdash_{\mathrm{T}_1} A$ and not $\vdash_{\mathrm{T}_2} A$, while $\vdash_{\mathrm{T}_2} B$ and not $\vdash_{\mathrm{T}_1} B$. Now $\vdash_{\mathbf{RM}} (A \rightarrow B) \vee (B \rightarrow A)$ (RM64 of §29.3.1); hence $\vdash_{\mathrm{T}_0} A \rightarrow B$ or $\vdash_{\mathrm{T}_0} B \rightarrow A$. The former would put B in T_1, and the latter would put A in T_2, contradicting our assumption.

LEMMA 3. Let T be an **RM**-containing **RM**-theory. Then $\vdash_{\mathrm{T}} A \rightarrow B$ iff both $A \vdash_{\mathrm{T}} B$ and $\sim B \vdash_{\mathrm{T}} \sim A$.

PROOF. The implication from left to right is obvious, since **RM**-theories are closed under both *modus ponens* and (hence, by contraposition) *modus tollens*.

In proving the converse implication we find it convenient to suppose that **RM** has been enriched (conservatively, in view of §45.1) with the primitive sentential constant f and its negation $t =_{\mathrm{df}} f \to f$. We note first that, as a substitution instance of the characteristic **RM** axiom scheme, we have $f \to (f \to f)$, i.e., $f \to t$. Also, t is useful in stating the following, which follows immediately from the corresponding result for **R** in Meyer, Dunn, and Leblanc 1974.

DEDUCTION THEOREM FOR **RM**. Let S be a set of sentences and let T be an **RM**-containing **RM**-theory. Then, if S, $A \vdash_{\mathrm{T}} B$ then $\mathrm{S} \vdash_{\mathrm{T}} A \& t \to B$.

This Deduction theorem is used in the following outline of the right-to-left part of Lemma 3:

1	$A \vdash_{\mathrm{T}} B$	Assumption
2	$\sim B \vdash_{\mathrm{T}} \sim A$	Assumption
3	$\vdash_{\mathrm{T}} A \& t \to B$	1 Deduction theorem
4	$\vdash_{\mathrm{T}} \sim B \& t \to \sim A$	2 Deduction theorem
5	$\vdash_{\mathrm{T}} A \to .\ B \vee f$	4 Contraposition, Double negation, De Morgan, Disjunction, Transitivity
6	$\vdash_{\mathrm{T}} A \to .\ B \vee t$	5 $\vdash_{\mathrm{RM}} f \to t$, Identity, Disjunction, Transitivity
7	$\vdash_{\mathrm{T}} A \to .\ A \& (B \vee t)$	6 Identity, Conjunction
8	$\vdash_{\mathrm{T}} A \to .\ (A \& B) \vee (A \& t)$	7 Distribution, Transitivity
9	$\vdash_{\mathrm{T}} A \to B$	8 Simplification, 3 Disjunction, Transitivity

REMARK. Note that $A \& t \to B$ is enthymematic implication in the form due to Meyer 1973 (see §36.2). Lemma 3 can also be viewed as a kind of deduction theorem (see §49.1.4).

LEMMA 4. $\vdash_{\mathrm{T}} A \to B$ iff, for all prime **RM**-theories T′ such that $\mathrm{T} \subseteq \mathrm{T}'$, $\vdash_{\mathrm{T}'} A$ only if $\vdash_{\mathrm{T}'} B$, and $\vdash_{\mathrm{T}'} \sim B$ only if $\vdash_{\mathrm{T}'} \sim A$; where T is any **RM**-containing **RM**-theory.

PROOF. Left-to-right is immediate. For right-to-left we argue contrapositively, assuming not $\vdash_{\mathrm{T}} A \to B$. Then, by Lemma 3, we know that either not $A \vdash_{\mathrm{T}} B$ or not $\sim B \vdash_{\mathrm{T}} \sim A$. Let $\mathrm{T}+A$ be the smallest **RM**-theory including $\mathrm{T} \cup \{A\}$, and similarly for $\mathrm{T} + \sim B$. Quite clearly either B is not a theorem of $\mathrm{T}+A$ or $\sim A$ is not a theorem of $\mathrm{T} + \sim B$. Using Lemma 1, either $\mathrm{T}+A$ can be extended to a prime **RM**-theory T′ so that not $\vdash_{\mathrm{T}'} B$, or $\mathrm{T}+B$ can be extended to a prime **RM**-theory T$''$ so that not $\vdash_{\mathrm{T}''} {\sim}A$, thus completing the proof.

Lemma 5. Let φ be the canonical model determined by some prime **RM**-containing **RM**-theory T_0. Then, for all sentences A of **RM** and all prime **RM**-theories $\mathrm{T} \supseteq \mathrm{T}_0$, (i) $\boldsymbol{T} \in \varphi(A, \mathrm{T})$ iff $\vdash_{\mathrm{T}} A$, and (ii) $\boldsymbol{F} \in \varphi(A, \mathrm{T})$ iff $\vdash_{\mathrm{T}} {\sim}A$.

Proof. By induction on the length of A.

Case 0. A is a sentential variable. The result holds by definition.

Case 1. $A = {\sim}B$. (i) $\boldsymbol{T} \in \varphi({\sim}B, \mathrm{T})$ iff $\boldsymbol{F} \in \varphi(B, \mathrm{T})$ iff (inductive hypothesis) ${\sim}B \in \mathrm{T}$. (ii) $\boldsymbol{F} \in \varphi({\sim}B, \mathrm{T})$ iff $\boldsymbol{T} \in \varphi(B, \mathrm{T})$ iff (inductive hypothesis) $\vdash_{\mathrm{T}} B$ iff (double negation) $\vdash_{\mathrm{T}} {\sim}{\sim}B$.

Case 2. $A = B\&C$. (i) $\boldsymbol{T} \in \varphi(B\&C, \mathrm{T})$ iff both $\boldsymbol{T} \in \varphi(B, \mathrm{T})$ and $\boldsymbol{T} \in \varphi(C, \mathrm{T})$ iff (inductive hypothesis) both $\vdash_{\mathrm{T}} B$ and $\vdash_{\mathrm{T}} C$ iff (simplification, adjunction) $\vdash_{\mathrm{T}} B\&C$. (ii) $\boldsymbol{F} \in \varphi(B\&C, \mathrm{T})$ iff $\boldsymbol{F} \in \varphi(B, \mathrm{T})$ or $\boldsymbol{F} \in \varphi(C, \mathrm{T})$ iff (inductive hypothesis) $\vdash_{\mathrm{T}} {\sim}B$ or $\vdash_{\mathrm{T}} {\sim}C$ iff (addition, primeness) $\vdash_{\mathrm{T}} {\sim}B \vee {\sim}C$ iff (De Morgan) $\vdash_{\mathrm{T}} {\sim}(B\&C)$.

Case 3. $A = B \vee C$. Similar to Case 2 (alternatively, dispense with $\vee$ as primitive, setting $A \vee B =_{\mathrm{df}} {\sim}({\sim}B\&{\sim}C)$).

Case 4. $A = B \rightarrow C$. Upon invoking the clause for $\rightarrow$ in the inductive definition of φ and applying the inductive hypothesis, (i) is just Lemma 4.

For (ii), we divide the equivalence into halves. Arguing left to right, let $\boldsymbol{F} \in \varphi(B \rightarrow C, \mathrm{T})$. Then either $\boldsymbol{T} \notin \varphi(B \rightarrow C, \mathrm{T})$ or else $\boldsymbol{T} \in \varphi(B, \mathrm{T})$ while $\boldsymbol{F} \in \varphi(C, \mathrm{T})$. In the first case, not $\vdash_{\mathrm{T}} B \rightarrow C$ (by (i), just established). But then $\vdash_{\mathrm{T}} {\sim}(B \rightarrow C)$ (excluded middle, primeness). In the second case, $\vdash_{\mathrm{T}} B$ and $\vdash_{\mathrm{T}} {\sim}C$ (inductive hypothesis). But then again $\vdash_{\mathrm{T}} {\sim}(B \rightarrow C)$, since $\vdash_{\mathbf{RM}} B\&{\sim}C \rightarrow. {\sim}(B \rightarrow C)$.

For right to left, suppose $\vdash_{\mathrm{T}} {\sim}(B \rightarrow C)$. We are to show that either $\boldsymbol{T} \notin \varphi(B \rightarrow C, \mathrm{T})$ or else $\boldsymbol{T} \in \varphi(B, \mathrm{T})$ while $\boldsymbol{F} \in \varphi(C, \mathrm{T})$. We have already argued (i); so the first alternative amounts to not $\vdash_{\mathrm{T}} B \rightarrow C$ whereas, by inductive hypothesis, the second alternative just amounts to $\vdash_{\mathrm{T}} B$ and $\vdash_{\mathrm{T}} {\sim}C$. It thus suffices to further assume $\vdash_{\mathrm{T}} B \rightarrow C$ for the purpose of showing $\vdash_{\mathrm{T}} B$ and $\vdash_{\mathrm{T}} {\sim}C$. Since $\vdash_{\mathbf{RM}} B \rightarrow C \rightarrow. {\sim}B \vee C \rightarrow. B \rightarrow C$ (see 2 in proof of Embedding theorem in §29.4), by *modus ponens*, $\vdash_{\mathrm{T}} {\sim}B \vee C \rightarrow. B \rightarrow C$. Then, by *modus tollens*, $\vdash_{\mathrm{T}} {\sim}({\sim}B \vee C)$. But then, by De Morgan, simplification, and double negation, $\vdash_{\mathrm{T}} B$ and $\vdash_{\mathrm{T}} {\sim}C$, as desired.

We can now prove the

Semantical Completeness theorem. If $\mathrm{S} \vDash_{\mathbf{RM}} A$ then $\mathrm{S} \vdash_{\mathbf{RM}} A$.

PROOF proceeds contrapositively. If not $S \vdash_{\mathbf{RM}} A$ then, by Lemma 1, there is a prime **RM**-theory T_0 with $S \subseteq T_0$ and not $\vdash_{T_0} A$ (letting the T of Lemma 1 be the smallest **RM**-containing **RM**-theory including S). Consider the canonical model φ_{T_0}.

Lemma 2 then contains the only nonevident information needed to see that the canonical model determined by a prime **RM**-theory is indeed an **RM**-model, and Lemma 5 assures that, for all $B \in S$, $\boldsymbol{T} \in \varphi_{T_0}(B, T_0)$, whereas $\boldsymbol{T} \notin \varphi_{T_0}(A, T_0)$.

§49.1.5. Decidability by filtration. That **RM** has the finite model property and is decidable was first shown by Meyer, using matrix methods (Corollary 3.2 of §29.3.2). It is nonetheless of some minimal interest that the method of filtration (see Segerberg 1968) can be applied to the binary Kripke-style semantics. Part of the interest lies in the fact that the ternary semantics of Routley and Meyer has so far resisted filtration (at least for the stronger systems like **E**, **R**, and **RM**).

PROOF proceeds by straightforward modifications of the arguments of Segerberg 1968, and the interested reader can work it out him/herself. To be sure to get started on the right track it may be well to define the essential equivalence relation. Where φ is an **RM**-model on an **RM** model structure (G, K, R) and where ψ is a set of sentences closed under subformulas, define for $H, H' \in K$, $H \equiv \psi H'$ iff $\varphi(B, H) = \varphi(B, H')$ for all $B \in \psi$. Thus the reader should be sensitized to the fact that the indiscernibility of H and H′ with respect to which sentences in ψ they make true does not suffice (as it does in Segerberg 1968) to support the appropriate equivalence. H and H′ here must also be indiscernible with respect to which sentences in ψ they make false.

§49.1.6. RM models and Sugihara matrices. It turns out that there are connections between the present model theory for **RM** and the earlier model theory in terms of Sugihara matrices. We merely sketch the main outlines of these connections, leaving many details of proof and even statement to the interested reader.

We make free use of definitions relating to Sugihara matrices from §29.3–4, but first we take this opportunity to correct an error in Dunn 1970. On page 2 it was intended that a "Sugihara matrix" be a chain with involution $^-$, i.e., that $^-$ be a one-one order-inverting mapping *of period two* of the chain onto itself, with the designated elements and operations specified as indicated. The italicized words were inadvertently omitted, although they were in the abstract Dunn 1968. Acknowledgment is expressed to M. Tokarz for pointing out this error.

It would be quite well known, if we were working on Kripke models for the intuitionist sentential calculus, that we could trade a Kripke model

$\varphi(A, H)$ for its equivalent valuation $\|A\|_\varphi = \{H \in K: \varphi(A, H) = \boldsymbol{T}\}$. We want to do something similar, but, because of the ambivalent nature of **RM** models, we must think of the equivalent valuation as $\|A\|_\varphi = \langle\{H \in K: \boldsymbol{T} \in \varphi(A, H)\}, \{H \in K: \boldsymbol{F} \in \varphi(A, H)\}\rangle$.

The set of all such ordered pairs of subsets of K forms a Sugihara matrix in a natural way. Note that, because of the Hereditary lemma of §49.1.3, each such pair $\langle X_1, X_2\rangle$ is such that both X_1 and X_2 are closed upward under R. This assures that these pairs form a chain under the natural ordering $\langle X_1, X_2\rangle \leq \langle Y_1, Y_2\rangle$ iff $X_1 \subseteq Y_1$ and $Y_2 \subseteq X_2$, and that $\overline{\langle X_1, X_2\rangle} = \langle X_2, X_1\rangle$ is the desired involution. Further, upon designating the pairs $\langle X_1, X_2\rangle$ such that $G \in X_1$ and defining operations on the pairs in the way naturally induced by the clauses for each connective in the inductive definition of φ in §49.1.2, the result is a Sugihara matrix. And the original model φ is *equivalent* to the valuation $v(A) = \|A\|_\varphi$ in this Sugihara matrix in the obvious sense that $\boldsymbol{T} \in \varphi(A, G)$ iff $v(A)$ is designated.

Conversely, every valuation in a Sugihara matrix is equivalent to some **RM**-model. This may be shown by a construction somewhat akin to that used in our completeness result, substituting prime filters of a certain kind for the prime theories. Note, incidentally, that, since Sugihara matrices are chains, all filters are prime filters.

Thus, where M is a Sugihara matrix with D its set of designated elements, let $G = D$, $K = \{F: F$ is a proper filter of M with $D \subseteq F\}$, and let FRF' iff $F \subseteq F'$. It is easy to see that (G, K, R) is an **RM** model structure. Further, if v is a valuation in M, define φ_v so that, for each sentential variable p, (i) $\boldsymbol{T} \in \varphi_v(p, F)$ iff $v(p) \in F$, and (ii) $\boldsymbol{F} \in \varphi_v(p, F)$ iff $\overline{v(p)} \in F$. An induction shows that (i) and (ii) hold for complex sentences as well, and (i) then has as a special case the desired statement of the equivalence of φ_v with v, i.e., $\boldsymbol{T} \in \varphi_v(A, G)$ iff $v(A) \in D$.

§49.1.7. The binary semantics with "star operation".

In §49.1.2 we made a comparison of our semantics for **RM** with Thomason's semantics for constructible falsity on the issue of "gluts" versus "gaps." It turns out that Routley 1974a contains an alternative semantical analysis of constructible falsity using a "star operation" in place of the Thomason device of truth-value gaps. Since in the context of the ternary semantics a star operation is producing both gaps and gluts (see §48.5), it seems plausible that we could employ a star operation in our binary semantics for **RM** in place of our truth-value gluts; i.e., we could let $\varphi(A, H)$ be a single one of the truth values $\boldsymbol{T}$ and $\boldsymbol{F}$ rather than any nonempty subset of them. This, indeed, turns out to be the case, and it seems just worth setting down. We were helped in this by discussions with each other, mostly carried out while riding a bus. There must be several puzzled riders of the number 15 omnibus in Canberra who wondered at the intense conversation regarding "positive and negative worlds." (The reader may well wish to read §§48.2 and 48.5 first before continuing, since many of the ideas of this subsection are cribbed from there, and we shall be brief.)

We define an **RM** model structure* to be an ordered quadruple (G, K, R, *), where (G, K, R) is an **RM** model structure in the sense of §49.1.1 (but not requiring that G be the R-least element of K) and * is a unary function on K satisfying

(1) G^*RG;
(2) $H^{**} = H$; and
(3) H_1RH_2 only if $H_2^*RH_1^*$.

We introduce the notations K^+ for $\{H \in K\colon H^*RH\}$ and K^- for $K - K^+$ (these are, respectively, the "positive and negative worlds" of the anecdote). We note that $G \in K^+$ in virtue of (1).

An **RM** *model** on such an **RM** model structure* is a function $\varphi(p, H)$ assigning to each sentential variable p precisely one of the truth values $\boldsymbol{T}$ and $\boldsymbol{F}$ at each $H \in K$. We also require the

HEREDITARY CONDITION*. For all $H, H' \in K$, if HRH' and $\varphi(p, H) = \boldsymbol{T}$ then $\varphi(p, H') = \boldsymbol{T}$.

We then extend φ inductively to complex sentences as follows:

($\sim$*) $\varphi({\sim}A, H) = \boldsymbol{T}$ iff $\varphi(A, H^*) = \boldsymbol{F}$.
(&*) $\varphi(A\&B, H) = \boldsymbol{T}$ iff $\varphi(A, H) = \boldsymbol{T}$ and $\varphi(B, H) = \boldsymbol{T}$.
($\vee$*) $\varphi(A\vee B, H) = \boldsymbol{T}$ iff $\varphi(A, H) = \boldsymbol{T}$ or $\varphi(B, H) = \boldsymbol{T}$.
($\rightarrow$*+) For $H \in K^+$, $\varphi(A\rightarrow B, H) = \boldsymbol{T}$ iff for all H' such that HRH',
  (i) $\varphi(A, H') = \boldsymbol{T}$ only if $\varphi(B, H') = \boldsymbol{T}$, and
  (ii) $\varphi(A, H'^*) = \boldsymbol{T}$ only if $\varphi(B, H'^*) = \boldsymbol{T}$.
($\rightarrow$*−) For $H \in K^-$, $\varphi(A\rightarrow B, H) = \boldsymbol{T}$ iff
  (i) $\varphi(A\rightarrow B, H^*) = \boldsymbol{T}$ and
  (ii) $\varphi(A, H^*) = \boldsymbol{T}$ only if $\varphi(B, H) = \boldsymbol{T}$.

At least a few comments are in order. First, note that, by virtue of linearity and (2), we have that, for each $H \in K$, if $H \in K^-$ then $H^* \in K^+$. This removes any appearance of vicious circularity in clause (i) of ($\rightarrow$*−), since the truth of $A\rightarrow B$ at H^* is thus referred back to the condition ($\rightarrow$*+).

Second, note that ($\rightarrow$*+) and ($\rightarrow$*−) are best understood as the vestigial remains of the double-entry bookkeeping needed for our ambivalent semantics. The reader might well re-express ($\rightarrow$*−) as a condition for $(A\rightarrow B, H) =$

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* We intend that "*" be read "star." There are of course no footnotes in this book.

F and then compare it with the ($\rightarrow F$) condition of the ambivalent semantics.

Third, the key to the translation in all this is to read $\boldsymbol{T} \in \varphi(A, \mathrm{H})$ in the ambivalent semantics as $\varphi(A, \mathrm{H}) = \boldsymbol{T}$ in the semantics* and $\boldsymbol{F} \in \varphi(A, \mathrm{H})$ as $\varphi({\sim}A, \mathrm{H}) = \boldsymbol{T}$. The star operation cunningly provides for the possibility that, for some A and H, both $\varphi(A, \mathrm{H}) = \boldsymbol{T}$ and $\varphi({\sim}A, \mathrm{H}) = \boldsymbol{T}$, which corresponds in the ambivalent semantics to both $\boldsymbol{T}, \boldsymbol{F} \in \varphi(A, \mathrm{H})$.

Fourth, the completeness proof goes by letting into the canonical model structure prime **RM**-theories that are not **RM**-containing. We are forced into this by the fact that, where T is an **RM**-containing prime **RM**-theory, T* (defined as in §48.5 as $\{A\colon {\sim}A \notin \mathrm{T}\}$) is not usually an **RM**-containing prime **RM**-theory. Indeed, where T has gluts (both A and ${\sim}A$ theorems), T* will have gaps (neither A nor ${\sim}A$ theorems). Indeed, T* is **RM**-containing only in the "extreme" case that T is a truthlike **RM**-theory and hence is negation-consistent as well as negation-complete, and then T* = T. Being specific, where T is a prime **RM**-containing **RM**-theory, we define *the canonical model structure* determined by* T to be $(\mathrm{G_T}, \mathrm{K_T}, \mathrm{R_T}, {*}_\mathrm{T})$, where $\mathrm{G_T} = \mathrm{T}$, $\mathrm{R_T}$ is subset, ${*}_\mathrm{T}$ is defined as above, and $\mathrm{K_T} = \{\mathrm{T}'\colon \mathrm{T}'$ is a prime **RM**-theory and $\mathrm{T} \subseteq \mathrm{T}'\} \cup \{\mathrm{T}'^*\colon \mathrm{T}'$ is a prime **RM**-theory and $\mathrm{T} \subseteq \mathrm{T}'\}$. The canonical model determined by T, φ_T, is then defined on this structure so that $\varphi_\mathrm{T}(p, \mathrm{T}') = \boldsymbol{T}$ iff $p \in \mathrm{T}'$.

We leave to the reader all the rest of the details of statement and proof relating to completeness and soundness of **RM** in the semantics*.

§49.1.8. Limitations of the binary semantics. It would be nice to report something like the result that dropping the assumptions that R is connected and that $\varphi(A, \mathrm{H})$ is nonempty would yield the system **R** of relevant implication. This seems plausible at first. Thus, dropping connectedness is the way to weaken Dummett's **LC** to Heyting's intuitionist logic (whose Meyer translation into **R** is the same as **LC**'s into **RM**; see Dunn and Meyer 1971). Further, dropping the assumption that $\varphi(A, \mathrm{H})$ is nonempty is the way to characterize the first degree implications of **R**, as in §50 below.

Unfortunately, however, such a report would be stunningly false, for the characteristic **RM** axiom $A \rightarrow (A \rightarrow A)$ is very difficult to reject on any reasonable variation of the binary semantics (see the remark at the end of its verification in §49.1.3). Indeed, the real break-through of the ternary semantics for **R** of Routley and Meyer (§48) is that it provides a way of making $A \rightarrow A$ not always true (even when A is true), which is the honest way to reject $A \rightarrow (A \rightarrow A)$.

But it would be of at least some small formal interest to investigate the sentential calculuses, unfamiliar as they might be, which arise by weakening requirements on an **RM** model in various ways, including the two mentioned above.

§49.2. Quantification and RM.

In §48.9 we discussed the following question: Is the calculus $\mathbf{R}^{\forall\exists x}$ of relevant implication with quantifiers (§38) complete in the semantics (explained in §48.9) derived by adding simple-minded clauses for the quantifiers to the underlying propositional semantics (of §48.3 and §48.5) for **R**?

We here give to an analogous question for **RM** an affirmative answer: the calculus $\mathbf{RM}^{\forall x}$ defined in §38 by adding a standard batch of quantificational postulates to the propositional calculus **RM** of §R2 is indeed complete with respect to adding to the *binary* semantics for **RM** of §49.1 some simple-minded clauses for the quantifiers.

§49.2.1. Grammar and proof theory of $\mathbf{RM}^{\forall x}$.

The language of $\mathbf{RM}^{\forall x}$ is as described in §38.1, except that disjunction ($\vee$) is defined in the usual De Morgan manner from conjunction (&), and the existential quantifier ($\exists$) is defined from the universal ($\forall$) in the equally standard way. The primitives are thus implication, conjunction, negation, and the universal quantifier. We will be making the standard distinction between parameters (a, b, c, etc.) and variables (x, y, z, etc.), with only the latter bindable by quantifiers, as in §38.1. A *sentence* is a formula with no free variables.

As in §R1 and §38.2, the proof theory of $\mathbf{RM}^{\forall x}$ is like that of $\mathbf{R}^{\forall\exists x}$, except that the "mingle" axiom scheme

RM0 $\quad A\rightarrow.A\rightarrow A$

is to be inserted at the end of the list of axioms R1–R13 given in §R2 for **R**. Also, our choice of primitives allows for the deletion of the disjunction axioms R7–9 of §R2 and of the existential quantifier axioms IQ7–9 of §38.2. The reader is reminded that all closures of instances of axioms are also axioms, so that generalization (while admissible) is dispensed with, the only remaining rules being *modus ponens* and adjunction.

We shall be speaking of $\mathbf{RM}^{\forall x}$*-containing* $\mathbf{RM}^{\forall x}$*-theories* as in Defs. 2 and 3 of §42.1, and we shall say that some of them are *prime* as in Def. 5 of §42.1, reminding the reader that this includes appropriate properties not only for disjunction (as in §49.1) but also for both quantifiers, as given in Def. 4 of §42.1. We recall that a prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theory is almost truth-like (Def. 3), missing only negation consistency. We use the subscripted turnstile exactly as in §49.1.3, except of course now referring to $\mathbf{RM}^{\forall x}$ instead of **RM**.

The propositional constant $\boldsymbol{t}$, intuitively the conjunction of all true sentences, can be added conservatively to **RM** with the axiom scheme $A\leftrightarrows(\boldsymbol{t}\rightarrow A)$ or else defined contextually as in §45.1. Although it is not an official part of $\mathbf{RM}^{\forall x}$, we shall not hesitate to make use of $\boldsymbol{t}$ when convenient, and we do so in the definition:

$A \supset B =_{\mathrm{df}} A\&t \to B$ *(enthymematic implication).*

The idea of "enthymematic implication" comes from §§35 and 36, but this particular form derives from Meyer 1973, where it is seen to have in **R** the behavior of intuitionist implication. Indeed, the superintuitionistic system **LC** of Dummett 1959 may be translated into **RM** using that definition of implication (see Dunn and Meyer 1971).

The following carry over directly from §49.1.4:

Deduction theorem, first version. Let Δ be an $\mathbf{RM}^{\forall x}$-theory, Θ a set of sentences, and A, B sentences. Then

$\Theta, A \vdash_{\Delta} B$ iff $\Theta \vdash_{\Delta} A \supset B$.

Deduction theorem, second version. With Δ, Θ, A, and B as above,

($\Theta, A \vdash_{\Delta} B$ and $\Theta, {\sim}B \vdash_{\Delta} {\sim}A$) iff $\Theta \vdash_{\Delta} A \to B$.

§49.2.2. Semantics. **RM** *model structures* were introduced in §49.1 as structures (G, K, R), where K is a nonempty set, R is a (weak) linear order on K, and G is the R-least member of K. For a *constant-domain* model structure we add a nonempty set D (the domain) so as to obtain a structure (G, K, R, D).

In the sequel, we adopt for convenience a "quasi-substitutional" interpretation of the quantifiers of the sort favored by A. Robinson, R. Smullyan, and others for classical first-order logic. This uses the notion of a U-*sentence* ($\mathrm{U} \neq \varnothing$), which is exactly like a sentence except that actual elements of U have been substituted for some or all occurrences of real variables. (Since, on the abstract approach to language common to most recent logical work, "symbols" can just as well be shoes and ships and sealing wax (or prime numbers for that matter) as marks on paper, U-sentences are just sentences in the linguistic extension of the given language by the set of "parameters" U.)

An $\mathbf{RM}^{\forall x}$-*model* will be a structure (G, K, R, D, φ), where (G, K, R, D) is a constant-domain model structure and φ (the *valuation*) is a function assigning to each parameter a member of D and to each atomic D-sentence, relative to a member of K, a nonempty subset of $\{\boldsymbol{T}, \boldsymbol{F}\}$.

The obvious "Rose-by-any-other-name" requirement is made that, if P is an atomic D-sentence and $P(\varphi)$ is the result of replacing each parameter a of P in every occurrence by $\varphi(a)$, then $\varphi(P, \mathrm{H}) = \varphi(P(\varphi), \mathrm{H})$ for each $\mathrm{H} \in \mathrm{K}$. Finally, for each atomic D-sentence P, we require the

Hereditary condition. If HRH$'$, then $\varphi(P, \mathrm{H}) \subseteq \varphi(P, \mathrm{H}')$.

We regard φ as extended inductively to compound D-sentences according to the following rules:

$(\sim T)$ $\quad T \in \varphi(\sim A, \mathrm{H})$ iff $F \in \varphi(A, \mathrm{H})$;
$(\sim F)$ $\quad F \in \varphi(\sim A, \mathrm{H})$ iff $T \in \varphi(A, \mathrm{H})$;
$(\wedge T)$ $\quad T \in \varphi(A\&B, \mathrm{H})$ iff $T \in \varphi(A, \mathrm{H})$ and $T \in \varphi(B, \mathrm{H})$;
$(\wedge F)$ $\quad F \in \varphi(A\&B, \mathrm{H})$ iff $F \in \varphi(A, \mathrm{H})$ or $F \in \varphi(B, \mathrm{H})$;
$(\rightarrow T)$ $\quad T \in \varphi(A \rightarrow B, \mathrm{H})$ iff, for all $\mathrm{H}' \in \mathrm{K}$ such that HRH'
(i) $T \in \varphi(A, \mathrm{H}')$ only if $T \in \varphi(B, \mathrm{H}')$, and
(ii) $F \in \varphi(B, \mathrm{H}')$ only if $F \in \varphi(A, \mathrm{H}')$;
$(\forall T)$ $\quad T \in \varphi(\forall x A x, \mathrm{H})$ iff $T \in \varphi(A\alpha, \mathrm{H})$ for all $\alpha \in \mathrm{D}$;
$(\forall F)$ $\quad F \in \varphi(\forall x A x, H)$ iff $F \in \varphi(A\alpha, \mathrm{H})$ for some $\alpha \in \mathrm{D}$.

The following may be verified along the lines of the inductive proof of the corresponding lemma in §49.1.3 (the new case $A = \forall x B x$ may be done by the reader in his mind's eye).

HEREDITARY LEMMA FOR $\mathbf{RM}^{\forall x}$. Let $(\mathrm{G}, \mathrm{K}, \mathrm{R}, \mathrm{D}, \varphi)$ be an $\mathbf{RM}^{\forall x}$-model. For any D-sentence A, if HRH', then $\varphi(A, \mathrm{H}) \subseteq \varphi(A, \mathrm{H}')$.

§49.2.3. Soundness. A sentence A is a *logical consequence* of a set of sentences Δ in $\mathbf{RM}^{\forall x}$ iff, for all $\mathbf{RM}^{\forall x}$ models with first component G and last component φ, $T \in \varphi(B, \mathrm{G})$ for all $B \in \Delta$ implies $T \in \varphi(A, \mathrm{G})$. We write this in symbols as $\Delta \vDash_{\mathbf{RM}^{\forall x}} A$.

The reader may establish the

SOUNDNESS THEOREM FOR $\mathbf{RM}^{\forall x}$. If $\Delta \vdash_{\mathbf{RM}^{\forall x}} A$, then $\Delta \vDash_{\mathbf{RM}^{\forall x}} A$.

In doing so he will find the Hereditary lemma useful and also a lemma extending the "Rose requirement" to compound sentences.

§49.2.4. Completeness of $\mathbf{RM}^{\forall x}$. We make a conscious adaptation of the argument of Gabbay 1974 (which the reader should have at hand) for the completeness of the second-order intuitionistic propositional calculus augmented with PQ5 of §32. Gabbay's argument makes frequent use of the Deduction theorem for intuitionist implication, which just does not hold for $\mathbf{RM}^{\forall x}$-implication. But it does hold for $\mathbf{RM}^{\forall x}$ *enthymematic* implication (our Deduction theorem in the first version), and therein lies the secret of the adaptation. An analysis of Gabbay's argument reveals that only rarely is it important that the implication involved in an application of the Deduction theorem be the primitive implication of the system, and then our Deduction theorem in the second version can be made to serve instead.

Though our argument is adapted from Gabbay, we obtain our terminology mostly from Def. 6 of §42.2. We take over directly the notion of a *pair* (of sets of sentences) and of one pair *extending* another. We modify the concept of $\mathbf{RM}^{\forall x}$-*exclusivity* to refer to the enthymematic $\supset$ instead of $\rightarrow$ (an argument based on → would have been possible). We relativize *exhaustiveness* to the language we are speaking of, and we take over intact the concept of *quantifier-prime*. Lastly, introducing now a concept distinctive to the Gabbay argument, we say that a pair (Δ, Θ) is of *constant domain* in a certain language iff whenever the pair $(\Delta, \Theta\cup\{\forall xBx\})$ is $\mathbf{RM}^{\forall x}$-exclusive then so is $(\Delta, \Theta\cup\{\forall xBx, Ba\})$ for some parameter a of the language.

We prepare by means of a series of lemmas.

Lemma 1. With Δ nonempty, let (Δ, Θ) be an $\mathbf{RM}^{\forall x}$-exclusive pair omitting infinitely many parameters. Then (Δ, Θ) can be extended to an $\mathbf{RM}^{\forall x}$-exclusive, exhaustive, and quantifier-prime pair (Δ^*, Θ^*). Furthermore, Δ^* is a prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theory that is disjoint from Θ^* and, accordingly, disjoint from Θ.

Proof. This is an immediate consequence of the Pair Extension lemma of §42.2; we note that $\mathbf{RM}^{\forall x}$ is Up-Down acceptable in the sense of Def. 1 of §42.1, having adjusted that concept to enthymematic implication. ($\mathbf{RM}^{\forall x}$ is of course Up-Down acceptable also with respect to its arrow; it is just that Lemma 1 implicitly refers to enthymematic implication via our altered notion of $\mathbf{RM}^{\forall x}$-exclusivity.) The only thing that needs to be added to the proof in §42.2 of the Pair Extension lemma is the remark that reliance on enthymematic implication guarantees that Δ^* must be $\mathbf{RM}^{\forall x}$-containing: since $t\rightarrow A$ is a theorem of $\mathbf{RM}^{\forall x}$ whenever A is, every formula enthymematically implies every theorem; so a combination of exhaustiveness and $\mathbf{RM}^{\forall x}$-exclusivity (and the nonemptiness of Δ) is bound to put no theorems in Θ^* and every theorem in Δ^*.

Lemma 2. Suppose that the pair (Δ, Θ) is $\mathbf{RM}^{\forall x}$-exclusive and of constant domain in a certain language and that Θ is finite. Let A, B be two sentences of this language so that $(\Delta', \Theta') = (\Delta\cup\{A\}, \Theta\cup\{B\})$ is $\mathbf{RM}^{\forall x}$-exclusive. Then (Δ', Θ') is of constant domain.

Proof is exactly like that of Gabbay's Lemma 2, but with enthymematic implication in the role of intuitionistic implication.

Lemma 3. Let (Δ, Θ) be a $\mathbf{RM}^{\forall x}$-exclusive, exhaustive, quantifier-prime pair in a certain language, and let $(A\rightarrow B)\in\Theta$. Then at least one of $(\Delta\cup\{A\}, \{B\})$, $(\Delta\cup\{\sim B\}, \{\sim A\})$ is a $\mathbf{RM}^{\forall x}$-exclusive pair of constant domain in the same language.

Proof. By the Deduction theorem (second version) at least one of $(\Delta\cup\{A\}, \{B\})$, $(\Delta\cup\{\sim B\}, \{\sim A\})$ is $\mathbf{RM}^{\forall x}$-exclusive, since otherwise $\Delta \vdash_{\mathbf{RM}^{\forall x}} A\supset B$, contradicting the $\mathbf{RM}^{\forall x}$-exclusivity of (Δ, Θ).

If $(\Delta\cup\{A\}, \{B\})$ is $\mathbf{RM}^{\forall x}$-exclusive, the argument proceeds as for Gabbay's Lemma 3, using enthymematic implication in the definition of the key sentence

$$F' = \forall y(A \supset (B \vee \forall xFx \vee Fy)).$$

If $(\Delta\cup\{\sim B\}, \{\sim A\})$ is the $\mathbf{RM}^{\forall x}$-exclusive one, then there is a precisely parallel argument with $F' = \forall y(\sim B \supset (\sim A \vee \forall xFx \vee Fy))$.

Lemma 4. Let (Δ, Θ) be an $\mathbf{RM}^{\forall x}$-exclusive, exhaustive, quantifier-prime pair. Assume that $F = (A \rightarrow B) \in \Theta$. Then there exists an $\mathbf{RM}^{\forall x}$-exclusive, exhaustive, quantifier-prime pair (Δ_F, Θ_F) such that $\Delta \subseteq \Delta_F$ and either (1) $A \in \Delta_F$, $B \in \Theta_F$, or (2) $\sim B \in \Delta_F$, $\sim A \in \Theta_F$. Furthermore, Δ_F is a prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theory.

Proof. By Lemma 3, at least one of $(\Delta\cup\{A\}, \{B\})$, $(\Delta\cup\{\sim B\}, \{\sim A\})$ is $\mathbf{RM}^{\forall x}$-exclusive and of constant domain. Set one such to be (Δ_0^*, Θ_0^*). Define $(\Delta_{n+1}^*, \Theta_{n+1}^*)$ in precisely the same inductive fashion in which $(\Delta_{n+1}, \Theta_{n+1})$ was defined in the proof of the Pair Extension lemma in §42.2—the very lemma upon which we relied for our proof of Lemma 1.

There is, however, a difference. That lemma requires, and Lemma 1 supplies, a promise that denumerably many parameters will not occur, so that there is no problem about securing quantifier-primeness by adding "witnesses" to existential sentences on the left and "hostile witnesses" to universal sentences on the right, as in §42.2. The hypothesis of Lemma 4, however, makes no such promise, and so we must exploit the constant-domain property instead.

We argue then that if (Δ_n^*, Θ_n^*) is defined, $\mathbf{RM}^{\forall x}$-exclusive, and of constant domain, so is $(\Delta_{n+1}^*, \Theta_{n+1}^*)$. It follows readily from (Δ_n^*, Θ_n^*) being of constant domain that if $B_{n+1} = \forall xFx$, then $(\Delta_{n+1}^*, \Theta_{n+1}^*)$ is defined and $\mathbf{RM}^{\forall x}$-exclusive. The case when $B_{n+1} = \exists xFx$ is more interesting, but is argued precisely like Gabbay's Case (1d) in the proof of his Lemma 4, again putting enthymematic implication in to do the job of intuitionist implication.

That $(\Delta_{n+1}^*, \Theta_{n+1}^*)$ continues to be of constant domain follows from Lemma 2.

Finally, set $\Delta_F = \bigcup_{n\in\omega}\Delta_n^*$, and $\Theta_F = \bigcup_{n\in\omega}\Theta_n^*$. A routine argument as in §42.2 shows that (Δ_F, Θ_F) has all the desired properties of the lemma.

Before stating the last lemma we need a definition. Where Δ is an $\mathbf{RM}^{\forall x}$-theory, the *canonical* $\mathbf{RM}^{\forall x}$ *model determined by* Δ, $(\mathrm{G}_\Delta, \mathrm{K}_\Delta, \mathrm{R}_\Delta, \mathrm{D}_\Delta, \varphi_\Delta)$, is as follows: $\mathrm{G}_\Delta = \Delta$; $\mathrm{K}_\Delta = \{\Delta'\colon \Delta'$ is a prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theory and $\Delta \subseteq \Delta'\}$; $\Delta_1\mathrm{R}_\Delta\Delta_2$ iff $\Delta_1 \subseteq \Delta_2$; $\mathrm{D}_\Delta = \{a\colon a$ is a parameter$\}$; $\varphi_\Delta(a) = a$ for each parameter a; and, for atomic sentences P, (i) $\boldsymbol{T} \in \varphi_\Delta(P, \Delta')$ iff $P \in \Delta'$, and

(ii) $\boldsymbol{F} \in \varphi_\Delta(P, \Delta')$ iff $\sim P \in \Delta'$. That $(G_\Delta, K_\Delta, R_\Delta)$ is an $\mathbf{RM}^{\forall x}$-model structure follows directly from Lemma 2 of §49.1.4.

LEMMA 5. Let Δ be a prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theory. Then, in the canonical $\mathbf{RM}^{\forall x}$-model determined by Δ, (i) $\boldsymbol{T} \in \varphi_\Delta(A, \Delta')$ iff $A \in \Delta'$, and (ii) $\boldsymbol{F} \in \varphi_\Delta(A, \Delta')$ iff $\sim A \in \Delta'$.

PROOF. By induction on the length of A, much as in Lemma 5 of §49.1.4; but the case $A = B \rightarrow C$ deserves some new attention, and of course there is the completely new but routine case $A = \forall x Bx$.

As for the first, we need to show that, for prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theories Δ, $B \rightarrow C \in \Delta$ iff, for all prime $\mathbf{RM}^{\forall x}$-containing $\mathbf{RM}^{\forall x}$-theories $\Delta' \supseteq \Delta$, $B \in \Delta'$ only if $C \in \Delta'$, and $\sim C \in \Delta'$ only if $\sim B \in \Delta'$. We showed the analogous thing in §49.1.4 with "**RM**" in place of "$\mathbf{RM}^{\forall x}$", but we observe that "prime" has changed its meaning, by Def. 5 of §42.1: before, it referred only to the sense of primeness appropriate for disjunction, but now it refers also to appropriate properties of the quantifiers. No matter: the gut of what we want is just our Lemma 4.

THEOREM. $\mathbf{RM}^{\forall x}$ is complete.

PROOF. Suppose not $\Delta \vdash_{\mathbf{RM}^{\forall x}} A$. We may assume that infinitely many parameters occur in neither Δ nor A; since both derivability and consequence are invariant under permutation of parameters, this loses no generality. It also costs nothing to assume that $\boldsymbol{t}$ is in Δ. Now apply Lemma 1 to $(\Delta, \{A\})$ to obtain an $\mathbf{RM}^{\forall x}$-containing prime $\mathbf{RM}^{\forall x}$-theory Δ^* excluding A, and consider the canonical $\mathbf{RM}^{\forall x}$-model determined by Δ^*; by Lemma 5 (since $\Delta \subseteq \Delta^*$ and $A \notin \Delta^*$), not $\Delta \vDash_{\mathbf{RM}^{\forall x}} A$.

COROLLARIES. The usual corollaries, e.g. the Compactness theorem and the Löwenheim-Skolem theorem, follow in the usual ways.

§50. Intuitive semantics for first degree entailments and "coupled trees". Classically, an argument: A therefore B, is "valid" (or A is said to "entail" B) iff each situation (model) is such that either A is false or B is true. This fits well with so-called "tableau" methods for showing that A entails B by working out the mutual inconsistency of A and $\sim B$. But both the classical notion of validity and the corresponding tableau methods allow that A may entail B because of some feature of A alone, irrespective of B, and *vice versa.* Thus, if A is a contradiction, each situation is such that A is false and so *a fortiori* is such that A is false or B is true. And, if A is a contradiction, when a tableau construction will show that A is inconsistent and so *a fortiori* that

A and $\sim B$ are inconsistent. Of course, the same points can be made dually when B is a logical truth.

The competing theory of "entailment" developed in this book requires that for A to entail B there must be some relation of real relevance between A and B, e.g., they share some sentence letter. In the present section we shall develop a notion of "relevant validity" and a corresponding tableau method that tie in with this theory.

We remark that it is only through circumstances that this section appears in the second rather than the first volume of this book, for some of its ideas date back to the sixties and all of it was worked out before the completion of Volume I or the advent of the operational and relational semantics described elsewhere in this chapter.

§50.1. Introduction. Jeffrey 1967 introduced "coupled trees" as a modified tableau method for testing an argument for validity. In §50.2 we shall describe the formalism of the coupled-tree method and explain how by pruning it of complications (needed by Jeffrey to get precisely the classically valid arguments) we get a well-motivated syntactical characterization of when an entailment holds relevantly between truth-functional sentences. In §50.3 an "intuitive" semantical characterization is presented and motivated using inconsistent and incomplete "situations." In §50.4 these two characterizations are connected by completeness and soundness results. In §50.5 the semantical characterization is connected similarly with the syntactical characterization ("tautological entailment") of Chapter III, which was connected in §24.2 to the provable "first degree entailments" (formulas of the form $A \rightarrow B$, where A and B contain no occurrences of $\rightarrow$) of the system **E**. In §50.6 another semantical characterization using "topics" and having an information-theoretic flavor is related to the semantics of §50.3. So we have the happy circumstance that all these characterizations coincide. In §50.7 we ruminate.

Except for the specific connections with coupled trees, these various results basically stem from Dunn 1966, many of them having seen "semi-publication" prior to Dunn 1976 (in which they found finished form), as specific references below will indicate. The presentation of the "intuitive semantics" in §50.3 comes essentially verbatim from Dunn 196+, and so its apparatus of incomplete and inconsistent "situations" antedates the fashionable and fruitful "situation semantics." See Barwise and Perry 1981 for incomplete situations, and Barwise and Perry 1983 for inconsistent (abstract) situations. Of course we do not mean to claim too much here. The Barwise-Perry semantics is clearly independent, and its application to natural-language constructions is rich and novel. But we like to think that at least first degree (relevant) entailments have a home there.

It should also be mentioned that there are in the literature at least two other set-theoretical modelings of the first degree entailments besides the modeling in this section, the one due to van Fraassen 1969 as presented in §20.3 of this book, the other due to Routley and Routley 1972 as discussed in §48.2 above. Both of these have certain similarities to the present semantics and to each other, and should be consulted. In fact the semantics of Routley and Routley 1972 is "isomorphic" to the present semantics (see §48.2, Meyer 1979b, and Meyer and Martin 1986).

§50.2. Relevantly coupled trees. Jeffrey's logic text 1967 provides an excellent introductory treatment of the method of "analytic tableaux" of Smullyan 1968. (See §60 for an altogether different use of such tableau ideas.) Jeffrey (p. 92) compares the method of "truth trees" (his suggestive name for analytic tableaux) with indirect proof, the essential point being that in order to test an argument, say, A therefore B (in symbols $A \vdash B$), for validity one uses the method of truth trees to test for the mutual inconsistency of A and $\sim B$. The idea of a truth tree is that it diagrams in a branching tree-like fashion (so as to keep track of various alternatives) all the truth conditions of a set of sentences. Each path represents a way in which the given sentences might become true, and when testing for inconsistency we search to see whether all these paths are "closed" by virtue of containing both a sentence and its denial.

Jeffrey (p. 93) provides a modification of the basic method of truth trees, a modification which he calls the method of "coupled trees" and which he compares to direct proof. The basic idea is that, in order to test the validity of an argument, $A \vdash B$, one constructs two truth trees, one for A (coming down) and one for B below (going up). Since each path in the tree for A represents an alternative set of truth conditions for A, and similarly for B, it is natural to require that every path in the upper tree "cover" some path in the lower tree in the sense that every sentence letter or denial of a sentence letter (the term *atom* will henceforth cover both of these, as in §15.1) that appears in the covered path appears also in the covering path. Thus every way in which A is true is also a way in which B is true.

There are, however, two technical complications that Jeffrey needs in order to get precisely the classically valid arguments. We shall explain these complications after we describe with more precision the formalism of the coupled-tree method.

Jeffrey's formalism includes sentence letters (we shall suppose they are p, q, r, etc.) and connectives for negation, conjunction, and disjunction (we suppose these are $\sim$, &, and $\vee$) as well as for the truth-functional conditional and biconditional. We shall ignore these last two since they are not primitive in the standard formulations of the system **E** (though they can of course be introduced as abbreviatory devices via their ordinary contextual definitions). There are then schematically the following five rules (Jeffrey actually avoids formal recognition of the first rule by a practice of erasing pairs of juxtaposed negation signs, but the rule is formally in Smullyan 1968):

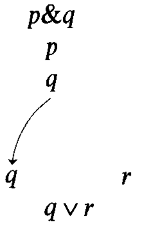
$$(\sim\sim)\quad \begin{array}{c} \sim\sim A \\ \hline A \end{array}$$

$$(\&)\quad \begin{array}{c} A\&B \\ \hline A \\ B \end{array} \qquad (\sim\&)\quad \begin{array}{cc} \multicolumn{2}{c}{\sim(A\&B)} \\ \hline \sim A & \sim B \end{array}$$

$$(\vee)\quad \begin{array}{cc} \multicolumn{2}{c}{A\vee B} \\ \hline A & B \end{array} \qquad (\sim\vee)\quad \begin{array}{c} \sim(A\vee B) \\ \hline \sim A \\ \sim B \end{array}$$

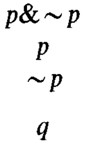
The rules are reasonably self-explanatory. Note that $(\sim\&)$ and $(\vee)$ have branching conclusions representing alternatives and are the source of the "tree" structure.

The basic idea of Jeffrey's coupled-tree method is illustrated by the argument $p\&q \vdash q\vee r$, for which we can construct the following coupled trees (the arrow indicates covering):



The corresponding entailment $p\&q \rightarrow q\vee r$ not only is a theorem of **E** but is basic to the motivation of **E** presented in §15.1, being paradigmatic of what is there called an "explicitly tautological primitive entailment."

The first of the two technical complications needed by Jeffrey is nicely illustrated by the argument $p\&\sim p \vdash q$. The coupled tree, if any, for this argument would be the following:



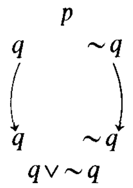
But there is a conspicuous absence of covering. This fits nicely with the §15.1 intuition that there is no relevance between premiss and conclusion. Jeffrey, though, is concerned to capture this classically valid inference. Thus he complicates the basic idea that every path above must cover some path below by excepting the closed paths above. (Of course this appears as no complication at all in the (classical) context Jeffrey has working for him, where "closed" paths are discountenanced in just the way their name suggests.) Thus, trivially, the above diagram (with a cross written under ${\sim}p$ to indicate that the path is closed) counts as a coupled tree.

The second complication is nicely illustrated by the dual of the last argument, namely, $p \vdash q \vee {\sim}q$. It would seem that the following would represent a failed attempt at constructing an appropriate coupled tree:

$$\begin{array}{ccc} & p & \\ q & & {\sim}q \\ & q \vee {\sim}q & \end{array}$$

Again the lack of covering can be taken to be in accord with §15.1 intuitions about irrelevance.

Jeffrey's device to wash this one through is to permit in the construction of the tree coming down from the premiss a simultaneous branching with a sentence and its negation. Thus the following counts as a coupled tree:



This amounts to adding tacitly as a rule (in constructing upper trees only) the following, which we shall call "punt":

$$\frac{A}{B \qquad {\sim}B}$$

Let us close this section with the definition it has been motivating: an argument $A \vdash B$ *passes the relevantly-coupled-tree test* iff, in constructing truth trees for A and B, every path in the tree for A (including the closed paths) covers some path in the tree for B (not allowing use of "punt").

§50.3. Intuitive semantics. Wittgenstein 1921 says in the *Tractatus* (Pears and McGuinness translation):

> 4.461. Propositions show what they say: tautologies and contradictions show that they say nothing.
>
> A tautology has no truth-conditions, since it is unconditionally true: and a contradiction is true on no condition.
>
> Tautologies and contradictions lack sense.

This Tractarian view survives today in some of the best logic texts. Jeffrey 1967 says a little more than most authors to justify the view that the truth-table rules of valuation give meaning to the connectives. Thus he says (p. 15):

> The rules of valuation make no mention of the meanings of sentences; they are couched entirely in terms of truth-values. Nevertheless, the rules of valuation determine the meanings of compound sentences in terms of the meanings of their ingredient sentence letters, for we know the meaning of a sentence (we know what statement the sentence makes) if we know what facts would make it true and what facts would make it false. Now if we have this information about the letters that occur in a sentence, the truth conditions supply the corresponding information about the whole sentence.

A little later (pp. 30–31) in discussing contradictions, Jeffrey says:

> The sentence
>
> *It is and is not raining*
>
> is only apparently about the weather, just as the sentence
>
> $2+2 = 4$ and $2+2 \neq 4$
>
> is only apparently about numbers. In fact, the two sentences have exactly the same meaning since they have exactly the same truth conditions: in all possible cases, both are false.

We think that we can avoid the necessity of Jeffrey's conclusion while yet agreeing, in a trivial sense, that the meaning of a sentence is determined by its truth conditions. Thus let p be the sentence "It is raining" and let q be the sentence "$2+2 = 4$." By standard truth-table considerations it follows that $p\&{\sim}p$ is true iff p is true and ${\sim}p$ is true, that is, iff p is true and p is false. Similarly, $q\&{\sim}q$ is true iff q is true and q is false. The question, put bluntly, then is whether the condition that p is true and p is false is the same condition as that q is true and q is false. We think it is not.

Notice that it is no argument against us to reply that the first is a contradiction meaning p is true and p is not true, while the second is also a contradiction meaning q is true and q is not true, and that of course any two contradictions have the same meaning. This only pushes the question with which we began up into the metalanguage.

Intuitively, $p\&{\sim}p$ and $q\&{\sim}q$ describe different situations, granted that neither situation is realizable. What we need is a semantics that is sensitive to this intuition.

As we observed in §48.1, it is now orthodox to realize a proposition as a function from possible worlds (or indices, reference points, situations, cases, whatever) to truth values (for explicitness see Montague 1970, who credits the idea to Kripke; see also the articles by Lewis and Stalnaker in the same volume as the Montague paper). This corresponds to the principle that different meanings can be distinguished by different situations with different truth values, i.e., by different truth conditions. It too has the untoward consequence that (relative to a given set of situations) there is only one contradictory proposition, simply because there is only one constant false function.

However, we need modify this picture only slightly to provide a kind of extensional apparatus that allows us to distinguish contradictory propositions from one another. Starting from the intuition expressed above that a contradiction can be true in some situations (of course, impossible situations) in which some other contradiction is not true, we can identify a proposition with a *relation* (instead of a function) from a set of situations into the set $\{T, F\}$. A contradictory proposition is then such a relation where F is in the image of every situation. There can then be many different contradictory propositions. These can be distinguished by a situation such that one proposition has T in its image while the other does not.

We could go on to develop the notion of an "interpretation" as an assignment of propositions to (truth-functional) sentences, setting down straightforward rules by which the propositions assigned to complex sentences are determined inductively from the propositions assigned to their ingredient sentence letters. However, such an assignment of a proposition to a sentence is obviously interchangeable with a rule telling us whether the sentence is true or false for each situation. We allow of course the option that the rule sometimes tells us neither as well as both, so as to be faithful to our construal of a proposition as relational but not necessarily functional in character. So such a rule, or *valuation*, could be identified with a three-placed relation φ relating sentences, situations, and truth values. A *situation model* then is an ordered pair $\langle \mathrm{K}, \varphi\rangle$, where K is a nonempty set (its members' being called "situations") and φ is such a relation which relates sentences, situations, and truth values in a natural recursive manner soon to be tied down.

As a matter of simplification in working up a semantics to assess the validity of *first degree* entailments, we can forget situations and deal just with two-placed relations simply relating sentences to truth values. This is for the familiar reason that, in assessing the validity of an entailment, every variety of situation whatsoever must be considered in which the antecedent is true, in order to see whether it is also a situation in which the consequent is true, and this is in effect done by looking at just the specifications of truth and falsity given the antecedent by situations and discarding the situations themselves.

We thus define a *relevance valuation* as a certain kind of inductively determined relation whose domain is the set of truth-functional sentences and whose range is a subset of $\{T, F\}$. Before going on to state the exact inductive clauses that must be satisfied, we introduce some conventions useful for their statement and useful also in the sequel. For a given such valuation V, we let $V^*(A)$ be the image of A under V (i.e., the set of truth values to which A is related by V). We can then indicate that V relates A to $\boldsymbol{T}$ $\{\boldsymbol{F}\}$ by $\boldsymbol{T}$ $\{\boldsymbol{F}\} \in V^*(A)$. In context we simply say that A is $\boldsymbol{T}$ $\{\boldsymbol{F}\}$, or A is true {false}. We can adopt similar conventions regarding the more complicated situation-relative kind of valuation φ of a situation model described above, letting for a given situation H, $\varphi^*(A, \mathrm{H})$ be the set of truth values X ("X" ranges over $\boldsymbol{T}$ and $\boldsymbol{F}$) such that $\varphi(A, \mathrm{H}, \mathrm{X})$. Also in context we allow ourselves to say things like "A is $\boldsymbol{T}$ in H" or, when H is fixed, even "A is $\boldsymbol{T}$."

With these understandings behind us, we require of a relevance valuation V and also of a valuation φ on a situation model (relative to each situation H) the following:

(i) $\sim A$ is $\boldsymbol{T}$ iff A is $\boldsymbol{F}$,
$\boldsymbol{F}$ iff A is $\boldsymbol{T}$;
(ii) $A\&B$ is $\boldsymbol{T}$ iff A is $\boldsymbol{T}$ and B is $\boldsymbol{T}$,
$\boldsymbol{F}$ iff A is $\boldsymbol{F}$ or B is $\boldsymbol{F}$;
(iii) $A \vee B$ is $\boldsymbol{T}$ iff A is $\boldsymbol{T}$ or B is $\boldsymbol{T}$,
$\boldsymbol{F}$ iff A is $\boldsymbol{F}$ and B is $\boldsymbol{F}$.

Note that, in each of (i)–(iii), we need two clauses, one giving truth conditions and the other giving falsity conditions. We cannot rely upon the standard intuition that a sentence that has been given the value $\boldsymbol{T}$ is not $\boldsymbol{F}$.

(There is no difficulty in extending all this to quantifiers. The basic idea is that, instead of doing what is sometimes done classically, to wit, interpreting an n-ary predicate as a *function* from the ordered n-tuples of individuals in the domain into $\{\boldsymbol{T}, \boldsymbol{F}\}$, one rather allows a *relation* from the n-tuples to the truth values. We could base on this insight a semantics for the first degree entailments (with quantifiers) of §40.)

We can already give a semantical explication of one of the principal features of entailment, namely, that $p\&\sim p$ need not entail q. For there is a valuation in which $p\&\sim p$ receives the value $\boldsymbol{T}$ and yet q does not. This is a valuation in which p receives both the values $\boldsymbol{T}$ and $\boldsymbol{F}$, while q receives the single value $\boldsymbol{F}$.

We can also give a semantical explication of perhaps the most controversial feature of entailment, namely, that $\sim p\&(p \vee q)$ need not entail q (the failure of the so-called "rule of disjunctive syllogism," or (γ)). Let us give this explication in the context of examining the supposed proof of Lewis and Langford 1932 that a contradiction entails everything (see also §16.1).

The proof starts out by supposing that $p\&\sim p$ is true. We then detach p by the rule of simplification (&E), and from p we obtain $p \vee q$ by the rule of addition ($\vee$I). Next we obtain $\sim p$ from our supposition of $p\&\sim p$ by another use of the rule of simplification. So far, O.K. But finally we claim that q follows from $\sim p$ and $p\vee q$ by disjunctive syllogism. In producing this proof for a class, this last step is often motivated by telling the following story. "So on our assumption that $p\&\sim p$ is true, we have obtained that one of p and q is true. But we have also obtained $\sim p$, which says that p is not the true one. So q must be the true one." Once when this story was told, some wise guy yelled out, "But p was the true one—look again at your assumption."

That wise guy was right. If we assume that $p\&\sim p$ is true, we are thereby assuming that p is both true and false; hence it should not be surprising that $p\&(\sim p\vee q)$ comes out true under that assumption, while q might still be false.

Do not take this wrong—we are not claiming that there are sentences that are in fact both true and false. We are merely pointing out that there are plenty of situations where we suppose, assert, believe, are told, etc., contradictory sentences to be true, and we therefore need a semantics that expresses the truth conditions of contradictions in terms of the truth values that the ingredient sentences would have to take for the contradictions to be true. (See §81 below for an extended analysis of the "told" case.)

The careful reader will by this time have noticed that in making a valuation a relation from sentences to truth values rather than a function we have thereby allowed a sentence be neither true nor false as well as both true and false. We have seen that the latter move is connected with invalidating inferences like $p\&\sim p\vdash q$ or $p\&\sim p\vdash q\&\sim q$. The former move is dually connected with invalidating inferences like $p\vdash q\vee\sim q$ or $p\vee\sim p\vdash q\vee\sim q$. And the two moves in concert succeed in invalidating inferences like $p\&\sim p\vdash q\vee\sim q$.

Just to be explicit, let us record that we do *not* know how to extend the "ambivalent" point of view to the whole of the systems **R** and **E** (etc.) in a fashion either intuitively satisfying or mathematically attractive. We *do* know how to do a pretty good job for **RM** (see §49), and Meyer 1979b shows how the "ambivalent" framework can in fact be extended to all of **R** and **E** (etc.), but as we noted in §48.5, his extension, though it most certainly solves an outstanding problem, seems to lose both intuitive punch and mathematical simplicity. For example, the three-termed R relation itself needs to be made four-valued. (Routley 1984a removes the need to make the relation four-valued, but with the trade-off of requiring *two* two-valued relations.) These judgments are, however, historical, and we are prepared to change our minds. In the meantime, although, as we said in §48.2, we have some reservations about the intuitive force of the Routleys' * operator, there is no doubt of the mathematical smoothness with which its use generalizes to the context of all of **R** and **E** (etc.), as indicated in §48.5. See also §51.5(5).

To return to our enterprise, how do we go about motivating allowing sentences to be assigned no truth value? The answer is, of course, "dually" to our motivation for both truth values. Rather than think about the (*per impossible*) truth conditions for contradictions, we think about the (*per impossible*) "non-truth conditions" for tautologies. The classical truth-table considerations of (i)–(iii) above tell us that the only way that $p \vee {\sim} p$ could possibly (better, *impossibly*?) be non-true is for p to have neither truth value.

Here again we are not arguing that there are sentences that are in fact neither true nor false. (We are not saying that there are not, either. There may be "truth-value gaps" due, for example, to failure of denotations of singular terms, but such things seem to us to be "irrelevant" to present concerns.) It is just that in working out the truth conditions for a compound sentence it is very easy to overlook the condition that an ingredient sentence have some truth value. (To consider this a "background assumption" seems to amount to the same thing.) Once such conditions are made explicit by relevance valuations, it no longer looks as if all tautologies have the same truth conditions, to wit, any.

By the way, we are painfully aware of the strangeness of some of our remarks motivating the semantics. Phrases like "impossible conditions" strain, if not break. It is at least as difficult to speculate about the impossible without seeming to talk of it as possible as it is to speculate about the merely possible without seeming to talk of it as actual. Frege 1893 had a (very roughly) similar problem in telling us about his concepts and seeming to talk of them as objects. We are tempted to join with him and say: "By a kind of necessity of language, my expressions, taken literally, sometimes miss my thought.... I fully realize that in such cases I was relying upon a reader who would be ready to meet me halfway—who does not begrudge a pinch of salt."

Let us end this section by officially defining some notions implicit in the preceding motivations. An argument $A \vdash B$ is *relevantly valid* iff, for every relevance valuation V, either $T \notin \mathrm{V}^*(A)$ or $T \in \mathrm{V}^*(B)$. This disjunction may be more naturally expressed (speaking in the material-conditional tone of voice that seems characteristic of metatheoretical investigations of the relevance logics) as: if $T \in \mathrm{V}^*(A)$, then $T \in \mathrm{V}^*(B)$.

We next introduce some cognate notions using situation models. We shall say that *A entails B in a situation model* $\langle \mathrm{K}, \varphi \rangle$ iff, for all $\mathrm{H} \in \mathrm{K}$, if $\varphi(A, \mathrm{H}, T)$ then $\varphi(B, \mathrm{H}, T)$. We shall say that *A universally entails B in a set of situations* K iff, for all situation models $\langle \mathrm{K}, \varphi \rangle$, A entails B in $\langle \mathrm{K}, \varphi \rangle$. Finally, we shall say that *A logically entails B* iff A universally entails B in all sets of situations.

From our remark motivating the introduction of the relevance valuations V as applying directly to sentences without the intervention of situations, it should be obvious to the reader that A logically entails B iff $A \vdash B$ is relevantly valid. This makes the notions involving situations somewhat otiose as far as providing a semantical modeling of the first degree entailments, and indeed we shall make no further mention or use of such notions until §50.6.

§50.4. Coupled trees and the semantics. In this section we establish the following and its converse:

SOUNDNESS THEOREM FOR RELEVANTLY COUPLED TREES. If an argument $A \vdash B$ passes the relevantly-coupled-tree test, then it is relevantly valid.

PROOF. The rules are *downward correct* in the sense that every relevance valuation that makes the premiss true also makes the conclusions in at least one branch true. Also they are *upward correct* since every relevance valuation that makes the conclusions in at least one branch true also makes the premiss true. Suppose then that $A \vdash B$ passes the relevantly-coupled-tree test and that V is a relevance valuation in which A is true. It is easy to see on the basis of the downward correctness of the rules that in the tree constructed downward from A there must be some path P such that every sentence in P is true in V. But the tree constructed upward from B must have some path Q such that P covers Q. All the atomic sentences or their negates that appear in Q must then be true in V, and it is easy to see on the basis of the upward correctness of the rules that B must be true in V as well.

We next attack the converse:

COMPLETENESS THEOREM FOR RELEVANTLY COUPLED TREES. If an argument $A \vdash B$ is relevantly valid, then it passes the relevantly-coupled-tree test.

PROOF. Suppose the argument fails to pass the relevantly-coupled-tree test. Then there exists a tree constructed downward from A and one constructed upward from B such that the tree for A has a path P that fails to cover each path in the one for B. Let the paths in the tree for B be $Q_1, \ldots, Q_n$, and let $\alpha_1, \ldots, \alpha_n$ be, respectively, atoms that keep P from covering $Q_1, \ldots, Q_n$. Define the relevance valuation V such that, for each sentence letter p,

(i) $T \in V^*(p)$ iff p is in P,
(ii) $F \in V^*(p)$ iff $\sim p$ is in P.

It is easy to see, because of upward correctness, that every sentence in P is true in V and, hence, in particular that A is true in V. Also, it is easy to see, because of downward correctness, that if B is true in V then every sentence in at least one of the paths Q_i must be true. But it is easy to see that α_i, a member of Q_i, is not true in V. So A is true in V while B is not, and so the argument fails to be relevantly valid.

§50.5. Tautological entailments and the semantics. We submit that the relevantly-coupled-tree test provides a plausible proof-theoretic characterization of when two truth-functional sentences A and B relevantly entail each other. In this section we shall briefly rehearse an earlier such characterization, the "tautological entailments" of §15.1, and then provide soundness and completeness results for the tautological entailments with respect to the intuitive semantics. Since §24.2 shows that a first degree entailment $A \rightarrow B$ is a theorem of the system **E** (or **R**) iff it is a tautological entailment, the results of this section and the last provide the hookup via the semantics between coupled trees and entailment promised in §50.1.

According to §15.1, a *primitive conjunction* {*disjunction*} is a conjunction {disjunction} of atoms, and a *primitive entailment* is of the form $A \rightarrow B$, where A is a primitive conjunction and B is a primitive disjunction. A primitive entailment $A \rightarrow B$ is *explicitly tautological* iff some {conjoined} atom of A is the same as some {disjoined} atom of B. A first degree entailment $A \rightarrow B$ is a *tautological entailment* iff, when A is put into disjunctive normal form $A_1 \vee \ldots \vee A_m$ and B is put into conjunctive normal form $B_1 \& \ldots \& B_n$, it turns out that each $A_i \rightarrow B_j$ is an explicitly tautological entailment.

The replacement rules that are permitted in reducing sentences to the normal forms are the familiar ones (see e.g., §15.2; we use the notation $\rightleftarrows$ to indicate mutual replaceability):

Commutativity	$A\&B \rightleftarrows B\&A,$	$A \vee B \rightleftarrows B \vee A;$
Associativity	$A\&(B\&C) \rightleftarrows (A\&B)\&C,$	$A \vee (B \vee C) \rightleftarrows (A \vee B) \vee C;$
Distributivity	$A\&(B \vee C) \rightleftarrows (A\&B) \vee (A\&C),$	
		$A \vee (B\&C) \rightleftarrows (A \vee B)\&(A \vee C);$
Double negation	$\sim\sim A \rightleftarrows A;$	
De Morgan	$\sim(A\&B) \rightleftarrows \sim A \vee \sim B,$	$\sim(A \vee B) \rightleftarrows \sim A \& \sim B.$

We now prove in one swat the

SOUNDNESS AND COMPLETENESS THEOREM FOR TAUTOLOGICAL ENTAILMENTS. A sentence $A \rightarrow B$ is a tautological entailment if (completeness) and only if (soundness) $A \vdash B$ is relevantly valid.

PROOF. Now $A \rightarrow B$ is a tautological entailment iff there is a disjunctive normal form of A, $A_1 \vee \ldots \vee A_m$, and a conjunctive normal form of B, $B_1 \& \ldots \& B_n$, such that each $A_i \rightarrow B_j$ is explicitly tautological. Since the rules of replacement are truth-preserving, the question of whether $A \vdash B$ is relevantly valid amounts to the question of whether $A_1 \vee \ldots \vee A_m \vdash B_1 \& \ldots \& B_n$ is relevantly valid. We next observe that it may routinely be argued that this last is relevantly valid iff $A_i \vdash B_j$ is relevantly valid for each A_i and B_j. So the whole theorem reduces by chains of equivalences to the question of whether, for a primitive entailment $A_i \rightarrow B_j$, $A_i \rightarrow B_j$ is explicitly tautological iff $A_i \vdash B_j$ is relevantly valid.

Pursuing this, let A_i be the conjunction of atoms $\alpha_1 \& \ldots \& \alpha_k$, and let B_j be the disjunction of atoms $\beta_1 \vee \ldots \vee \beta_l$. If $A_i \rightarrow B_j$ is explicitly tautological, then some α_s is the same as some β_t. It is obvious for a given relevance valuation that if a conjunction is true then so is each conjunct and that if a disjunct is true then so is the disjunction. So if a given relevance valuation makes A_i true it will also make α_s, i.e., β_t true, and hence B_j true. So $A_i \vdash B_j$ is relevantly valid. On the other hand if $A_i \rightarrow B_j$ fails to be explicitly tautological it is easy to construct a relevance valuation that invalidates $A_i \rightarrow B_j$. For a given sentence letter p, let $T \in \mathrm{V}^*(p)$ iff p is a conjunct of A_i, and let $F \in \mathrm{V}^*(p)$ iff $\sim p$ is a conjunct of A_i. Note that this has the effect of making an atom true iff it is a conjunct of A_i. So, since each conjunct of A_i is true in V, obviously A_i is true in V. But also clearly B_j is not true in V; for if it were true then some disjunct β_t would be true and hence would be a conjunct of A_i, contrary to our assumption that A_i and B_j fail to share an atom.

§50.6. An earlier semantical gloss of essentially the same mathematics. This was contained in Dunn 1966 (and also reported in Dunn 1971). There a *proposition surrogate* was defined to be an ordered pair $\langle \mathrm{X}, \mathrm{Y} \rangle$, where X and Y are sets. The members of X are to be thought of as the "topics" that a given proposition gives definite information about, and the members of Y as the "topics" that the negated proposition gives definite information about. These "topics" are members of some arbitrary but fixed set U called "the universe of discourse," and $\langle \mathrm{X}, \mathrm{Y} \rangle$ is said to be a proposition surrogate "in" U. An *interpretation* in U is, then, a function I assigning a proposition surrogate in U to each truth-functional sentence A in accord with the rules (Ii)–(Iiii) next given (in our statement of these rules we adopt the notational convention that if $\mathrm{I}(A) = \langle X, Y \rangle$ then $\mathrm{I}^+(A) = \mathrm{X}$ and $\mathrm{I}^-(A) = \mathrm{Y}$):

(Ii) $\mathrm{I}(\sim A) = \langle \mathrm{I}^-(A), \mathrm{I}^+(A) \rangle$;

(Iii) $\mathrm{I}(A \& B) = \langle \mathrm{I}^+(A) \cup \mathrm{I}^+(B), \mathrm{I}^-(A) \cap \mathrm{I}^-(B) \rangle$;

(Iiii) $\mathrm{I}(A \vee B) = \langle \mathrm{I}^+(A) \cap \mathrm{I}^+(B), \mathrm{I}^-(A) \cup \mathrm{I}^-(B) \rangle$.

It is natural to say that *$I(A)$ implies $I(B)$* iff both $\mathrm{I}^+(B) \subseteq \mathrm{I}^+(A)$ and $\mathrm{I}^-(A) \subseteq \mathrm{I}^-(B)$ and to regard a first degree entailment $A \rightarrow B$ as *valid in a universe of discourse* U iff, for every interpretation I in U, $\mathrm{I}(A)$ implies $\mathrm{I}(B)$, and as *valid* simpliciter iff it is valid in every universe of discourse. The basic result about all this, proved in Dunn 1966, is that a first degree entailment is a theorem of **E** iff it is valid, indeed iff it is valid in a universe of discourse with but a single topic.

The above semantical treatment of first degree entailments was explicitly designed as a generalization of the Carnap and Bar-Hillel 1952 theory of semantic information. In that theory to each sentence A is assigned a certain set representing the informational "content" of A (in symbols, $\mathrm{C}(A)$). This assignment turns out to satisfy the following rules:

(Ci) $\mathrm{C}(\sim A) = -\mathrm{C}(A)$;

(Cii) $\mathrm{C}(A \& B) = \mathrm{C}(A) \cup \mathrm{C}(B)$;

(Ciii) $\mathrm{C}(A \vee B) = \mathrm{C}(A) \cap \mathrm{C}(B)$.

The basic nature of the generalization may be seen by comparing (Ii) and (Ci). Clause (Ci) says that the information given by the negation of a sentence may be determined in a very simple way, i.e., as the set-theoretical complement, from the information given directly by the sentence, whereas clause (Ii) leaves open the relationship between the information given by a sentence and the information given by its negation. These may overlap, they may not be exhaustive; hence the need for the "double-entry bookkeeping" done by proposition surrogates.

Carnap and Bar-Hillel are quite explicit about the particular nature of the elements of C(A). They first define a *content-element* as the negation of a state description, and then define C(A) as the set of all content-elements L-implied by A. They might just as well (see Bar-Hillel 1964, Ch. 17) have defined C(A) as the set of all state descriptions (§16.2) in which A is false, i.e., which L-imply $\sim A$, and it will be convenient for our purposes to suppose this is what they did. It is also useful to mention the dual concept of the *range* of A, in symbols, R(A), defined by Carnap and Bar-Hillel as the set of state descriptions in which A is true, i.e., which L-imply A.

Now we could go on to give a similarly explicit syntactical account of I(A). We would define a "situation description" (a stylistic variant of "set-up" as in §16.2.1) as a conjunction of atoms (formed from some fixed finite set of atomic sentences, as is typical with state descriptions), *not* requiring (as does a state description) that precisely one of every atomic sentence and its negate occur in the conjunction. Thus, e.g., if the atomic sentences are p, q, and r, the following would all be situation descriptions, although only the first would be a state description:

$$p\&\sim q\&r, \qquad p\&\sim p\&\sim q\&r, \qquad p\&\sim q, \qquad p\&\sim p\&\sim q.$$

Since situation descriptions differ from state descriptions in that they may be inconsistent or incomplete, there are in general many more of them. Thus from n atomic sentences only 2^n state descriptions may be formed, whereas there are 2^{2n} situation descriptions (counting the void situation description; see the picture in §16.2.1, which, however, omits the void situation description (set-up)).

Relative to a fixed finite set of atomic sentences including all the atomic sentences ingredient in a given truth-functional sentence A, we can analogously define notions of the "relevant content" of A ($\mathrm{C}_{\mathrm{rel}}(A)$) and the "relevant range" of A ($\mathrm{R}_{\mathrm{rel}}(A)$), defining $\mathrm{C}_{\mathrm{rel}}(A)$ as the set of situation descriptions that tautologically entail $\sim A$ and $\mathrm{R}_{\mathrm{rel}}(A)$ as the set of situation descriptions that tautologically entail A. We can then set $\mathrm{I}(A) = \langle \mathrm{C}_{\mathrm{rel}}(A), \mathrm{R}_{\mathrm{rel}}(A)\rangle$. We leave to that unsung hero of logical-mathematical researches, "the interested reader," the task of working out the detail of all this. We content ourselves with observing that in general $\mathrm{C}_{\mathrm{rel}}(A) \neq -\mathrm{R}_{\mathrm{rel}}(A)$, since sometimes $\mathrm{C}_{\mathrm{rel}}(A) \cap \mathrm{R}_{\mathrm{rel}}(A) \neq \varnothing$, e.g., $p\&\sim p$ is a member of both $\mathrm{C}_{\mathrm{rel}}(p\&\sim p)$ and $\mathrm{R}_{\mathrm{rel}}(p\&\sim p)$, and also since sometimes $C_{rel}(A) \cup R_{rel}(A) \neq U$, e.g., q is not a member of either $C_{rel}(p \& {\sim} p)$ or $R_{rel}(p \& {\sim} p)$.

In semantical treatments of logic these days, possible worlds are much more used than their syntactical alter egos, the state descriptions. We have already introduced in §50.4 the fashionable identification of a proposition with the set of worlds in which it is true (see also §48.1). This corresponds to $R(A)$, and there is also the option of introducing a possible-world analogue of $C(A)$, namely, the set of worlds in which A is false. Similarly, we can give situation analogues of $C_{rel}(A)$ and $R_{rel}(A)$, respectively the set of situations in which A is false and the set of situations in which A is true.

It should by now be obvious to the reader that the situation models of §50.3 and the interpretations in universes of discourse come very close to being mere stylistic variants of one another. Through the moves motivated in the last paragraph as natural and even "familiar," it is possible to construct from a given situation model (K, φ) a corresponding interpretation I in a universe of discourse U, setting $U = K$, and $I(A) = \langle\{H \in K: \varphi(A, H, \boldsymbol{F})\}, \{H \in K: \varphi(A, H, \boldsymbol{T})\}\rangle$. It is easy to see that I so defined really is an interpretation, i.e., satisfies clauses (Ii)–(Iiii) above. And, of course, one can also go the other way. Thus, given an interpretation I in a universe of discourse U, one can set $K = U$, defining $\varphi(A, H, \boldsymbol{T})$ to hold iff $H \in I^{-}(A)$, and $\varphi(A, H, \boldsymbol{F})$ to hold iff $H \in I^{+}(A)$. The fact that we were able to pull off completeness and soundness results in §50.5 without the full apparatus of situation models (needing only the relevance valuations V) can be regarded as a direct analogue of the result of Dunn 1966, already cited, that a universe of discourse with but a single member is all that is needed to determine the valid first degree entailments. For, as has already been explained in §50.4, to do things with the relevance valuations V is in effect to work with, in turn, various single situations.

There is one subtle defect in this story of stylistic variance, and that has to do with the fact that we required for $A \rightarrow B$ to be valid in a universe of discourse U that, for every interpretation I in U, *both* $I^{+}(B) \subseteq I^{+}(A)$ and $I^{-}(A) \subseteq I^{-}(B)$. The *last* conjunct corresponds nicely to our requirement that, for A to imply B in a situation model $\langle K, \varphi\rangle$, for all $H \in K$, if $\varphi(A, H, \boldsymbol{T})$ then $\varphi(B, H, \boldsymbol{T})$. However, the *first* conjunct corresponds to a requirement that we *never* made, to wit, that, for all $H \in K$, if $\varphi(B, H, \boldsymbol{F})$ then $\varphi(A, H, \boldsymbol{F})$. Putting the same basic point another way, it was not required for the relevant validity of $A \vdash B$ that whenever $\boldsymbol{F} \in V^{*}(B)$ then $\boldsymbol{F} \in V^{*}(A)$, but only that whenever $\boldsymbol{T} \in V^{*}(A)$ then $\boldsymbol{T} \in V^{*}(B)$.

We think it is odd in fact that no requirements were made about B's passing $\boldsymbol{F}$ to A, but only requirements about A's passing $\boldsymbol{T}$ to B. In a rough and ready fashion it would seem that implication, entailment, validity, etc. should have as much to do with falsity preservation as with truth preservation—it is just that the direction is reversed. But for now we think that basically what needs to be said is that, when it comes to assessing relevant validity and logical entailment, a situation where a sentence is both true and false is nicely symmetrical with a situation where it is neither true nor false. Thus, fixing ideas to the case of relevant validity, corresponding to every relevant valuation V is its "dual" $\bar{\mathrm{V}}$ such that $\boldsymbol{T} \in \bar{\mathrm{V}}^*(A)$ iff $\boldsymbol{F} \notin \mathrm{V}^*(B)$, and also $\boldsymbol{F} \in \bar{\mathrm{V}}^*(A)$ iff $\boldsymbol{T} \notin \bar{\mathrm{V}}^*(B)$. (Technically it needs to be argued inductively that the dual of a relevance valuation is also indeed a relevance valuation, but this is fairly obvious.) It then happens that, if there is a relevance valuation V such that $\boldsymbol{F} \in \mathrm{V}^*(B)$ and yet $\mathbf{F} \notin \mathrm{V}^*(A)$, then there is (in general) *another* relevance valuation (its dual) $\bar{\mathrm{V}}$, such that $\mathbf{T} \in \bar{\mathrm{V}}^*(A)$ and yet $\mathbf{T} \notin \bar{\mathrm{V}}^*(B)$. Thus, in assessing relevant validity, where we consider *all* relevance valuations, we can merely by requiring truth preservation also get as a "spin-off" falsity preservation in the converse direction.

§50.7. Ruminations. We suppose the dominant trend of this section has been to support the view that co-entailment in the system **E** might serve as the foundation of synonymy. We are not sure how much we are willing to go along with that trend, in view of the fact that $p \rightleftarrows p\&(p\vee q)$ is such a provable co-entailment. As the reader can easily see, both entailments are tautological, and the one from left to right holds "mainly" because of the primitive entailment $p \rightarrow p\vee q$.

We find ourselves in an inconsistent triad of intuitions here. We do believe that p entails $p\vee q$. We also believe that co-entailment (*properly understood*) should suffice for synonymy (again *properly understood*). But we find it difficult to accept the idea that an atomic sentence that talks just about the rain is synonymous with some compound sentence that seems to talk about the number 2 as well.

We are not going to solve this problem here, but it seems worth saying that, unless one is willing to give up on the relation of synonymy altogether (except perhaps as holding only between each sentence and itself), one should be prepared to buy sentences as synonymous which may differ from each other in their grammatical visage. Perhaps this is just an extreme such case. More moderate cases, e.g., that p is synonymous with $p\&p$, might even raise some eyebrows ("the last talks about conjunction, whereas the first does not"). If, however, one wished to seek an intermediate position, it might be worth beginning with mutual "conjunctive containment" in the sense of §82.4 below as the appropriate standard of synonymy; as is made plain in §82, taking it in some such way would by no means imply that one had to give up the entailment from p to $p\vee q$.

§51. Models for entailment: Relational-operational semantics for relevance logics (by Kit Fine). This section provides a detailed working out of a semantics for the various relevance logics. These semantics were developed independently, within a few months of those described in §47 and §48, to which they are equivalent or similar. Indeed, Routley and Meyer developed their semantics to a great extent (see Routley with Plumwood and Meyer and Brady 1982 for the most current presentation) and already knew most of the results of this section. As noted in more detail in Fine 1974, after the author read Routley and Meyer 1973, 1972a, 1972b, and Urquhart 1972, a few changes were made in the interests of uniformity and completeness, but the bulk of this section remains as originally worked out.

In the context of this book, one distinctive property of the modeling of this section is that it features as primitive both (i) a binary relational concept like that defined in §48, and (ii) an operational concept like that primitive in §47; these two do together the work done by the three-termed relational primitive of §48. A second feature is that we combine (i) a basic primitive one-place predicate which may be thought of as applying to *arbitrary* theories (or set-ups or pieces of information, depending on vocabulary preference) as in §47 rather than as applying only to "prime" theories (that is, only to theories all of whose disjunctions are decided) as in §48, with (ii) a second primitive one-place predicate which can be thought of as applying only to the "prime" theories as in §48. The reader who has gone through the proofs of the preceding sections will see that this second feature, too, amounts to a segmentation of functions for the purpose of allowing easier discernment of the essential conceptual structures involved. Some detailed relationships between the approaches are indicated in §51.5 (3,4).

[Note by the principal authors. Choices of fonts, terminology, and occasionally notation (e.g., "$\wedge$" for conjunction) in this section and in §53 have largely been left as in the original papers.]

The *first two sections present the deductive-semantic framework*: §51.1 specifies the models, and §51.2 the logics. The following two sections establish completeness; §51.3 for a minimal logic **B**, and §51.4 for **Π′**, **Π″**, **E** and the several subsystems. §51.5 outlines various alternative versions of the modeling. The last two sections contain applications of the modeling: §51.6 to the admissibility of *modus ponens*; and §51.7 to the finite model property and decidability. Many of the systems considered are shown to have these properties; see §63 for a further survey on decidability, and §65 for fundamental undecidability results.

§51.1. Models. This section introduces the models: first of all, the technical notions of formula, model, commitment, frame, and validity are defined; secondly, an intuitive account of these notions is given; and, thirdly, all models are shown to satisfy a natural requirement.

The set *Fml* (*formulas*) is constructed in the usual way from the set SL (sentence letters) = $\{p_0, p_1, \ldots\}$, the truth-functional connectives $\wedge$ (and) and $\sim$ (not), the entailment operator $\rightarrow$, and parentheses ( ). We follow the standard shorthand conventions; and, in particular, we write $(A \supset B)$ for $(\sim A \vee B)$.

A *model* $\mathfrak{A}$ is a septuple $(T, S, l, \circ, -, \geq, \varphi)$ such that:

(i) T is a set, $S \subseteq T$, $l \in T$, $\circ\colon T^2 \rightarrow T$, $-\colon S \rightarrow S$, $\geq$ is a reflexive, transitive, and antisymmetric relation on T, and $\varphi \subseteq T \times \mathrm{SL}$;
(ii) $(\forall t,u,v \in T)(t \geq u \Rightarrow (t \circ v) \geq (u \circ v))$;
(iii) $(\forall a \in S)(\forall t,u \in T)(a \geq (t \circ u) \Rightarrow$
$(\exists b,c \in S)(b \geq t$ and $c \geq u$ and $a \geq (b \circ u)$ and $a \geq (t \circ c))$;
(iv) $(\forall t \in T)((l \circ t) \geq t)$;
(v) $(\forall t \in T)(t \geq (l \circ t))$;
(vi) $(\forall a \in S)(a \geq l \Rightarrow a \geq -a)$;
(vii) $(\forall a \in S)(--a = a)$;
(viii) $(\forall a,b \in S)(a \geq b \Rightarrow -b \geq -a)$; and
(ix) $(\forall t \in T)(\forall p \in SL)(\varphi tp \Leftrightarrow (\forall a \in S)(a \geq t \Rightarrow \varphi ap))$.

Where there is no possibility of confusion, we write tu for $(t \circ u)$ and let t, u, v, etc., range over members of T and a, b, c, etc., range over members of S. Thus condition (iii) can be rewritten as: $(\forall a)(\forall t,u)(a \geq tu \Rightarrow (\exists b \geq t)(\exists c \geq u)(a \geq bu$ and $a \geq tc))$.

The relation $\vDash$ of *commitment* holds among models, theories, and formulas. Given a model $\mathfrak{A}$, commitment is defined by the following clauses:

(i) $t \vDash p \Leftrightarrow \varphi tp$, $p \in \mathrm{SL}$;
(ii) $t \vDash (B \wedge C) \Leftrightarrow t \vDash B$ and $t \vDash C$;
(iii) $t \vDash \sim B \Leftrightarrow (\forall a \geq t)($not $-a \vDash B)$; and
(iv) $t \vDash B \rightarrow C \Leftrightarrow (\forall u)(u \vDash B \Rightarrow tu \vDash C)$.

We use prefixes 'M' and 'C', respectively, to refer to the conditions (i)–(ix) for a model and to the conditions (i)–(iv) for commitment. To make the model $\mathfrak{A}$ explicit, we write $(\mathfrak{A}, t) \vDash A$ for $t \vDash A$.

A model $\mathfrak{A} = (T, S, l, \circ, -, \geq, \varphi)$ is *based on* $\mathfrak{F}$ if $\mathfrak{F} = (T, S, l, \circ, -, \geq)$, i.e., $\mathfrak{A}$ without its valuation. A *frame* is any sextuple on which a model is based. For Δ a set of formulas and $\mathfrak{A}$ a model as above, $\mathfrak{A} \vDash \Delta$, Δ is *valid in* $\mathfrak{A}$, if $(\forall A \in \Delta)((\mathfrak{A}, l) \vDash A)$. For $\mathfrak{F}$ a frame, $\mathfrak{F} \vDash \Delta$, Δ is *valid in* $\mathfrak{F}$, if Δ is valid in each model based on $\mathfrak{F}$. We write $\mathfrak{A} \vDash A$ and $\mathfrak{F} \vDash A$ for $\mathfrak{A} \vDash \{A\}$ and $\mathfrak{F} \vDash \{A\}$ respectively.

Suppose that X is a class of frames. Δ is *sound for* X if Δ is valid in each frame of X. Δ is *sufficient for* X if no formula outside Δ is valid in each frame of X. Δ is *complete for* X if it is both sound and sufficient for X; and Δ is *complete* if it is complete for some X.

Let us interpret the foregoing notions. T is to be the set of all *theories*, i.e., of all sets of propositions closed under commitment. Other entities that commit one to propositions—subjects, beliefs, suppositions and the like—may be identified with the set of propositions they commit one to.

S is to be the set of all theories that contain a disjunct of any contained disjunction, that answer every either-or question they raise. Such theories may be called *consolidated* or *saturated* or *prime.*

l is to be *logic*, i.e., the theory that comprises all, and only, the logical truths. Thus a formula is valid if and only if it belongs to logic.

Given any two theories, t and u, the *closure* (tu) *of* u *under* t is the set of propositions P such that t commits one to the proposition that u commits one to P. Thus (tu) is the commitment of u for t. Given a saturated theory a, the *co-theory* $-a$ of a is the set of propositions whose negations do not belong to a (so $-a$ here is the same as a^* in §48.5). $\geq$ is the relation of *inclusion* on theories, and φ, of course, is a *valuation.*

Given this interpretation, the conditions M(i)–(ix) and C(i)–(iv) are very reasonable. Thus M(ix) states that a theory commits one to a proposition if each of its saturated extensions commits one to the proposition. This is a form of $\vee$-elimination. C(ii) states that a theory commits one to a conjunction if it commits one to each of its conjuncts. C(iv) states that a theory commits one to an entailment if it commits a theory to the consequent of the entailment whenever that theory is committed to the antecedent.

The following result generalizes condition M(ix). It states that any theory is the lower limit of the saturated theories that contain it.

THEOREM 1. $t \vDash A \Leftrightarrow (\forall a \geq t)(a \vDash A)$.

PROOF. By induction on the construction of A.

(1) $A = p$. $t \vDash p \Leftrightarrow \varphi tp$ (C(i)) $\Leftrightarrow (\forall a \geq t)(\varphi ap)$ (M(ix)) $\Leftrightarrow$ $(\forall a \geq t)(a \vDash p)$ (C(i)).

(2) $A = B \wedge C$. $t \vDash B \wedge C \Leftrightarrow t \vDash B$ and $t \vDash C$ (C(ii)) $\Leftrightarrow$ $(\forall a \geq t)(a \vDash B$ and $a \vDash C)$ (IH) $\Leftrightarrow (\forall a \geq t)(a \vDash B \wedge C)$ (C(ii)).

(3) $A = {\sim}B$. $\Rightarrow$. $(\exists a \geq t)(a \nvDash {\sim}B) \Rightarrow (\exists b \geq a)(-b \vDash B)$ (C(iii)) $\Rightarrow -a \geq -b$ and $-a \vDash B$ (M(viii) and IH) $\Rightarrow t \vDash {\sim}B$ (C(iii)).

$\Leftarrow$. $t \nvDash {\sim}B \Rightarrow (\exists a \geq t)(-a \vDash B$ (C(iii))) $\Rightarrow (\exists b \geq a)(-b \vDash B)$ (by $\geq$ reflexive) $\Rightarrow a \nvDash {\sim}B$ (C(iii)).

(4) $A = (B \rightarrow C)$. $\Rightarrow$. Suppose $t \vDash (B \rightarrow C)$, $a \geq t$ and $u \vDash B$. Then $(tu) \vDash C$ (C(iv)). By M(iv), $(au) \geq (tu)$. So, by IH and transitivity of $\geq$, $(au) \vDash C$. Hence $a \vDash (B \rightarrow C)$.

$\Leftarrow$. Suppose $t \nvDash (B \rightarrow C)$. Then $(\exists u)(u \vDash B$ and $(tu) \nvDash C)$ (C(iv)). By IH, $(\exists c \geq (tu))(c \nvDash C)$; and, by M(iii), $(\exists a \geq t)$ $(c \geq (au))$. So, by IH, $(au) \nvDash C$. Hence, $a \nvDash B \rightarrow C$ (C(iv)).

COROLLARY.

(i) $t \vDash A$ and $t \leq u \Rightarrow u \vDash A$.

(ii) $a \vDash {\sim}B \Rightarrow -a \nvDash B$.

(iii) $a \vDash B \vee C \Leftrightarrow a \vDash B$ or $a \vDash C$.
(iv) $(\forall a)(a \vDash B \Rightarrow ta \vDash C) \Rightarrow t \vDash B \to C$.
(v) $(\forall a)(a \vDash B \Rightarrow a \vDash C) \Leftrightarrow \mathfrak{U} \vDash B \to C$.

PROOF. (i) From Theorem 1 and the transitivity of $\geq$. (We shall refer to this theorem hereafter as Theorem 1.1, i.e., Theorem 1 of §51.1).

(ii) $a \vDash {\sim}B \Leftrightarrow (\forall b \geq a)(-b \nvDash B)$. But $(\forall b \geq a)(-b \vDash B) \Rightarrow -a \nvDash B$, by $\geq$ reflexive; and $-a \nvDash B \Rightarrow (\forall b \geq a)(-b \vDash B)$, by M(viii) and (i) above.

(iii) $a \vDash {\sim}({\sim}B \wedge {\sim}C) \Leftrightarrow -a \nvDash {\sim}B \wedge {\sim}C$ ((ii)) $\Leftrightarrow -a \nvDash {\sim}B$ or $-a \nvDash {\sim}C$ (C(ii)) $\Leftrightarrow --a \vDash B$ or $--a \vDash C$ ((ii) again) $\Leftrightarrow a \vDash B$ or $a \vDash C$ (M(vii)).

(iv) Suppose $t \nvDash B \to C$; so $(\exists u)(u \vDash B$ and $tu \nvDash C)$. By Theorem 1 $(\exists b \geq tu)(b \nvDash C)$. By M(iii), $(\exists a \geq u)(b \geq ta)$. So, by (i) and $a \geq u$, $a \vDash B$; and, by (i) and $b \geq ta$, $ta \nvDash C$.

(v) $\mathfrak{U} \vDash B \to C \Leftrightarrow l \vDash B \to C \Leftrightarrow (\forall a)(a \vDash B \Rightarrow la \vDash C)$ ((iv)) $\Leftrightarrow (\forall a)(a \vDash B \Rightarrow a \vDash C)$, since M(iv), M(v), and the fact that $\geq$ is antisymmetric imply that $(\forall a)(la = a)$.

Use of the theorem and its corollary will often be tacit.

§51.2. Logics. This section introduces the logics that are the syntactic companions to the models. An appropriate notion of deduction is defined; the deduction theorem is proved; and, finally, it is used to establish some theorems common to all logics.

A *logic* is any set of formulas that contains the following theorems and is closed under the following rules:

Theorems

1 $p \wedge q \to p$
2 $p \wedge q \to q$
3 $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$
4 $(q \to p) \wedge (r \to p) \to (q \vee r \to p)$
5 $p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$
6 ${\sim}{\sim}p \to p$
7 ${\sim}p \vee p$

Rules

8 $A, B \,/\, A \wedge B$
9 $A, A \to B \,/\, B$
10 $A \to B \,/\, (B \to C) \to (A \to C)$
11 $B \to C \,/\, (A \to B) \to (A \to C)$
12 $A \to {\sim}B \,/\, B \to {\sim}A$.

We use ‘P1’–‘P12’ to refer to the theorems and rules listed just above (‘P’ is for “postulates”). Use of P13, the rule of substitution, will nearly always be tacit.

Although this postulate set is the most natural for the completeness proof to follow, various simplifications can be made. P10–11 might be wrapped up into the single rule $A \to B,\ C \to D \,/\, (B \to C) \to (A \to D)$. Also, P5 might be replaced by the slightly more economical axiom $p \wedge (q \vee r) \to (p \wedge q) \vee r$. If both $\vee$ and $\wedge$ were taken as primitives, one would require the additional axioms $p \to p \vee q$ and $q \to p \vee q$.

For any logic **L**, we say $\Delta \vdash_{\mathbf{L}} B$, A is **L**-*deducible from* Δ, if there is a sequence of formulas $A_0, A_1, \ldots, A_n$ such that $A_n = B$ and $(\forall i \geq n)$ $(A_i \in \Delta$ or $(\exists j,k < i)(A_i = A_j \wedge A_k)$ or $(\exists j < i)(A_j \to A_i \in L)$. Note that **L**-theorems do not figure *as* theorems in a deduction. For $\mathbf{L} = \mathbf{E}$ and $\Delta = \{A\}$, **L**-deduction coincides with *proof that A entails B* (§23.6), and, for $\Delta \supseteq \mathbf{L}$, **L**-deduction coincides with Official deduction (§22.2.1).

Lemma 1. For any logic **L**, $A \to A \in \mathbf{L}$.

Proof. $\sim\sim\sim A \to \sim A \in \mathbf{L}$, by P6; and so $A \to \sim\sim\sim\sim A \in \mathbf{L}$, by P12. $\sim\sim\sim\sim A \to \sim\sim A \in L$, by P6 again, and so $A \to \sim\sim A \in \mathbf{L}$, by P10. But $\sim\sim A \to A \in \mathbf{L}$, by P6 for the last time; and so $A \to A \in \mathbf{L}$, by P10 and P9.

Theorem 1 (Deduction). For any logic **L**, $A \vDash_{\mathbf{L}} B \Rightarrow A \to B \in \mathbf{L}$.

Proof. By induction on the construction of the deduction of B from A.

(1) $B = A$. By the lemma above, $A \to A \in \mathbf{L}$.

(2) $B = C \wedge D$. By IH, $A \to C$, $A \to D \in \mathbf{L}$; by P8, $(A \to C) \wedge (A \to D) \in \mathbf{L}$; by P3, $(A \to C) \wedge (A \to D) \to (A \to C \wedge D) \in \mathbf{L}$. So, by P9, $A \to C \wedge D \in \mathbf{L}$.

(3) $C \to B \in \mathbf{L}$. By IH, $A \to C \in \mathbf{L}$; and so, by P10, $A \to B \in \mathbf{L}$.

The next result packs in all the proof-theoretical work required for the proof of completeness. Given the Deduction theorem, its proof is straightforward.

Theorem 2. For any logic **L**,

(i) $A \to A \vee B$, $B \to A \vee B \in \mathbf{L}$,

(ii) $(\Delta, A \vdash_{\mathbf{L}} C$ and $\Delta, B \vdash_{\mathbf{L}} C) \Rightarrow (\Delta, A \vee B \vdash_{\mathbf{L}} C)$,

(iii) $\sim(A \wedge B) \to . \sim A \vee \sim B \in \mathbf{L}$,

(iv) $\sim B, \sim C \vdash_{\mathbf{L}} \sim(B \vee C)$,

(v) $A \to B \in \mathbf{L} \Rightarrow \sim B \to \sim A \in \mathbf{L}$.

§51.3. The minimal logic. This section gives a post-Henkin completeness proof for the smallest logic, **B**. It may be axiomatized by taking P1–7 as axiom-schemes and P8–13 as rules of inference. Since we have our eye on bigger game than **B**, the preliminary lemmas will be stated with reference to all logics.

Suppose that **L** is a logic. A set of formulas Δ is an **L**-*theory* if it is closed under **L**-deduction, i.e., if $(\forall A)(\Delta \vdash_{\mathbf{L}} A \Rightarrow A \in \Delta)$. Δ is **L**-*prime* if $(\forall A,B)(\Delta \vdash A \vee B \Rightarrow (A \in \Delta \text{ or } B \in \Delta))$. Note that if Δ is **L**-prime then Δ is also an **L**-theory. For if Δ is **L**-prime and $\Delta \vdash A$ then $\Delta \vdash A \vee A$, by Theorem 2.2(i).

LEMMA 1 (Lindenbaum's). Suppose that Γ is closed under disjunction, i.e., that $(\forall A,B \in \Gamma)(A \vee B \in \Gamma)$, and that Δ is an **L**-theory that does not intersect Γ. Then there is an **L**-prime $\Delta' \supseteq \Delta$ that also does not intersect Γ.

PROOF. Let $B_0 \vee C_0$, $B_1 \vee C_1, \ldots$ be an enumeration of all formulas of the form $B \vee C$. For $i, j = 0, 1, \ldots$, define sets Δ_{ij} as follows:

$$\begin{array}{rll}
\Delta_{00} &= \varnothing & \\
\Delta_{i(j+1)} &= \Delta_{ij} & \text{if } \Delta_{ij} \nvdash_{\mathbf{L}} B_i \vee C_j \\
&= \Delta_{ij} \cup \{B_j\} & \text{if } \{A\colon \Delta_{ij}, B_j \vdash_{\mathbf{L}} A\} \text{ does not intersect } \Gamma \\
& & \quad \text{and } \Delta_{ij} \vdash_{\mathbf{L}} B_j \vee C_j \\
&= \Delta_{ij} \cup \{C_j\} & \text{otherwise; and} \\
\Delta_{(i+1)0} &= \bigcup_{j=0}^{\infty} \Delta_{ij}. &
\end{array}$$

Finally, let $\Delta' = \bigcup_{i=0}^{\infty} \Delta_{i0}$.

1. Δ' is saturated.

PROOF. Suppose $\Delta' \vdash_{\mathbf{L}} B \vee C$. Then $(\exists j)(B \vee C = B_j \vee C_j)$. Since deductions are finite, $(\exists A_1, \ldots, A_n \in \Delta')(A_1, \ldots, A_n \vdash_{\mathbf{L}} B_j \vee C_j, n \geq 0)$. So clearly $(\exists i)(\Delta_{i0} \vdash_{\mathbf{L}} B_j \vee C_j)$. But then, by the definition of $\Delta_{i(j+1)}$, $B_j \in \Delta_{i(j+1)}$ or $C_j \in \Delta_{i(j+1)}$, and so B_j or $C_j \in \Delta$.

2. Δ' does not intersect Γ.

PROOF. Suppose otherwise, and choose minimum i, j such that $\Delta_{i(j+1)}$ intersects Γ. Since Δ does not intersect Γ, $j \geq 0$. So, by the definition of $\Delta_{i(j+1)}$, $\Delta_{ij} \vdash_{\mathbf{L}} B_j \vee C_j$, $(\exists B \in \Gamma)(\Delta_{ij}, B_j \vdash_{\mathbf{L}} B)$ and $(\exists C \in \Gamma)(\Delta_{ij}, C_j \vdash_{\mathbf{L}} C)$. By Theorem 2.2(i), $\Delta_{ij}, B_j \vdash B \vee C$ and $\Delta_{ij}, C_j \vdash B \vee C$; by Theorem 2.2(ii), $\Delta_{ij}, B_j \vee C_j \vdash B \vee C$; and, therefore, $\Delta_{ij} \vdash B \vee C$. But $B \vee C \in \Gamma$, contrary to i and j minimum.

For a logic **L**, the *canonical model* $\mathfrak{U}_{\mathbf{L}} = (T, S, l, \circ, -, \geq, \varphi)$, where:

$T = \{t\colon t \text{ is an } \mathbf{L}\text{-theory}\}$;

$S = \{a\colon a \text{ is } \mathbf{L}\text{-prime}\}$;

$l = \mathbf{L}$;

$t \circ u = \{B\colon (\exists A \in u)(A \rightarrow B \in t)\}$;

$-a = \{A\colon {\sim} A \notin a\}$;

$t \geq u \Leftrightarrow (\forall A)(A \in u \Rightarrow A \in t)$; and

$\varphi t p \Leftrightarrow p \in t$, where $t, u \in T$ and $a \in S$.

The *canonical frame* $\mathfrak{F}_{\mathbf{L}}$ is the frame on which $\mathfrak{U}_{\mathbf{L}}$ is based.

Lemma 2 (Modelhood). For any logic **L**, $\mathfrak{U}_{\mathbf{L}}$ is a model.

Proof. We must verify the conditions M(i)–(ix).

(i) $S \subseteq T$. Since **L**-prime ⇒ **L**-theory.

$l \in T$. By P8 and P9.

$\circ$: $T^2 \to T$. Suppose $t, u \in T$. To show $(t \circ u) \in T$. First we show that tu is closed under conjunction. Suppose $C, D \in tu$. Then

$$(\exists A,B \in u)(A \to C,\ B \to D \in t).$$

So, by P1, 2, 10, and 3, $A \wedge B \to C \wedge D \in t$. But $A \wedge B \in u$, and so $C \wedge D \in tu$.

Now we show that if $B \in tu$ and $B \to C \in \mathbf{L}$ then $C \in tu$. Suppose $B \in tu$ and $B \to C \in \mathbf{L}$. Then $(\exists A \in u)(A \to B \in t)$. Since $B \to C \in \mathbf{L}$, $A \to C \in t$, by P11; and so $C \in tu$.

$-$: $S \to S$. Suppose B, $C \in -a$ but $B \wedge C \notin -a$. Then ${\sim}(B \wedge C) \in a$. So ${\sim}B \vee {\sim}C \in a$, by Theorem 2.2(iii), and ${\sim}B$ or ${\sim}C \in a$ since a prime. But then B or $C \notin -a$. A contradiction.

Now suppose $B \vee C \in -a$, but B, $C \notin -a$. Then ${\sim}B$, ${\sim}C \in a$. By Theorem 2.2, ${\sim}(B \vee C) \in a$, and so $B \vee C \notin -a$. A contradiction.

Finally, suppose $B \in -a$, $B \to C \in \mathbf{L}$, but $C \notin -a$. Then ${\sim}C \in a$. Now ${\sim}C \to {\sim}B \in \mathbf{L}$, by Theorem 2.2(iv), and so ${\sim}B \in -a$. But then $B \notin -a$. A contradiction.

The other cases under (i) are straightforward.

(ii) Also straightforward.

(iii) Suppose $a \geq (t \circ u)$. First we show $(\exists c \geq u)(a \geq (t \circ c))$. Let $\Gamma = \{A\colon (\exists B)(B \notin a \text{ and } A \to B \in t)\}$. By Lemma 1, it suffices to show that Γ is closed under disjunction. So suppose A_1, $A_2 \in \Gamma$. Then

$$(\exists B_1, B_2 \notin a)(A_1 \to B_1,\ A_2 \to B_2 \in t).$$

By P11 and Theorem 2.2(i) and (ii), $A_1 \vee A_2 \to B_1 \vee B_2 \in t$.

Since a is prime, $B_1 \vee B_2 \in a$, and so $A_1 \vee A_2 \in \Gamma$.

Now we show $(\exists b \geq t)(a \geq bu)$. Let $\Gamma = \{A\colon (\exists B,C)(A \vdash_{\mathbf{L}} B \to C \text{ and } B \in u \text{ and } C \notin a)\}$. By Lemma 3.1, it suffices to show that Γ is closed under disjunction. So suppose A_1, $A_2 \notin \Gamma$. Then $(\exists B_1,B_2,C_1,C_2)(A_1 \vdash_{\mathbf{L}} B_1 \to C_1$, $A_2 \vdash_{\mathbf{L}} B_2 \to C_2$, B_1, $B_2 \in c$ and C_1, $C_2 \notin a)$. By Theorem 2.2(i), P10, and P11,

$$A_1 \vdash B_1 \wedge B_2 \to C_1 \vee C_2 \text{ and } A_2 \vdash B_1 \wedge B_2 \to C_1 \vee C_2;$$

and so, by Theorem 2.2(ii) $A_1 \vee A_2 \vdash_{\mathbf{L}} B_1 \wedge B_2 \to C_1 \vee C_2$. But $B_1 \wedge B_2 \in u$ and $C_1 \vee C_2 \notin a$ since a is prime and so $A_1 \vee A_2 \in \Gamma$.

(iv) Suppose $A \in t$. By Lemma 2.1, $A \to A \in \mathbf{L} = l$; and so $A \in lt$.

(v) Suppose $B \in lt$. Then $(\exists A \in t)(A \to B \in l = \mathbf{L})$. But, since t is an **L**-theory, $B \in t$.

(vi) Suppose $a \geq l$ and $A \in -a$. Then $(\sim A) \notin a$. But, by P7, $A \vee \sim A \in l \geq a$; and so, since a is prime, $A \in a$.

(vii) $A \in --a \Leftrightarrow \sim A \notin -a \Leftrightarrow \sim\sim A \in a$. But $\sim\sim A \in a \Rightarrow A \in a$, by P6, and $A \in a \Rightarrow \sim\sim A \in a$, by P6, 12, and 10.

(viii) Suppose $a \geq b$ and $A \in -a$. Then $\sim A \notin a$; so $\sim A \notin b$; and so $A \in -b$.

(ix) From Lemma 3.1.

Note that the canonical model $\mathfrak{U}_{\mathbf{L}}$ possesses some supererogatory properties. For example, $(T, \geq)$ is a complete lattice with a greatest and least member. Also, a right-handed version of M(ii) holds, viz. $(\forall t,u,v)\ (u \geq v \Rightarrow tu \geq tv)$.

Lemma 3 (Truth for the Canonical Model). For $\mathbf{L}$ a logic, $(\mathfrak{U}_{\mathbf{L}}, t) \models A \Leftrightarrow A \in t$.

Proof. Since C(i)–(iv) define $\models$, it suffices to show that membership satisfies the four conditions.

(i) $p \in t \Leftrightarrow \varphi tp$. By definition of φ.

(ii) $B \wedge C \in t \Rightarrow B, C \in t$, by P1 and P2. $B, C \in t \Rightarrow B \wedge C \in t$, since t is a theory.

(iii) $\sim B \in t \Rightarrow (\forall a \geq t)(\sim B \in a) \Rightarrow (\forall a \geq t)(B \notin -a)$, by definition of "$-$". By Lemma 3.1, $\sim B \notin t \Rightarrow (\exists a \geq t)(\sim B \notin a) \Rightarrow (\exists a \geq t)(B \in -a)$.

(iv) $B \rightarrow C \in t \Rightarrow (\forall u)(B \in u \Rightarrow C \in (t \circ u))$, by definition of $\circ$. Now suppose $B \rightarrow C \notin t$. Let $u = \{D\colon B \vdash_{\mathbf{L}} D\}$. Clearly, u is a theory and $B \in u$. Now $C \notin (tu)$. For otherwise, $(\exists D)(B \vdash_{\mathbf{L}} D$ and $D \rightarrow C \in t)$. But then, by P10, $B \rightarrow C \in t$ after all.

Theorem 1 (Completeness). **B** is complete for the class of all frames.

Proof. *Soundness*. By induction on the construction of proofs.

P1–5. Straightforward, given the corollary to Theorem 1.1.

P6. $a \models \sim\sim p \Rightarrow -a \not\models \sim p \Rightarrow --a \models p \Rightarrow a \models p$ (M(vii)).

P7. $a \not\models \sim p \Rightarrow -a \models p \Rightarrow a \models p$ for $a \geq l$ (M(vi)).

P8, 10 and 11. Straightforward.

P9. Suppose $\mathfrak{U} \models A$ and $\mathfrak{U} \models A \rightarrow B$; i.e., $l \models A$ and $l \models A \rightarrow B$. Then $ll = l \models B$; i.e., $\mathfrak{U} \models B$.

P12. Suppose $\mathfrak{U} \models A \rightarrow \sim B$. Then $a \models B \Rightarrow --a \models B$ (M(vii)) $\Rightarrow -a \not\models \sim\sim B \Rightarrow -a \not\models A \Rightarrow a \models \sim A$.

Sufficiency. Suppose $A \notin \mathbf{L}$. By Lemma 3.3, $(\mathfrak{U}_{\mathbf{L}}, l) \not\models A$; and so, by Lemma 3.2, A is not valid in all frames.

For a logic $\mathbf{L}$, a set of formulas Δ is $\mathbf{L}$-*consistent* if, for any formula $\sim A \in \mathbf{L}$, $\Delta \not\vdash_{\mathbf{L}} A$. $\mathbf{L}$ is *compact* if, for any $\mathbf{L}$-consistent set of formulas Δ, there is a model $\mathfrak{U}$ and a point $-a \geq l$ in $\mathfrak{U}$ such that $\mathfrak{U}$ is based on a frame that validates **L** and $(\mathfrak{U}, a) \models B$ for each $B \in \Delta$. The set $\Gamma = \{A\colon \sim A \in l\}$ is closed under disjunction. So, by Lemma 3.1, if Δ if **L**-consistent then ($\exists$**L**-prime a) ($a \geq \Delta$ and a does not intersect Γ). But then $-a \geq \mathbf{L}$, and so the proof of the theorem establishes

Corollary 1. **B** is compact.

§51.4. The systems E and R. This subsection is in three main parts. First, we prove completeness for **E**, **Π′**, and several of their subsystems. Second, we extend this proof to logics with a constant $\boldsymbol{t}$ for "maximum necessity" or a unary connective for necessity. Third, we consider the systems and their simplifications in more detail.

The main idea of the completeness proof is to set up a correspondence between added axioms and conditions on frames. Suppose A is a formula, R a rule of inference, and X a class of frames. *A corresponds to X* if (i) $\mathfrak{F}$ validates A whenever $\mathfrak{F} \in X$ and (ii) $\mathfrak{F}_{\mathbf{L}} \in X$ whenever $A \in \mathbf{L}$. *R corresponds to X* if (i) R is $\mathfrak{F}$-validity-preserving, i.e., $RA_1 \ldots A_n$ and $\mathfrak{F}$ validates $A_1 \ldots,$ A_{n-1} implies that $\mathfrak{F}$ validates A_n, and (ii) $\mathfrak{F}_{\mathbf{L}} \in X$ whenever **L** is closed under R. Correspondences give rise to completeness proofs in the following way. Given a list of postulates and matching conditions, as for example below, we let "n" refer to the nth postulate and "n'" refer to the nth condition. (Warning: we continue to use "Pn" to refer to the list of theorems and rules in §51.2.) Suppose that $\mathbf{B}n_1 \ldots n_k$ is the smallest logic with the Postulates (axioms or rules) $n_1, \ldots, n_k$. Then:

Lemma 1. If Postulates $n_1, \ldots, n_k$ correspond to Conditions $n'_1, \ldots, n'_k$, respectively, then $\mathbf{B}n_1 \ldots n_k$ is complete for Conditions $n'_1, \ldots, n'_k$.

Proof. As for Theorem 4.1.

We consider the following correspondences:

	Postulate	*Condition*
1	$(p\to q)\wedge(q\to r) \to (p\to r)$	$t(tu) \leq tu$
2	$(p\to q) \to ((q\to r)\to(p\to r))$	$t(uv) \leq (ut)v$
3	$(q\to r) \to ((p\to q)\to(p\to r))$	$t(uv) \leq (tu)v$
4	$(p\to\sim q) \to (q\to\sim p)$	$a \geq tb \Rightarrow b \geq t-a$
5	$p\wedge(p\to q) \to q$	$tt \leq t$
6	$(p\to q) \to \sim p\vee q$	$a-a \leq a$
7	$(p\to(q\to r)) \to (p\wedge q\to r)$	$(tu)u \leq tu$
8	$A\,/\,(A\to B)\to B$	$tl \leq t$
9	$p \to ((p\to q)\to q)$	$tu \leq ut$

All these conditions are fairly reasonable for the proposed interpretations of the functions and $-$. In the next section, we consider alternative and possibly more natural formulations of the conditions.

LEMMA 2. Postulate n corresponds to the class of frames satisfying Condition n', $n = 1, \ldots, 9$.

PROOF. We run through each postulate in turn.

Postulate 1. First we establish part (i) of the definition of "corresponds." Suppose $t \vDash (p \to q) \wedge (q \to r)$ and $u \vDash p$ (to show $(tu) \vDash r$). $t \vDash p \to q$, and so $(tu) \vDash q$. $t \vDash q \to r$, and so $t(tu) \vDash r$. Therefore, $tu \vDash r$, by Condition 1$'$.

Now we establish part (ii) of the definition of "corresponds." Suppose $C \in t(tu)$ in the canonical model $\mathfrak{U}_{\mathbf{L}}$ with Postulate $1 \in \mathbf{L}$. Then $(\exists B)(B \in tu$ and $B \to C \in t)$. So $(\exists A)(A \in u$ and $A \to B \in t)$. Since Postulate $1 \in \mathbf{L}$, $A \to C \in t$, and so $C \in tu$. But then $t(tu) \leq tu$, and $\mathfrak{F}_{\mathbf{L}}$ satisfies Condition 1$'$.

Postulate 2. Suppose $u \vDash p \to q$ (to show $u \vDash (q \to r) \to (p \to r)$). So also suppose $t \vDash (q \to r)$ (to show $ut \vDash p \to r$). And so, finally, suppose $v \vDash p$ (to show $(ut)v \vDash r$). Since $u \vDash p \to q$, $uv \vDash q$; and, since $t \vDash q \to r$, $t(uv) \vDash r$. But then, by Condition 2$'$, $(ut)v \vDash r$.

Now suppose $C \in t(uv)$ in the canonical model. Then $(\exists B)(B \in uv$ and $B \to C \in t)$. So $(\exists A)(A \in v$ and $A \to B \in u)$. By Postulate 2, $(B \to C) \to (A \to C) \in u$; so $A \to C \in ut$; and so $C \in (ut)v$.

Postulate 3. Similar to Postulate 2.

Postulate 4. Suppose $t \nvDash q \to {\sim}p$. Then $(\exists b)(b \vDash q$ and $t - b \nvDash {\sim}p)$; and, by Theorem 1.1, $(\exists a \leq tb)(a \nvDash {\sim}p)$. By Condition 4$'$, $-b \geq t - a$. Since $a \nvDash {\sim}p$, $-a \vDash p$; and, since $b \vDash q$, $-b \nvDash {\sim}q$ and $t - a \nvDash {\sim}q$. But then $t \nvDash p \to {\sim}q$.

Now suppose $B \in t - a$ and $B \notin -b$. Then $({\sim}B) \in b$ and $(\exists A)(A \in -a$ and $A \to B \in t)$. Given that Postulate $3 \in \mathbf{L}$, it is easy to show that $(A \to B) \to ({\sim}B \to {\sim}A) \in \mathbf{L}$; so ${\sim}B \to -A \in t$; and so ${\sim}A \in tb$. But ${\sim}A \notin a$, and so $a \nleq tb$.

Postulate 5. Suppose $t \vDash p \wedge (p \to q)$. Then $t \vDash p \to q$ and $(tt) \vDash q$. So, by Condition 5$'$, $t \vDash q$.

Now suppose $B \in (tt)$ in the canonical model. Then $(\exists A)(A \in t$ and $A \to B \in t)$. So, by Postulate 5, $B \in t$.

Postulate 6. Suppose $a \vDash p \to q$ and $a \nvDash {\sim}p$. Then $-a \vDash p$. So $(a - a) \vDash q$; and so, by Condition 6$'$, $a \vDash q$.

Suppose $B \in (a - a)$ in the canonical model. Then $(\exists A)(A \in -a$ and $A \to B \in a)$. By Postulate 6, ${\sim}A \vee B \in a$. But, since ${\sim}A \notin a$, $B \in a$.

Postulate 7. Suppose $t \vDash p \to (q \to r)$ (to show $t \vDash p \wedge q \to r$). So suppose $u \vDash p \wedge q$ (to show $tu \vDash r$). Then $u \vDash p$ and $tu \vDash q \to r$; but then $u \vDash q$ and $(tu)u \vDash r$; and so, by Condition 7′, $tu \vDash r$.

Now suppose $C \in (tu)u$ in the canonical model. Then $(\exists B)(B \in u$ and $B \to C \in tu)$. So $(\exists A)(A \in u$ and $A \to (B \to C) \in t)$. By Postulate 7, $A \wedge B \to C \in t$. But $A \wedge B \in u$; and so $C \in tu$.

Postulate 8. Suppose $\mathfrak{U} \vDash A$ (to show $\mathfrak{U} \vDash (A \to B) \to B$). So suppose $t \vDash A \to B$ (to show $t \vDash B$). Since $ll \vDash A$, $tl \vDash B$. But $tl \leq tl$, by Condition 8′, and so $t \vDash B$.

$tl \leq t$ in the canonical model $\mathfrak{U}_{\mathbf{L}}$. For suppose $A \in \mathrm{L}$ and $A \to B \in t$. Then $B \in t$, by Postulate 8.

Postulate 9. Suppose $t \vDash p$ (to show $t \vDash (p \to q) \to q$). So suppose $u \vDash p \to q$ (to show $tu \vDash q$). Then $ut \vDash q$; and so, by Condition 9′, $tu \vDash q$.

Now suppose $B \in ut$ in the canonical model. Then $(\exists A \in t)(A \to B \in u)$. By Postulate 9, $(A \to B) \to B \in t$; and so $B \in tu$.

We shall also consider the rule γ: $A,\ A \supset B \,/\, B$ and the Condition γ': $(\forall a \geq l)(\exists b \geq l)(b \leq a$ and $b \leq -b)$.

LEMMA 3. (i) If $\mathfrak{F}$ satisfies γ', then γ is $\mathfrak{F}$-validity-preserving. (ii) If **L** has γ but is not *Fml*, the set of all formulas, then $\mathfrak{F}_{\mathbf{L}}$ satisfies γ'.

PROOF. (i) Suppose $\mathfrak{U} \vDash A$, $\mathfrak{U} \vDash {\sim}A \vee B$ and $a \geq l$. By Condition γ', $(\exists b \geq l)(b \leq a$ and $b \leq -b)$; and so $b \vDash A$. Also, $b \vDash B$; for if $b \vDash {\sim}A$ then $-b \nvDash A$, contrary to $b \leq -b$. But then $a \vDash B$. Hence $(\forall a \geq l)(a \vDash B)$, and so $l \vDash B$.

(ii) Suppose $a \geq l$ in the canonical model $\mathfrak{U}_L$. Let $\Gamma = \{A_1 \vee \ldots \vee A_n$: ${\sim}A_i \in \mathbf{L}$ or $A_i \notin a$, for each i with $1 \leq i \leq n\}$. Now Γ does not intersect $\mathbf{L} = l$. For suppose $A = A_1 \vee \ldots \vee A_n \in \mathbf{L} \cap \Gamma$. We may suppose that ${\sim}A_1, \ldots, {\sim}A_k \in \mathbf{L}$ and $A_{k+1}, \ldots, A_n \notin a$, $0 \leq k \leq n$. Then there are three cases. (a) $k = 0$. But then, $A_1 \vee \ldots \vee A_n \in a$ since $a \supseteq \mathbf{L}$, and $(\exists i \geq 0)(A_i \in a)$, since a is **L**-prime. (b) $k = n$. Then ${\sim}A \in \mathbf{L}$. So, for any formula B, both ${\sim}A$ and $A \vee B \in \mathbf{L}$; so, by the rule γ, $B \in \mathbf{L}$; and **L** is the set of all formulas. (c) $0 < k < n$. Then ${\sim}(A_1 \vee \ldots \vee A_k) \in \mathbf{L}$. So, by the rule γ, $A_{k+1} \vee \ldots \vee A_n \in \mathbf{L}$; and so, by $a \geq l$, $(\exists i < k)(A_i \in a)$.

Γ is closed under disjunction. So, by Lemma 3.1, $(\exists b \geq \mathbf{L})$ (b does not intersect Γ). By the definition of Γ, $b \leq a$. Also $b \leq -b$. For suppose $A \in b$ and $A \notin -b$. Then ${\sim}A \in b$. So $A \wedge {\sim}A \in b$, contrary to $A \wedge {\sim}A \in \Gamma$.

The exclusion under (ii) is essential, but not important. It is essential because γ is not $\mathfrak{F}_{\mathbf{L}}$-validity-preserving for $\mathbf{L} = \mathit{Fml}$, and it is unimportant because *Fml* is complete for the null class of frames.

We now come to the central completeness result.

THEOREM 1. Any logic $\mathbf{L} = \mathbf{B}n_1 \ldots n_k$ is complete for the class of frames satisfying Conditions $n'_1, \ldots, n'_k$, where $n_1, \ldots, n_k \in \{1, \ldots, 9, \gamma\}$, $k \geq 0$.

PROOF. Interpreting $\rightarrow$ as material implication shows that $\mathbf{L} \neq Fml$. So the result follows from Lemmas 1, 2, and 3 of this subsection.

We note three corollaries. As a special case of Theorem 1, we have

COROLLARY 1. $\mathbf{B}1 \ldots 8$, $\mathbf{B}1 \ldots 8\gamma$, and $\mathbf{B}1 \ldots 89$ are complete.

The significance of this corollary is that it can be directly shown that $\mathbf{B}1 \ldots 8$ is equivalent to the system **E**, that $\mathbf{B}1 \ldots 8\gamma$ is equivalent to the system $\mathbf{\Pi}'$, and that $\mathbf{B}1 \ldots 89$ is equivalent to the system **R**. In the sequel, we shall appropriate these labels for the present axiomatizations of these logics.

In analogy to Corollary 2 to Theorem 3.1, we also have

COROLLARY 2. Any logic $\mathbf{L} = \mathbf{B}n_1 \ldots n_k$, where $n_1, \ldots, n_k \in \{1, \ldots, 9, \gamma\}$, $k \geq 0$, is compact.

Our previous results still hold for a language enriched with the constant $\boldsymbol{t}$ for maximum necessity.

$\boldsymbol{t}$ should receive the following clause in the definition of commitment:

(iv) $t \vDash \boldsymbol{t} \Leftrightarrow t \geq l$.

$\Box A$ (necessarily A) may be defined as $(\boldsymbol{t} \rightarrow A)$. Thus $t \vDash \Box A \Leftrightarrow (\forall u)((u \vDash \boldsymbol{t}) \Rightarrow (tu \vDash A)) \Leftrightarrow (\forall u \geq l)(tu \vDash A)$. So t commits one to necessarily A if, for any theory u containing logic, it commits one to the proposition that u commits one to A.

The minimal logic **B** should contain two new postulates:

14 $\boldsymbol{t}$
15 $A \,/\, \boldsymbol{t} \rightarrow A$.

The completeness proofs go through as before, but in the proof of Lemma 3.3 one must show that $\boldsymbol{t} \in t \Leftrightarrow t \geq l$. But $\Rightarrow$ follows from Postulate 15 and $\Leftarrow$ follows from Postulate 14.

Our previous results (with or without $\boldsymbol{t}$) also hold for a language enriched with another unary connective N for necessity.

Models should now contain a new component $'$ mapping T into T. (Warning: we are using the same "prime" symbol here as the one we use to mark the correspondence between postulates and conditions.) Intuitively, t' consists of those propositions which are necessary in t. The commitment clause for N is

(vi) $t \vDash NB \Leftrightarrow t' \vDash B$.

The minimal logic **B** should contain two new postulates:

16 $Np \wedge Nq \rightarrow N(p \wedge q)$
17 $A \rightarrow B \,/\, NA \rightarrow NB.$

Completeness is now straightforward. In the canonical model, t' should be defined as $\{A\colon NA \in t\}$. C(vi) above is automatic, and Postulates 16 and 17 are required only to show that t' is a theory if t is.

One could, of course, consider additional axioms on N and additional restrictions on $'$. There follows a list of some correspondences:

	Postulate	*Condition*
10	$Np \rightarrow p$	$t' \leq t$
11	$Np \rightarrow NNp$	$t' \leq t''$
12	$N(p \rightarrow q) \rightarrow (Np \rightarrow Nq)$	$t'u' \leq (tu)'$
13	$A \,/\, NA$	$l \leq l'$

Proofs of correspondence are straightforward. For example, to establish Condition 13′, one merely observes that $A \in l \Rightarrow A \in \mathbf{L}$ (definition of l) $\Rightarrow$ $NA \in \mathbf{L}$ (Postulate 13) $\Rightarrow$ $A \in \mathbf{L}'$ (definition of $'$).

Let us now consider the various subsystems and their simplifications in more detail. (Reminder: "P*n*" refers to the theorems and rules listed at the beginning of §51.2.)

1. Postulate 1 with 2 makes rule P10 redundant, and 1 with 3 makes rule P11 redundant. Moreover, in the presence of Postulates 1 and 4, Postulates 2 and 3 are equivalent.

2. Postulates 5 and 6 relate entailment to truth. There are various simplifications. In the presence of Postulate 5, 2 or 3 yields 1. In the presence of Postulate 6, P7 is redundant. Postulate 6 itself may be replaced by $(p \rightarrow {\sim} p) \rightarrow {\sim} p$.

It is amusing to establish these results on axioms model-theoretically. For example, let us show that conditions 5′ and 2′ (or 3′) imply 1′. $t(tu) \leq (tt)u$, by 2′ (or 3′); $tt \leq t$, by 5′; and so $t(tu) \leq tu$, by M(ii).

3. **B**1–7 is equivalent to the system **T** of ticket entailment. Postulate 7 yields 5 and may itself be replaced by $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$. In the presence of Postulate 9, Postulates 7–8 and one of the Postulates 2–3 are redundant.

Postulate 8, in the presence of the other axioms, may be replaced by the two axioms $((p \rightarrow p) \rightarrow q) \rightarrow q$ and $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

All these simplifications show that **E** = **B**34678 and that **R** = **B**3469.

4. $\boldsymbol{f}$ (minimum absurdity) can be defined as ${\sim}\boldsymbol{t}$. Given this definition, **Π**′ with $\boldsymbol{t}$ is equivalent to Ackermann's system **Π**″. $\boldsymbol{t}$ (or $\boldsymbol{f}$) is not definable in the original language, but $\Box A$ can be defined as $(A \rightarrow A) \rightarrow A$ in the presence of the Postulates 2, 3, and 8 (see §4.3). This is, in a sense, a stroke of luck; for

$\Box A$ would not appear to be definable in any of the logics without these postulates.

In the presence of Postulates 12 and 13, Postulate 8 may be replaced by the axiom $(\mathbf{t}\rightarrow p) \rightarrow p$, i.e., by $\Box p \rightarrow p$.

Finally, let us note that the above methods apply to many other postulates. For example: $(p\rightarrow(q\rightarrow r)) \rightarrow (q\rightarrow(p\rightarrow r))$ corresponds to $(tu)v \leq (tv)u$; $(p\rightarrow(q\rightarrow r)) \rightarrow ((p\rightarrow q)\rightarrow(p\rightarrow r))$ to $(tu)(vu) \leq (tv)u$; and $p\rightarrow(\mathbf{t} \rightarrow p) = p\rightarrow\Box p$ corresponds to $\forall a(a \geq l \Rightarrow ta \leq t)$.

§51.5. Alternative models. In this subsection we consider various alternative formulations of our original modeling. The proofs of model equivalence are, for the most part, straightforward and are therefore omitted.

1. In terms of $\geq$ and $-$, one can define a completeness relation Jab by $a \geq -b$ and a compatibility relation Kab by $-a \geq b$. M(vii) and (i) and (ii) of the corollary to Theorem 1.1 imply that (Jab and $a \nvDash A) \Rightarrow b \vDash {\sim}A$ and (Kab and $a \vDash A) \Rightarrow b \nvDash A$. Thus Ja, or *a is self-complete*, can be defined as Jaa, with (Jaa and $a \nvDash A) \Rightarrow a \vDash {\sim}A$; and Ka, or *a is self-compatible*, can be defined as Kaa with (Kaa and $a \vDash A) \Rightarrow a \nvDash {\sim}A$. Either J or K could be taken as primitive instead of $\geq$, and $a \geq b$ could then be defined as $Ja{-}b$ or $K{-}ab$.

2. The single theory l could be replaced by a set N of *normal* theories. The transition between the original models and the new models would be given by $N = \{a: a \geq l\}$ and $l = \bigcap N$ for $(T, \geq)$ complete. Intuitively, N is the set of possible or "overpossible" worlds. Corresponding to the normalcy conditions M(iv)–(vi), one would need: $(\forall a{\in}N)(\forall t{\in}T)(at \geq t)$; $(\forall a{\in}S)(\exists b{\in}N)(a \geq ba)$; and $(\forall a{\in}N)(a \geq a)$. A is valid, $\mathfrak{U} \vDash A$, if $(\forall a{\in}N)$ $(\mathfrak{U}, a) \vDash A)$. Condition $8'$ becomes $(\exists b{\in}N)(ab \leq a)$. Condition γ' becomes $(\forall a{\in}N)(\exists b{\in}N)(b \leq a$ and $b \leq -b)$ and allows one to replace the coy condition M(vi) by the more sensible: $(\forall a{\in}N)(a = -a)$. Thus N can now be regarded as the set of possible worlds. Finally, the commitment clause for $\mathbf{t}$ becomes:

$$t \vDash \mathbf{t} \Leftrightarrow (\forall a \geq t)(a{\in}N).$$

3. The nonsaturated theories could be dispensed with altogether. This is the main approach of Routley and Meyer as reported in §48. The closure function could no longer be used to evaluate entailments, since it need not be closed in S. Instead, one could use a relativized inclusion relation Rabc (also written $b \leq_a c$) with the sense that c is as strong as b relative to a.

Thus a model is now a quintuple $\mathfrak{A} = (S, N, R, -, \geq, \varphi)$ such that:

(i) $N \subseteq S$, $R \subseteq S^3$, $-: S\rightarrow S$, $\geq$ is a partial ordering on S, and $\varphi \subseteq S \times \mathbf{SL}$;
(ii) $c \leq_b d$ and $a \leq b \Rightarrow c \leq_a d$;
(iii) $(\forall a{\in}N)(b \leq_a c \Rightarrow b \leq c)$;
(iv) $(\exists b{\in}N)(a \leq_b a)$;

(v) $(\forall a \in N)(a \geq -a)$;
(vi) $a = --a$;
(vii) $a \geq b = -b \geq -a$;
(viii) φap and $a \leq b \Rightarrow \varphi bp$.

Because there are no unsaturated theories, M(ix) is simplified and M(iii) is dropped altogether.

The definition of commitment takes the following form:

(i) $a \vDash p \Leftrightarrow \varphi ap$;
(ii) $a \vDash B \wedge C \Leftrightarrow a \vDash B$ and $a \vDash C$;
(iii) $a \vDash \sim B \Leftrightarrow -a \nvDash B$; and
(iv) $a \vDash B \rightarrow C \Leftrightarrow (\forall b,c)(b \leq_a c$ and $b \vDash B \Rightarrow c \vDash C)$.

Conditions 1′, . . . , 8′ of the last subsection may be re-expressed as:

1 $b \leq_a c \Rightarrow (\exists d)(b \leq_a d$ and $d \leq_a c)$;
2 $(b \leq_a c$ and $d \leq_c e) \Rightarrow (\exists f)(d \leq_a f$ and $f \leq_b e)$;
3 $(b \leq_a c$ and $d \leq_c e) \Rightarrow (\exists f)(d \leq_b f$ and $f \leq_a e)$;
4 $b \leq_a c \Rightarrow -c \geq_a -b$;
5 $a \leq_a a$;
6 $-a \leq_a a$;
7 $b \leq_a c \Rightarrow (\exists d)(b \leq_a d$ and $b \leq_b c)$; and
8 $(\exists b \in N)(b \leq_a a)$.

Given an original model, one may derive an equivalent new model by letting $b \leq_a c \Leftrightarrow (a \circ b) \leq c$. Conditions 1′–8′ will then also carry over. Let us take 3′ as an example. Suppose $b \leq_a c$ and $d \leq_c e$; i.e., $ab \leq c$ and $cd \leq e$. By M(ii) (and the transitivity of $\leq$), $(ab)d \leq e$; and so, by the original Condition 3′, $a(bd) \leq e$. By M(iii) and M(ii), $(\exists f)(bd \leq f$ and $af \leq e)$; i.e., $(\exists f)(d \leq_b f$ and $f \leq_a e)$.

The derived models also possess the property:

(ix) $b \leq_a c$ and $c \leq d \Rightarrow b \leq_a d$.

Given a new model with this property, one may obtain an equivalent original model $\mathfrak{U}^*$ by letting

$$\begin{aligned}
T^* &= \{S' \subseteq S \colon (\forall a \in S', b \in S)(a \leq b \Rightarrow b \in S')\};\\
S^* &= \{[a] \colon a \in S\}, \text{ where } [a] = \{b \in S \colon b \geq a\};\\
N^* &= \{[a] \colon a \in N\};\\
S_1 \circ^* S_2 &= \{c \colon (\exists a \in S_1, b \in S_2)(b \leq_a c)\};\\
-^*[a] &= [-a];\\
S_1 \leq^* S_2 &\Leftrightarrow S_1 \supseteq S_2;\\
\varphi^* S_1 p &\Leftrightarrow (\forall a \in S_1)(\varphi ap), \text{ where } a \in S \text{ and } S_1, S_2 \in T^*.
\end{aligned}$$

For models that satisfy (ix), one has

(x) $b \leq c \Leftrightarrow (\exists a \in N)(b \leq_a c)$.

Given this condition, (iii) and (iv) are then redundant. For (iii) is (x) $\Leftarrow$, and (iv) follows from (x) and the reflexivity of $\leq$.

4. $tu = v$ states that v is *exactly* the commitment of u by t. $b \leq_a c$ $(ab \leq c)$ states that c is *at least* the commitment of b by a. So it is natural to introduce a relation $u \geq_t v$ $(tu \geq v)$ with the sense that v is *at most* the commitment of u by t. (Note that $\geq_a$ is *not* the converse of $\leq_a$.)

It is not good to do model theory in terms of $\geq_t$. The evaluation clause for entailment is the clumsy:

$$t \vDash B \to C \Leftrightarrow (\forall u)(\exists v)(u \vDash B \Rightarrow u \geq_t v \text{ and } v \vdash C).$$

Also, $\geq_t$-models do not permit the elimination of nonsaturated theories and do not appear to permit the formulation of conditions corresponding to Postulates 2 or 3.

However, $\geq_t$-models do highlight the intriguing metamorphosis of conditions that can be induced by a different choice of primitives. For example, the $\geq_t$-version of Condition 1′ is:

$$u \geq_t v \text{ and } v \geq_t w \Rightarrow u \geq_t w.$$

Thus this condition becomes semi-associativity for $\circ$-models, relativized density for $\leq_a$-models, and relativized transitivity for $\geq_t$-models.

5. The co-function $-$ could be dropped from a model. [Note by principal authors: This in effect extends the four-valued semantics of §50 to nested implications.] One could then add a negative valuation $\bar{\varphi}$ and a negative relativized inclusion relation $\bar{R}$ (or closure function $\bar{\circ}$) and evaluate positive ($\vDash$) and negative ($\Dashv$) commitment independently. Thus:

(i) (a) $a \vDash p \Leftrightarrow \varphi ap$
    (b) $a \Dashv p \Leftrightarrow \bar{\varphi} ap$
(ii) (a) $a \vDash B \wedge C \Leftrightarrow a \vDash B$ and $a \vDash C$
    (b) $a \Dashv B \wedge C \Leftrightarrow a \Dashv B$ or $a \Dashv C$
(iii) (a) $a \vDash {\sim} B \Leftrightarrow a \Dashv B$
    (b) $a \Dashv {\sim} B \Leftrightarrow a \vDash B$
(iv) (a) $a \vDash B \to C \Leftrightarrow (\forall b,c)(Rabc$ and $b \vDash B \Rightarrow c \vDash C)$
    (b) $a \Dashv B \to C \Leftrightarrow (\exists b,c)(\bar{R}abc$ and $b \vDash B$ and $c \Dashv C)$

In order to reformulate Conditions 4′ and 6′, one would need an appropriate relation, say the completeness relation J. An advantage of these models is that the saturated theories need not be closed under a co-function $-$. One can thereby consider logics that do not have the rule $A \to {\sim} B \,/\, B \to {\sim} A$.

6. The combinations of Conditions 1′–9′ appear to lack any unity or underlying rationale. One can overcome this defect by so structuring the models that the appropriate combinations of conditions automatically hold. We illustrate this procedure for the logics **C** = **B3468**, **E** and **R**.

A **C**-articulated model is a model $\mathfrak{U} = (T, S, l, \circ, -, \geq, \varphi)$ satisfying Conditions 4′ and 6′ and such that for some set S_0,

$T = \{\alpha\colon \alpha$ is a finite (possibly empty) sequence of S_0-elements$\}$;
$S = \{\alpha \in T\colon \alpha$ is one-termed$\}$;
$l = *$, the null sequence; and
$\circ$ is the concatenation function restricted to T.

An **E**-articulated model is a model $\mathfrak{U} = (T, S, l, \circ, -, \geq, \varphi)$ satisfying Conditions 4′ and 6′ and such that, for some set S_0: $T = \{\alpha\colon \alpha$ is a finite sequence of S_0-elements without repeats, i.e., without consecutive and identical terms$\}$,

$S = \{\alpha \in T\colon \alpha$ is one-termed$\}$;
$l = *$;
$\alpha \circ \beta =$ the largest subsequence of $\alpha\beta$ in T, $\alpha, \beta \in T$.

It is easy to show that $\alpha \circ \beta$ is uniquely defined.

Finally, an **R**-articulated model is a model, $\mathfrak{U} = (T, S, l, \circ, -, \geq, \varphi)$ satisfying conditions 4′ and 6′ and such that for some set S_0:

$T = \{\alpha\colon \alpha$ is a finite subset of $S_0\}$;
$S = \{\alpha \in T\colon \text{card}\ \alpha = 1\}$;
$l = \varnothing$, the null set; and $\circ$ is set-theoretic union restricted to T.

It follows from the concrete specification of T, S, l and $\circ$, that **C**-models satisfy Conditions 3′ and 8′, that **E**-models satisfy Conditions 3′, 7′, and 8′, that **R**-models satisfy Conditions 3′, 7′, 8′, and 9′, and that all the articulated models satisfy M(iv) and M(v). Now Conditions M(vi) and M(viii) follow from Conditions 4′ and 6′. So these models can be defined as the appropriate structures satisfying M(i), (ii), (iii), (vii), and (ix) and Conditions 4′ and 6′.

C-, **E**-, and **R**-frames are, of course, the frames on which **C**-, **E**- and **R**-models are based. It can be shown, either directly or by a transformation on ordinary models, that **C**, **E**, and **R**, respectively, are complete for the class of **C**-, **E**-, and **R**-frames.

Intuitively, the elements of S_0 may be regarded as consolidated *assumptions* or assumption sets. T elements are assumption *complexes*, S elements are *unit* complexes identifiable with the original assumptions, and l is the *null* complex. The relation $\models$ is now *deducibility*. Validity is simple deducibility from the null complex. Clause (iv) for $\rightarrow$ is a form of the Deduction theorem: the complex α yields $B \rightarrow C$ iff whenever β yields B the combination of α and β yields C. Thus $\circ$ is now the combination operation on complexes which is appropriate to the Deduction theorem.

The different kinds of articulated models reflect different views on deducibility from a complex of assumptions. For **C**-articulated models, both the order and the repetition of assumptions within a complex are relevant to deduction. For **E**-articulated models, order is relevant, but repetitions are automatically collapsed. For **R**-models, neither order nor repetition is relevant.

7. Some of the ideas of our modeling should have other applications. For example, one could use theories with a limit evaluation in classical modal logic. This idea has been subsequently developed in Humberstone 1981. That account differs from unpublished work of the author of the present section in employing an accessibility relation on the partial possible worlds rather than an operation ′, as in clause (vi) of §51.4.

Also, one could obtain completeness proofs for abnormally weak conditional logics by relaxing some of the conditions on a model. **B** without P7 is complete for the class of frames that need not satisfy M(iv). The class of frames that need not satisfy M(v) gives rise to a logic in which no entailments are valid; and the class of frames that need not satisfy M(v) or M(vi) gives rise to the weakest of all logics, the null class of formulas.

§51.6. Finite models. A logic has the *finite model property* (*fmp*) if it is complete for a class of finite frames. In this subsection we show that a good many of our logics possess *fmp*. Without any loss of generality and with some gain in convenience, we may assume that the logics are formulated with the constant $\boldsymbol{t}$.

Fix on a logic $\mathbf{L}$ and a set of formulas Δ containing $\boldsymbol{t}$. Let Γ be the smallest set of formulas containing Δ which is closed under subformula and truth-functional composition, i.e., such that $(\forall A,B)$ $(\sim A \in \Gamma \Rightarrow A \in \Gamma$ and $(A \wedge B, A \rightarrow B \in \Gamma) \Rightarrow (A, B \in \Gamma))$ and $(\forall A,B)(A, B \in \Gamma \Rightarrow \sim A, A \wedge B \in \Gamma)$. Suppose that the canonical model $\mathfrak{U}_{\mathbf{L}}$ is $(T, S, l, \circ, -, \geq, \varphi)$. For $t \in T$, let $t_0 = \{A\colon t \cap \Gamma \vdash_{\mathbf{L}} A\}$. Note that $t_0 \in T$. We let the *canonical model* $\mathfrak{U}_{\mathbf{L},\Delta}$ *for* $\mathbf{L}$ *and* Δ be $(T', S', l', -', \circ', \geq', \varphi')$, where

$$T' = \{t_0\colon t \in T\}$$
$$S' = \{a_0\colon a \in S\}$$
$$l' = l_0$$
$$t \circ' u = (t \circ u)_0$$
$$-'(a_0) = (-a)_0$$
$$t \geq' u \Leftrightarrow t \geq u$$
$$\varphi' tp \Leftrightarrow \varphi tp, \text{ where } t, u \in T', \text{ and } p \in \mathrm{SL}.$$

$\mathfrak{F}_{\mathbf{L},\Delta}$ is the frame on which $\mathfrak{U}_{\mathbf{L},\Delta}$ is based.

The definition of $-'$ is correct. For assume $a_0 = b_0$. We must show that $-a \cap \Gamma = -b \cap \Gamma$. So suppose that $A \in -a \cap \Gamma$. Then $\sim A \notin a$. Since $\sim A \in \Gamma$, $\sim A \notin b$, and so $A \in -b \cap \Gamma$.

Where there is no ambiguity, we may omit the superscript "′" for $\mathfrak{U}_{\mathbf{L},\Delta}$. $\mathfrak{U}_{\mathbf{L},\Delta}$ is essentially the model induced by the congruence relation $t \cap \Gamma = u \cap \Gamma$. However, we have exploited the fact that $(T, \geq)$ is complete, and we have identified each equivalence class $|t|$ with $\bigcap |t| = t_0$.

LEMMA 1. If Δ is finite, then $\mathfrak{F}_{\mathbf{L},\Delta}$ (i.e., T') is finite.

PROOF. Let $\Delta' = \Delta \cup \{\sim A\colon A \in \Delta\}$. We may show by an easy induction that if $a \cap \Delta' = b \cap \Delta'$ then $a_0 = b_0$. So S' is finite. Now if $\{a_0\colon a \geq t\} = \{a_0\colon a \geq u\}$ then $t_0 = u_0$. For if $t_0 \neq u_0$, we may assume $(\exists A \in t \cap \Gamma)(A \notin u)$. So $(\exists a \geq u)(A \notin a)$. But $(\forall b \geq t)(A \in b)$, and so $b_0 \neq a_0$. Therefore, T' is also finite.

LEMMA 2. $a_0 \geq t \circ' u \Rightarrow (\exists a', b, c)(a_0 \geq a'_0,\ \ b_0 \geq t,\ \ c_0 \geq u,$ and $a' \geq t \circ c)$, where $t, u \in \Gamma$, $a \in S$.

PROOF. Assume $a_0 \geq t \circ' u$, i.e., $a_0 \geq (tu)_0$. Let $\Gamma' = \{A \in \Gamma\colon A \notin a\}$. Since $a \in S$ and Γ is closed under disjunction, so is Γ'. Now tu does not intersect Γ; otherwise $(tu)_0$ intersects Γ and $a_0 \ngeq t \circ' u$. Therefore $(\exists a' \geq tu)(a'$ does not intersect $\Gamma)$. But then $a_0 \geq a'_0$, and, by M(iii), $(\exists b \geq t)(\exists c \geq u)(a' \geq b \circ u$ and $a' \geq t \circ c)$.

LEMMA 3. $\mathfrak{U}_{\mathbf{L},\Delta}$ is a model.

PROOF. We must verify the conditions M(i)–(ix).

(i) $S' \subseteq T'$. Since $S \subseteq T$.
$l' \in T'$. Since $l \in T$.
$-'\colon T^2 \rightarrow T'$. By definition.
$-\colon S' \rightarrow S'$. Since $-\colon S \rightarrow S$.
$\geq'$ is reflexive transitive and antisymmetric. Since $\geq$ is, and $\geq'$ is a restriction of $\geq$.
$\varphi' \subseteq T' \times \mathrm{SL}$. By definition.

(ii) For $t, u, v \in T'$, $t \geq' u \Rightarrow t \circ v \geq u \circ v \Rightarrow (t \circ v)_0 \geq (uv)_0 \Rightarrow t \circ' v \geq u \circ' v$.

(iii) Suppose $a_0 \geq t \circ' u$. By Lemma 2, $(\exists a', b, c)(a_0 \geq a'_0, b_0 \geq t, c_0 \geq u, a' \geq bu$, and $a' \geq tc)$. But then $a_0 \geq a'_0 \geq (bu)_0 \geq (b_0 u)_0 = b_0 \circ' u$. Similarly, for $a_0 \geq t \circ' c_0$.

(iv) To show $l' \circ' t \geq t$. Suppose $A \in t \cap \Gamma$. Since $\mathbf{t} \in l \cap \Gamma$, $\mathbf{t} \in l'$, and so $A \rightarrow A \in l'$. But then $A \in (l't)$, and so $A \in (l't)$.

(v) $t \geq lt \geq l't \geq l' \circ' t$.

(vi) For $a \geq l$, $a \geq -a$. So $a_0 \geq (-a)_0 = -'(a_0)$.

(vii) $-'-'(a_0) = -'(-a_0) = (-a)_0 = a_0$.

(viii) Suppose $a_0 \geq b_0$ and $A \in -'(a_0) \cap \Gamma = (-a) \cap \Gamma$. Then $\sim A \notin a_0$, and so $\sim A \in b_0$. But $\sim A \in \Gamma$, and so $A \in (-b_0) = -'(b_0)$.

(ix) Suppose $t \in T'$. Then $\varphi' tp \Rightarrow \varphi tp \Rightarrow (\forall a_0 \in S')\ (a_0 \geq t \Rightarrow \varphi a_0 p$, i.e., $\varphi' a_0 p)$. Also, (not $-\varphi' tp) \Rightarrow (\exists a \in S)(a \geq t$ and not $-\varphi ap) \Rightarrow (\exists a_0 \in S')\ (a_0 \geq t$ and (not $-\varphi a_0 p)$, i.e., (not $-\varphi' a_0 p$)).

LEMMA 4. If $\mathfrak{F}_L$ satisfies 1′, 4′, 5′, 6′, 7′, 8′, 9′, or γ', then so does $\mathfrak{F}_{L,\Delta}$.

PROOF. 1′. For $t, u \in T'$, $t(tu)_0 \leq t(tu) \leq tu$. So $(t(tu)_0)_0 \leq (tu)_0$. But $t \circ' (t \circ' u) = t \circ' (tu)_0 = (t(tu)_0)_0$ and $t \circ' u = (tu)_0$. Hence $t \circ' (t \circ' u) \leq t \circ' u$.

4′. Suppose $c_0a_0 \geq t_0\circ' c_0$. By Lemma 2, $(\exists a',b',c')(a_0 \geq a'_0, b'_0 \geq t_0, c'_0 \geq c_0$, and $a' \geq b'c')$. Therefore, $-c' \geq b' - a'$. But $-'(c_0) \geq -'(c'_0)$ (M(viii)) $= (-c')_0 \geq (b'-a')_0 \geq (b'_0(-a')_0)_0 \geq (b'_0 -'(a'_0))_0 \geq (t_0 -'(a_0))$, since $b'_0 \geq t_0$ and $-'(a'_0) \geq -'(a_0)$ (by M(viii)).

5′. For $t \in T'$, $tt \leq t$, So $t\circ t = (tt)_0 \leq t$.

6′. $a - a \leq a$. So $(a_0 -'(a_0))_0 = (a_0(-a)_0)_0 \leq (a - a_0) \leq a_0$.

7′. For t, $u \in T'$, $(tu)u \leq tu$. So $(t\circ' u)\circ' u = ((tu)_0u)_0 \leq ((tu)u)_0 \leq (tu)_0 = t\circ' u$.

8′. For $t \in T'$, $tl \leq t$. So $(t\circ' l') \leq (t\circ l')_0 \leq (t\circ l)_0 \leq t$.

9′. For $t, u \in T'$, $tu \leq ut$. So $t\circ' u = (t\circ u)_0 \leq (ut)_0 \leq u\circ' t$.

γ'. $(\forall a \geq l)(\exists b \geq l)(b \leq a$ and $b \leq -b)$. Therefore, $(\forall a \geq l')(\exists b \geq l)(b_0 \leq a_0$, and $b_0 \leq (-b)_0 = -'(b_0))$.

LEMMA 5. For any formula $A \in \Gamma$ and theory $t \in T$, $(\mathfrak{A}_{\mathbf{L},\Delta}, t_0) \vDash A \Leftrightarrow (\mathfrak{A}_{\mathbf{L}}, t) \vDash A$.

PROOF. By induction on the construction of A.

(0) $A = \boldsymbol{t}$. $(\mathfrak{A}_{\mathbf{L},\Delta}, t_0) \vDash \boldsymbol{t} \Leftrightarrow t_0 \geq l' = l \Leftrightarrow t \geq l \Leftrightarrow (\mathfrak{A}_{\mathbf{L}}, t) \vDash \boldsymbol{t}$.

(1) $A = p$. $(\mathfrak{A}_{\mathbf{L},\Delta}, t_0) \vDash p \Leftrightarrow \varphi' t_0 p \Leftrightarrow \varphi t_0 p \Leftrightarrow \varphi t p$ (since $p \in \Gamma$) $\Leftrightarrow (\mathfrak{A}_{\mathbf{L}}, t) \vDash p$.

(2) and (3). $A = {\sim}B$ and $A = (B \wedge C)$. Straightforward, given that Γ is closed under subformulas.

(4) $A = B \rightarrow C$. $\Rightarrow$. Suppose $(\mathfrak{A}_{\mathbf{L}}, t) \vDash' B \rightarrow C$. Then $(\exists u)[(\mathfrak{A}_{\mathbf{L}}, u) \vDash B$ and $(\mathfrak{A}_{\mathbf{L}}, tu) \nvDash C$ and $t_0u_0 \leq tu]$ and so $(\mathfrak{A}_{\mathbf{L}}, t_0u_0) \nvDash C$. By induction hypothesis (IH), $(\mathfrak{A}_{\mathbf{L},\Delta}, u_0) \vDash B$ and $(\mathfrak{A}_{\mathbf{L},\Delta}, (t_0u_0)_0) \nvDash C$. But $(t_0u_0)_0 = t\circ' u_0$ and so $(\mathfrak{A}_{\mathbf{L},\Delta}, t_0) \nvDash B \rightarrow C$.

$\Leftarrow$. Suppose $(\mathfrak{A}_{\mathbf{L},\Delta}, t_0) \nvDash B \rightarrow C$. Then $(\exists u_0)[(\mathfrak{A}_{\mathbf{L},\Delta}, u_0) \vDash B$ and $(\mathfrak{U}_{\mathbf{L},\Delta}, (t_0u_0)_0) \nvDash C]$. By IH, $(\mathfrak{A}_{\mathbf{L}}, u_0) \vDash B$ and $(\mathfrak{A}_{\mathbf{L}}, t_0u_0) \nvDash C$; and so $(\mathfrak{U}_{\mathbf{L}}, t_0) \nvDash B \rightarrow C$. But since $B \rightarrow C \in \Gamma$, $(\mathfrak{A}_{\mathbf{L}}, t_0) \vDash B \rightarrow C \Leftrightarrow (\mathfrak{A}_{\mathbf{L}}, t) \vDash B \rightarrow C$; and so $(\mathfrak{A}_{\mathbf{L}}, t) \nvDash B \rightarrow C$.

THEOREM 1. Any logic $\mathbf{L} = \mathbf{B}n_1 \ldots n_k$ has *fmp*, where

$$n_1, \ldots, n_k \in \{1, 4, 5, 6, 7, 8, 9, \gamma\}.$$

PROOF. Let X be the class of finite frames satisfying the conditions $n'_1, \ldots, n'_k$. By Theorem 4.1, $\mathbf{L}$ is sound for X. Now suppose $A \notin \mathbf{L}$. By previous results, $(\mathfrak{A}_{\mathbf{L}}, l) \nvDash A$, where $\mathfrak{F}_{\mathbf{L}} \in X$. Let $\Delta = \{A\}$. By Lemma 1, $\mathfrak{F}_{\mathbf{L},\Delta}$ is finite; by Lemmas 3 and 4, $\mathfrak{F}_{\mathbf{L},\Delta}$ satisfies $n'_1, \ldots, n'_k$; and so $\mathfrak{F}_{\mathbf{L},\Delta} \in X$. By Lemma 4, $(\mathfrak{A}_{\mathbf{L}}, l') \nvDash A$ and so A is not valid in some frame of X.

We can now prove in the usual way

COROLLARY. Each of the logics $\mathbf{L} = \mathbf{B}n_1 \ldots n_k$ above is decidable.

Unfortunately, the argument of this section and its straightforward modifications do not work for $\mathbf{E}$ or, indeed, for any of our logics containing

Axioms 2 or 3. Consider Axiom 3. For $t, u, v \in T'$, $t\circ'(u\circ'v) = (t(uv)_0)_0 \leq (t(uv))_0 \leq ((tu)v)_0$. But here we must stop since $(t\circ'u)\circ'v = ((tu)_0 v)_0$. This breakdown in the proof is attributable to the undecidability results of §65.

§51.7. Admissibility of (γ). This section shows that many of our logics are closed under the rule γ. We shall consider simultaneously the logics with or without ***t*** and suppose, for convenience, that the logics are formulated with N from §51.5(2) in place of l. (The proof of this section is similar to that of Routley and Meyer as described in §48.8; see §25 and §42 for other proofs.)

Fix on a model $\mathfrak{A} = (T, S, N, \circ, -, \geq, \varphi)$ and an element $d \in N$ such that not $(-d \geq d)$. Suppose $e \notin T$ and let $T' = T \cup \{e\}$. Define $f\colon T' \to T$ and $g\colon T' \to T$ by $(\forall t \in T)(f(t) = g(t) = t)$ and $f(e) = d$ and $g(e) = -d$. Finally, let $\mathfrak{A}' = (T', S', N', \circ', -', \geq', \varphi')$, where

$$\begin{aligned}
S' &= S \cup \{e\}\\
N' &= N \cup \{e\}\\
t\circ' u &= f(t)\circ f(u) \text{ if } t \text{ or } u \in T\\
&= e \text{ if } t = u = e\\
-'a &= -a \text{ if } a \in T\\
&= e \text{ if } a = e\\
t \geq' u &\Leftrightarrow t = u = e \text{ or } g(t) \geq f(u)\\
\varphi' tp &\Leftrightarrow \varphi f(t)p, \text{ where } t, u \in T \text{ and } a \in S.
\end{aligned}$$

Note that $-e = e$. Our aim is that $\mathfrak{A}'$ should satisfy any of the conditions satisfied by $\mathfrak{A}$ and that it should agree with $\mathfrak{A}$ on the points of T. For ease of reading we may drop the superscript $'$ in referring to the components of $\mathfrak{A}'$. This should cause no ambiguity since $\mathfrak{A}'$ is an extension of $\mathfrak{A}$.

First, we note some elementary facts.

LEMMA 1.

(a) $g(t) \leq f(t)$
(b) $f(tu) = f(t)f(u)$
(c) $g(tu) = f(tu)$ for $t, u \in T$
(d) $g(tu) \geq g(t)f(u)$
(e) $-f(a) = g(-a)$ and $-g(a) = f(-a)$, where $t, u \in T$ and $a \in S$.

PROOF.

(a) If $t \in T$, $g(t) = t = f(t)$. If $t = e$, $g(t) = -d \leq d$ (M(vi)) $= f(t)$.

(b) and (c). If $t = u = e$, $f(tu) = f(ee) = f(e) = d$ and $f(t)f(u) = dd$. But $dd \geq d$, by M(iv), and $d \geq dd$, by Condition 5′. So since $\geq$ is antisymmetric, $f(tu) = d = dd = f(t)f(u)$. If t or $u \in T$, $tu = f(t)f(u)$, by definition. But $f(t)f(u) \in T$, and so $f(tu) = g(tu) = tu = f(t)f(u)$.

(d) If $t = u = e$, $g(tu) = g(ee) = g(e) = -d$ and $g(t)f(u) = -dd \leq -d$, by Condition 6′. If t or $u \in T$, $g(tu) = f(tu)$ ((c)) $= f(t)f(u)$ ((b)) $\geq g(t)f(u)$ (by (a) and M(ii)).

(e) If $a \in T$, $-f(a) = -a = -a = -g(a) = f(-a)$. If $a = e$, then $-f(a) = -d = g(e) = g(e) = g(-a)$ and $-g(a) = --d = d = f(e) = f(-e)$.

Lemma 2. $\mathfrak{U}'$ is a model that satisfies conditions 5′ and 6′.

Proof. (i) Straightforward except for $\geq'$ a partial ordering.

$\geq'$ is reflexive. For, if $t = e$ then $t \geq' t$, and if $t \in T$ then $f(t) = g(t) \geq f(t)$, and so $t \geq' t$.

$\geq'$ is antisymmetric. For suppose $t \geq' u$ and $u \geq' t$. If $t, u \in T$, then $t = u$. Otherwise we may suppose $t \in T$ and $u = e$. But then $t \geq d \geq -d \geq t$; and so $-d \geq d$, contrary to assumption.

$\geq'$ is transitive. For suppose $t \geq' u$ and $u \geq' v$. If any two of t, u, or $v = e$, then clearly $t \geq' v$. If any two of t, u, or $v \in T$, then $g(t) \geq f(u) \geq g(u)$ ((a)) $\geq f(v)$; and so $t \geq' v$.

(ii) Suppose $t \geq u$. If $t = u = e$, then $tv = uv$ and so $tv \geq tu$. If t or $u \in T$, then $g(t) \geq f(u)$; so $g(t \circ v) \geq g(t)f(v)$ ((d)) $\geq f(u)f(v) \geq f(uv)$ ((b)); and so $tv \geq uv$.

(iii) Suppose $a \geq tu$. If $t = u = e$, put $b = c = e$. If t or $u \in T$, then $g(a) \geq f(tu) = f(t)f(u)$ ((c)). So $(\exists b, c \in S)(g(b) = b \geq f(t)$, $g(c) = c \geq f(u)$, and $g(a) \geq bc = f(bc))$. But then $b \geq t$, $c \geq u$, and $a \geq bc$.

(iv) (for N') If $a = t = e$, then $at = ee = e \geq e$. If a or $t \in T$, then $g(at) \geq f(a)f(t)$ ((c)) $\geq f(t)$ (since $f(a) \in N$); and so $at \geq t$.

(v) (for N'). For $a = e$, put $b = e$.

(vi) and (vii). By $-e = e$.

(viii) Suppose $a \geq b$. If $a = b = e$, then $-b = e \geq e = -a$. If a or $b \in T$, then $g(a) \geq f(b)$; so $g(-b) = -f(b)$ ((e)) $\geq -g(a) = f(-a)$ ((e)); and so $-b \geq -a$.

(ix) Suppose $\varphi' tp$ and $t \leq u$. Then $\varphi f(t)p$ and $f(p) \leq f(u)$; and so $\varphi f(u)p$, i.e., $\varphi' up$.

5′ and 6′. Since $e = -e$ and $ee = e$.

Lemma 3. If $\mathfrak{U}$ satisfies any of the conditions 1, 2, 3, 4, 7, 8, 9 then so does $\mathfrak{U}'$.

Proof. Condition 1′. If $t = u = e$, then $t(tu) = e = tu$. If t or $u \in T$, $f(t(tu)) \leq f(t)f((tu))$ ((b)) $\leq f(t)(f(t)f(u))$ ((b)) $\leq f(t)f(u) \leq g(tu)$ ((b) and (c)); and so $t(tu) \leq tu$.

2′, 3′, 7′, and 9′. It should be clear that the proof for 1′ depends solely on the fact that, in the inequality "$\alpha \leq \beta$", α and β are terms in '$\circ$' and β contains all the variables of α. So similar proofs apply to 2′, 3′, 7′, and 9′.

4′. Suppose $a \geq b$. If $a = t = b = e$, then $-b = e \geq ee = t-a$. If one of a, t or $b \in T$, then $g(a) \geq f(tb) \geq f(t)f(b)$ ((b)). But then $g(-b) \geq -f(b)$ ((d)) $\geq f(t)-g(a) \geq f(t)f(-a)$ ((d)) $\geq f(t-a)$ ((b)); and so $-b \geq t-a$.

8′ (for N'). For $a{\in}T$, $(\exists b{\in}N)(ab \leq a)$. For $a = e$, $e{\in}N'$ and $ee = e \leq e$.

Lemma 4. For each $t{\in}T$, $(\mathfrak{U}, t) \models A \Leftrightarrow (\mathfrak{U}', t) \models A$.

Proof. By induction on the construction of A.

(0) $A = \boldsymbol{t}$ (for a logic with $\boldsymbol{t}$). $(\mathfrak{U}, t) \models \boldsymbol{t} \Leftrightarrow (\forall a{\geq}S)(a \geq t \Rightarrow a{\in}N) \Leftrightarrow (\forall a{\in}S')(a \geq t \Rightarrow a{\in}N')$(since $e{\in}N'$) $\Leftrightarrow (\mathfrak{U}', t) \models \boldsymbol{t}$.

(1) $A = p$. $(\mathfrak{U}, t) \models p \Leftrightarrow \varphi tp \Leftrightarrow \varphi' tp$ (since $t{\in}T$) $\Leftrightarrow (\mathfrak{U}', t) \models p$.

(2) $A = B \wedge C$. $(\mathfrak{U}, t) \models B \wedge C \Leftrightarrow (\mathfrak{U}, t) \models B$ and $(\mathfrak{U}, t) \models C \Rightarrow (\mathfrak{U}', t) \models B$ and $(\mathfrak{U}', t) \models C$ (IH) $\Leftrightarrow (\mathfrak{U}', t) \models B \wedge C$.

(3) $A = {\sim}B$. Suppose $(\mathfrak{U}, t) \not\models {\sim}B$. Then $(\exists a{\in}S)(a \geq t$ and $(\mathfrak{U}, -a) \models B)$. By IH, $(\mathfrak{U}', -a) \models B$ and so $(\mathfrak{U}', t) \not\models {\sim}B$. Now suppose $(\mathfrak{U}', t) \not\models {\sim}B$. Then $(\exists a{\in}S')(a \geq t$ and $(\mathfrak{U}', -a) \models B)$. If $a{\in}S$, then $(\mathfrak{U}, -a) \not\models B$ (IH); and so $(\mathfrak{U}, t) \not\models {\sim}B$. If $a = e$, then $(\mathfrak{U}', d) \models B$ (since $d \geq -a = e$); and so $(\mathfrak{U}, d) \models B$ (IH). But $d \geq -d = g(a) \geq f(t) = t$, and so $(\mathfrak{U}, t) \not\models B$.

(4) $A = B \rightarrow C$. Suppose $(\mathfrak{U}, t) \not\models B \rightarrow C$. Then: $(\exists u{\in}T)(\mathfrak{U}, u) \models B$ and $(\mathfrak{U}, tu) \not\models C)$. By IH, $(\mathfrak{U}', u) \models B$ and $(\mathfrak{U}', tu) \not\models C$; and so $(\mathfrak{U}', t) \not\models B \rightarrow C$. Now suppose $(\mathfrak{U}', t) \not\models B \rightarrow C$. Then $(\exists u{\in}T')((\mathfrak{U}', u) \models B$ and $(\mathfrak{U}', tu) \not\models C)$. If $u{\in}T$, then $(\mathfrak{U}, u) \models B$ and $(\mathfrak{U}, tu) \not\models C$ (IH), and so $(\mathfrak{U}, t) \not\models B \rightarrow C$. If $u{=}e$, then $(\mathfrak{U}', d) \models B$ (since $d \geq e$) and $(\mathfrak{U}', td) \not\models C$ (since $te = td$ for $t{\in}T$). So, by IH, $(\mathfrak{U}, d) \models B$ and $(\mathfrak{U}, td) \not\models C$; and so $(\mathfrak{U}, t) \not\models B \rightarrow C$.

Theorem 1. Suppose that $\mathbf{L} = \mathbf{B}n_1, \ldots, n_k$, where

$$\{5, 6\} \subseteq \{n_1, \ldots, n_k\} \subseteq \{1, \ldots, 9\}.$$

Then $\mathbf{L}$ is closed under the rule γ.

Proof. Suppose A, $A \supset B \in \mathbf{L}$ but $B{\notin}\mathbf{L}$. By Theorem 4.1, there is a model $\mathfrak{U}$ satisfying conditions $n_1', \ldots, n_k'$ and a normal theory d in $\mathfrak{U}$ such that $d \not\models B$, i.e., $-d \models {\sim}B$. We may suppose $d = -d$. For otherwise, by Lemma 4 of this subsection, $(\mathfrak{U}', -d) \models {\sim}B$; so $(\mathfrak{U}', e) \models {\sim}B$ (since $e \geq -d$); and so $(\mathfrak{U}', e) \not\models B$ (since $e \leq -e$), where $\mathfrak{U}'$ also satisfies conditions $n_1', \ldots, n_k'$, by Lemma 3. By Theorem 4.1 again, $d \models A$ and $d \models A \supset B$. Therefore, $d \models {\sim}A$ or $d \models B$. Since $d \not\models B$, $d \models {\sim}A$, contrary to $d \leq -d$.

An examination of the proof reveals that $\mathbf{B}n_1 \ldots n_k$ is closed under (γ) for $6 \in \{n_1, \ldots, n_k\} \subseteq \{1, 4, 5, 6, 7, 8, 9\}$. It would be of some interest to determine whether all the logics of Theorem 4.1 are closed under (γ).

§52. No fit between constant-domain semantics and $\mathbf{R}^{\forall\exists x}$. The background for this section, which reports an achievement of Fine, may be outlined as follows. Relevance logic began at the propositional level with

Ackermann's *strenge Implikation* as described in §§44–46, to which at some point we added propositional quantifiers as pioneered by Church (see §§30–37) and individual quantifiers (§§38–42) by means of both some Hilbert-style axioms and some Fitch-style natural deduction rules. All of this seemed to comprise a stable collection of proof-theoretically based logical insights.

Penetration was increased on the propositional level with Urquhart's operational semantics of §47, the Routley-Meyer relational semantics of §48, and the Fine operational-relational semantics of §51; and each of these suggested the same simple-minded extension to individual quantifiers, an extension of the sort called "constant domain," with the semantic extension also having the appearance of matching the rather simple-minded intuitive conception of the quantifiers that controlled our proof-theoretical explorations in the first place.

What this section reports is that appearance is mere appearance: there is in fact no fit between (1) the result of adding easy and plausible quantifier postulates to the underlying propositional logics and (2) the result of adding easy and plausible semantic clauses for the quantifiers to those for the underlying propositional logics. (The "semi-relevant" logic **RM**, however, is characterized by a constant-domain semantics, at least in the semantics with a binary accessibility relation; see §49.2.) In contrast, §53 following explains a totally different semantics answering to the calculuses presented in §§38–42.

Our aim here is to summarize some of Fine 1989, wherein it is demonstrated that the following formula is (1) valid in the $\mathbf{R}^{\forall\exists x}$ constant-domain semantics of §48.9, but (2) not provable in $\mathbf{R}^{\forall\exists x}$, thus providing a witness that $\mathbf{R}^{\forall\exists x}$ is not complete for the constant-domain semantics, and, equivalently, establishing by example that the constant-domain semantics is not apt for $\mathbf{R}^{\forall\exists x}$:

$$\begin{aligned}(K)\qquad &(p\rightarrow\exists xEx)\&\forall x((p\rightarrow Fx)\vee(Gx\rightarrow Hx))\rightarrow.\\ &\forall x((Ex\&Fx)\rightarrow q)\&\forall x((Ex\rightarrow q)\vee Gx)\rightarrow.\exists xHx\vee(p\rightarrow q).\end{aligned}$$

Showing that (K) above is valid in the $\mathbf{R}^{\forall\exists x}$ constant-domain semantics of §48.9 requires an argument whose complication is directly proportional to that of (K) itself. Choose an $\mathbf{R}^{\forall\exists x}$ constant-domain model (K, 0, R, *, D, φ, $\models$). Suppose for *reductio* that (K) does not hold at 0 in K. Then, because of the propositional form of (K), one must have

1. Rabx
2. Rxcd
3. a $\models_\alpha p\rightarrow\exists xEx$
4. a $\models_\alpha \forall x((p\rightarrow Fx)\vee(Gx\rightarrow Hx))$
5. b $\models_\alpha \forall x((Ex\&Fx)\rightarrow q)$
6. b $\models_\alpha \forall x((Ex\rightarrow q)\vee Gx)$
7. not x $\models_\alpha \exists xHx$

8. $\mathrm{c} \vDash_{\alpha} p$
9. not $\mathrm{d} \vDash_{\alpha} q$

By 1, 2, and Associativity from §48.3, for some $\mathrm{y} \in \mathrm{K}$,

10. Racy and
11. Rbyd.

First we require two intermediate facts: 10, 3, and 8 imply that, for some $\mathrm{d} \in \mathrm{D}$, where $\alpha' = \alpha[\mathrm{d}/x]$ (see §48.9),

12. $\mathrm{y} \vDash_{\alpha'} Ex$. Also
13. not $\mathrm{y} \vDash_{\alpha'} Fx$;

for otherwise 11, 5, and 9 would contradict. The rest of the argument relies on considering three cases arising from instantiating and distributing 4 and 6:

Case. $\mathrm{a} \vDash_{\alpha'} p \rightarrow Fx$.
Case. $\mathrm{b} \vDash_{\alpha'} Ex \rightarrow q$.
Case. $\mathrm{a} \vDash_{\alpha'} Gx \rightarrow Hx$ and $\mathrm{b} \vDash_{\alpha'} Gx$.

The first runs up against 10, 8, and 13; the second against 11, 12, and 9; and the third against 1 and 7. This, by *reductio*, finishes the argument that (*K*) is valid in the $\mathbf{R}^{\forall\exists x}$ constant-domain semantics of §48.9.

The other half of the argument to the conclusion that (*K*) witnesses the lack of fit between $\mathbf{R}^{\forall\exists x}$ and the constant-domain semantics requires that we show that (*K*) is not provable in $\mathbf{R}^{\forall\exists x}$. Fine 1989 offers a method of semantic modeling—similar to but not the same as the method of §48.9—which separates (*K*) from $\mathbf{R}^{\forall\exists x}$ by verifying all theorems of $\mathbf{R}^{\exists\forall x}$ while falsifying (*K*). Along the way Fine introduces several interesting semantic ideas, but here, at the cost of a sense of unexplained complexity, we limit ourselves to the description of what is needed for the immediate purpose.

The construction begins with an underlying six-element space $\mathrm{X} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*\}$, and an underlying operation * on X that maps each α_i onto its star-labeled mate, and vice versa. Let "α" range over X. $\mathrm{F}_0 = \langle 0, \mathrm{A}, \mathrm{R}, * \rangle$, which is an intermediate construct, is defined as follows: (1) A is the family of all subsets of X that contain no star pairs α and α^*. (2) 0 is some chosen item not in A. (3) R is the three-place relation on A defined by the condition that, for a, b, c, $\in$ A, Rabc iff (i) if $\alpha \in \mathrm{a} \cup \mathrm{b}$ but $\alpha^* \notin \mathrm{a} \cup \mathrm{b}$, then $\alpha \in \mathrm{c}$; and (ii) if $\alpha \in \mathrm{c}$ then $\alpha \in \mathrm{a} \cup \mathrm{b}$. (4) * is the projection onto A of the underlying * on X; so (because $\alpha^{**} = \alpha$) $\alpha \in \mathrm{a}^*$ iff $\alpha^* \in \mathrm{a}$. Six members of A will need names: $\mathrm{a}_1 = \{\alpha_1\}$, $\mathrm{a}_2 = \{\alpha_2\}$, $\mathrm{a}_3 = \{\alpha_3\}$, $\mathrm{a}_4 = \{\alpha_1, \alpha_2\}$, $\mathrm{a}_5 = \{\alpha_1, \alpha_3\}$, $\mathrm{a}_6 = \{\alpha_1, \alpha_2, \alpha_3\}$.

Leave F_0 aside for a moment, and choose three pairwise disjoint infinite sets I_1, I_2, I_3, letting I be their union. Subsets J and J′ of I are *finite variants* if their symmetric difference $(J-J')\cup(J'-J)$ is finite. Now define $\boldsymbol{A}$ as consisting of certain pairs (and some nonpairs) as follows:

1. $\langle 0, 0\rangle$
2. $\langle a_1, \langle J, K\rangle\rangle$ and $\langle a_1^*, \langle J, K\rangle\rangle$, for J and K finite variants of I_1 and I_2, respectively
3. $\langle a_2, \langle J, K\rangle\rangle$ and $\langle a_2^*, \langle J, K\rangle\rangle$ for J and K finite variants of I_2 and I_1, respectively
4. $\langle a_3, 0\rangle$ and $\langle a_3^*, 0\rangle$
5. $\langle a_4, J\rangle$ and $\langle a_4^*, J\rangle$ for J a finite subset of I
6. $\langle a_5, J\rangle$ and $\langle a_5^*, J\rangle$ for J a finite variant of I_3
7. $\langle a_6, 0\rangle$, $\langle a_6^*, 0\rangle$, $\langle a_6, 1\rangle$, and $\langle a_6^*, 1\rangle$.
8. $\langle a\rangle$ for every $a \in A$ *except* $a_4, a_5, a_6, a_4^*, a_5^*, a_6^*$.

Such is $\boldsymbol{A}$. We will let "$\boldsymbol{a}$", etc., range over $\boldsymbol{A}$, and we will use typography to pick out the first component of $\boldsymbol{a}$: given $\boldsymbol{a}$, a is the element of A on which (by 1–8) it is founded. If $\boldsymbol{a}$ comes into $\boldsymbol{A}$ by 1–7 (but not 8), it is said to be *interpreted*, and $i(\boldsymbol{a})$ is its second element. The operation * on $\boldsymbol{A}$ is defined in the obvious way: if $\boldsymbol{a} = \langle \boldsymbol{a}, x\rangle$ then $\boldsymbol{a}^* = \langle a^*, x\rangle$, and if $\boldsymbol{a} = \langle a\rangle$, then $\boldsymbol{a}^* = \langle a^*\rangle$.

The definition of $\boldsymbol{R}$ as a three-term relation on $\boldsymbol{A}$ requires preparation. First, let the following triples from A (not $\boldsymbol{A}$) be called *basic special*: $a_1a_2a_4$, $a_1a_3a_5$, $a_4a_3a_6$, $a_2a_5a_6$. Next, where a *variant* of a triple xyz from a set on which * is defined is any one of yxz, xz*y*, z*xy*, z*yx*, and yz*x*, let *special* triples of A be the basic special triples and their variants. This terminology is extended to $\boldsymbol{A}$: a triple $\boldsymbol{abc}$ from $\boldsymbol{A}$ is *special* if (1) each of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is interpreted and (2) the triple abc from A is special.

Among the special triples of $\boldsymbol{A}$ (not A), some are *basic admissible* ("suitable" below just reminds us that each member of the triple must belong to $\boldsymbol{A}$ in virtue of one of the appropriate clauses 1-7 helping to define $\boldsymbol{A}$):

1. $\langle a_1, \langle J_1, J_2\rangle\rangle\langle a_2, \langle K_1, K_2\rangle\rangle\langle a_4, L\rangle$ for suitable J_1, J_2, K_1, K_2, and any suitable subset of $J_2\cap K_2$.
2. $\langle a_1, \langle J_1, J_2\rangle\rangle\langle a_3, 0\rangle\langle a_5, L\rangle$ for suitable J_1, J_2, and any suitable L such that $J_1\cap L = \varnothing$.
3. $\langle a_4, J\rangle\langle a_3, K\rangle\langle a_6, L\rangle$ for suitable J, K, L.
4. $\langle a_2, \langle J_1, J_2\rangle\rangle\langle a_5, K\rangle\langle a_6, \pi\rangle$ for suitable J_1, J_2, and K, and for $\pi = 1$ or 0 according as $J_1\cap K$ is nonempty or empty.

Basic admissible triples and their variants are *admissible*.

These definitions prepare us to define a three-term relation $\boldsymbol{R}$ on $\boldsymbol{A}$. $\boldsymbol{Rabc}$ if either $\boldsymbol{abc}$ is not special and Rabc, or $\boldsymbol{abc}$ is admissible (Rabc will hold *a fortiori*).

Let $\boldsymbol{0}$ be an object not in $\boldsymbol{A}$.

Let $\boldsymbol{I}$ be that function with domain $A\cup\{\boldsymbol{0}\}$ such that
$\boldsymbol{I}(\boldsymbol{a}) = \mathrm{J}\cup\mathrm{K}$ if $\mathrm{a} \in \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_1^*, \mathrm{a}_2^*\}$ and $\mathrm{i(a)} = \langle \mathrm{J}, \mathrm{K}\rangle$, and
$\boldsymbol{I}(\boldsymbol{a}) = \mathrm{I}$ otherwise.

At this point we have defined $\boldsymbol{0}$, $\boldsymbol{A}$, $\boldsymbol{R}$, $*$, and $\boldsymbol{I}$. Evidently I is the union of all the $\boldsymbol{I}(\boldsymbol{a})$ for $\boldsymbol{a} \in A\cup\{\boldsymbol{0}\}$. These items are used to confer a value on each formula in a manner a little different from that of §48.9. An *assignment* assigns values in I to variables; let α be an assignment. Say that a formula A is *defined at* $(\boldsymbol{a}, \alpha)$ if α delivers a value in $\boldsymbol{I}(\boldsymbol{a})$ for every variable free in A. Then $\boldsymbol{a} \vDash_\alpha A$ (read something like "A is true at $\boldsymbol{a}$ on α") is characterized as follows, for $\boldsymbol{a} \in A\cup\{\boldsymbol{0}\}$ and α an assignment:

0. If A is not defined at $(\boldsymbol{a}, \alpha)$ then not $\boldsymbol{a} \vDash_\alpha A$.

For the remaining clauses, assume the formula in question is defined at $(\boldsymbol{a}, \alpha)$.

1*G*. $\boldsymbol{a} \vDash_\alpha Gx$ iff $\boldsymbol{a} = \langle \mathrm{a}_2, \langle \mathrm{J}, \mathrm{K}\rangle\rangle$ and $\alpha(x) \in \mathrm{K}$.
1*p*. $\boldsymbol{a} \vDash_\alpha p$ iff $\boldsymbol{a} = \langle \mathrm{a}_3, 0\rangle$.
1*H*. $\boldsymbol{a} \vDash_\alpha Hx$ iff $\boldsymbol{a} = \langle \mathrm{a}_4, \mathrm{L}\rangle$ and $\alpha(x) \in \mathrm{L}$.
1*E*. $\boldsymbol{a} \vDash_\alpha Ex$ iff $\boldsymbol{a} = \langle \mathrm{a}_5, \mathrm{L}\rangle$ and $\alpha(x) \in \mathrm{L}$.
1*F*. $\boldsymbol{a} \vDash_\alpha Fx$ iff $\boldsymbol{a} = \langle \mathrm{a}_5, \mathrm{L}\rangle$ and $\alpha(x) \in \mathrm{I} - \mathrm{L}$.
1*q*. $\boldsymbol{a} \vDash_\alpha q$ iff $\boldsymbol{a} = \langle \mathrm{a}_6, 1\rangle$.
2. $\boldsymbol{a} \vDash_\alpha B\&C$ iff $\boldsymbol{a} \vDash_\alpha B$ and $\boldsymbol{a} \vDash_\alpha C$.
3. $\boldsymbol{a} \vDash_\alpha \forall xB$ iff $\boldsymbol{a} \vDash_{\alpha[\mathrm{x}/x]} B$ for all $\mathrm{x} \in \boldsymbol{I}(\boldsymbol{a})$.
4. $\boldsymbol{0} \vDash_\alpha {\sim}B$ iff not $\boldsymbol{0} \vDash_\alpha B$.
5. $\boldsymbol{0} \vDash_\alpha B\rightarrow C$ iff $\boldsymbol{a} \vDash_\alpha C$ whenever C is defined at (a, α) and $\boldsymbol{a} \vDash_\alpha B$.
6. For $\boldsymbol{a} \neq \boldsymbol{0}$: $\boldsymbol{a} \vDash_\alpha {\sim}B$ iff not $\boldsymbol{a}^* \vDash_\alpha B$.
7. For $\boldsymbol{a} \neq \boldsymbol{0}$: $\boldsymbol{a} \vDash_\alpha B\rightarrow C$ iff, for every $\boldsymbol{b}, \boldsymbol{c} \in A$, if $\boldsymbol{Rabc}$, $\boldsymbol{b} \vDash_\alpha B$, and C is defined at $(\boldsymbol{c}, \alpha)$, then $\boldsymbol{c} \vDash_\alpha C$.

The point $\boldsymbol{0}$ of this "critical model" separates the formula (K) from $\mathbf{R}^{\forall\exists x}$. That is,

(I) $\boldsymbol{0} \vDash_\alpha A$ for every theorem A of $\mathbf{R}^{\forall\exists x}$ and every assignment α; and
(II) Not $\boldsymbol{0} \vDash_\alpha (K)$. ($\alpha$ is irrelevant, since (K) is closed.)

Since the critical model was cooked up with (II) in mind, it is not surprising that (I) is the hard part, and the part of this part that is most difficult is the verification that the axiom distributing the universal quantifier over the arrow holds at point $\boldsymbol{0}$ of the "critical model":

$$\boldsymbol{0} \vDash_\alpha \forall x(Ax\rightarrow Bx)\rightarrow.\forall xAx\rightarrow\forall xBx.$$

Thus since, by (I) and (II), (K) is not provable in $\mathbf{R}^{\forall\exists x}$, even though we saw earlier that it is valid in the constant domain semantics of §48.9, the incompleteness/inaptness result is established. What to make of it? See §53 just following for the most interesting proposal.

§53. Semantics for quantified relevance logic (by Kit Fine). This section is a companion piece to §52. That section showed that $\mathbf{R}^{\forall\exists x}$ (see §38) and various other systems of quantified relevance logic were not complete for the standard semantics. The present section provides a semantics with respect to which they are complete. (We should like to thank R. K. Meyer and NDB for several helpful remarks. Adrian Abraham, in unpublished work, has provided a semantics for the system $\mathbf{CR}^{\forall\exists x}$ which results from enriching $\mathbf{R}^{\forall\exists x}$ with Boolean negation. His approach, however, is very different from the present and, regrettably, calls for a condition on models that can be stated only with the help of the truth predicate.)

It appears to follow from the result of §52 that $\mathbf{R}^{\forall\exists x}$ is incomplete not only for the standard semantics but also for any reasonable semantics, i.e., one that corresponds to our intuitive ideas of logical truth. It will be worthwhile to explain exactly what the difficulties are, since the present semantics can be seen to be motivated by the desire to avoid them. Our considerations apply more generally to almost any system that contains $(p\rightarrow q)\rightarrow((q\rightarrow r)\rightarrow(p\rightarrow r))$ and $(q\rightarrow r)\rightarrow((p\rightarrow q)\rightarrow(p\rightarrow r))$ as theorems; but, for simplicity, we shall confine our attention to the single case of $\mathbf{R}^{\forall\exists x}$.

[Note by principal authors. Choices of fonts, terminology, and occasionally notation (e.g., "$\wedge$" for conjunction) in this section, as in §51, have largely been left as in the original paper.]

Let us use A, B, C, etc. for the formulas of $\mathbf{R}^{\forall\exists x}$ and φ, ψ, χ, etc, for the corresponding concrete formulas, obtained by using meaningful predicates in place of predicate-letters. It seems resonable to assume that the universal quantifier $\forall x$ is used in such a way that, for each appropriate specification of the domain, there will be "canonical" names $n_1, n_2, \ldots$ of which it will hold that: $\forall x\psi(x)$ is true iff $\psi(n_1)$, $\psi(n_2), \ldots$ are true, for any formula $\psi(x)$ of the closure of the base language under the names $n_1, n_2, \ldots$

The right-hand side of the biconditional holds just in case the conjunction $\psi(n_1)\wedge\psi(n_2)\wedge \ldots$ is true. So our assumption amounts to the material equivalence of the universally quantified statement $\forall x\psi(x)$ to the conjunction $\bigwedge\psi(n_i) = \psi(n_1)\wedge\psi(n_2)\wedge \ldots$ of all of its "canonical" instances $\psi(n_1)$, $\psi(n_2), \ldots$.

It should be noted that nothing more than material equivalence is required. In particular, it is not required that the equivalence be relevant or somehow necessary. It should also be noted that we do not require that $\psi(n)$ be true for each name n of an individual in the domain; we do not even require that the specified names $n_1, n_2, \ldots$ name anything. Our assumption is, therefore, compatible with all the standard interpretations of the quantifier.

In the system $\mathbf{R}^{\forall\exists x}$, the formula $\forall x(p\rightarrow Fx)\rightarrow(p\rightarrow\forall xFx)$ is a theorem (assuming, of course, that the requisite schematic letters p and F are available in the language of $\mathbf{R}^{\forall\exists x}$). If $\mathbf{R}^{\forall\exists x}$ is to be sound for some intuitive semantics, then every concrete instance of this formula should be true. Let $\psi(x)$ be used for Fx and $\bigwedge\psi(n_i)$ for p. Then, in particular, the sentence $\forall x(\bigwedge\psi(n_i)\rightarrow\psi(x))\rightarrow(\bigwedge\psi(n_i)\rightarrow\forall x\psi(x))$ should be true.

The antecedent of this conditional is true. For take any canonical name n_i (under a given specification of the domain). Then clearly $\bigwedge\psi(n_i)\rightarrow\psi(n_i)$ is true. So, by our assumption concerning the universal quantifier, $\forall x(\bigwedge\psi(n_i)\rightarrow\psi(x))$ is true.

Given that the antecedent is true, so is the consequent $\bigwedge\psi(n_i)\rightarrow\forall x\psi(x)$. Now $\forall x(\forall xFx\rightarrow Fx)$ is also a theorem of $\mathbf{R}^{\forall\exists x}$. So, by our assumption again, but in the other direction, each of the conditionals $\forall x\psi(x)\rightarrow\psi(n_i)$ is true. It therefore follows that $\forall x\psi(x)\rightleftarrows\bigwedge\psi(n_i)$ is true. Thus the *material* equivalence of $\forall x\psi(x)$ to $\bigwedge\psi(n_i)$, together with the validity of principles from $\mathbf{R}^{\forall\exists x}$, yields the *relevant* equivalence of $\forall x\psi(x)$ to $\bigwedge\psi(n_i)$.

Given a concrete sentence φ, let φ' be the result of replacing each of its subformulas $\forall x\psi(x)$ with the conjunction $\bigwedge\psi(n_i)$. It can then be shown that φ is materially equivalent to φ'. For in $\mathbf{R}^{\forall\exists x}$ can be derived the substitution scheme $p\rightleftarrows q\supset(A(p)\equiv A(q))$, where $A(q)$ is the result of replacing all occurrences of p in $A(p)$ with q. Take an outermost occurrence of a formula of the form $\forall x\psi(x)$ in φ. Let p be $\forall x\psi(x)$, q be $\bigwedge\psi(n_i)$, $A(p)$ be $\varphi=\varphi(\forall x\psi(x))$, and $A(q)$ be $\varphi_1=\varphi(\bigwedge\psi(n_i))$. The antecedent $\forall x\psi(x)\rightleftarrows\bigwedge\psi(n_i)$ of the resulting sentence is true; therefore, so is the consequent $\varphi\equiv\varphi_1$. Let us continue this process until all the occurrences of universal subformulas are removed. We then obtain a sequence of formulas $\varphi=\varphi_0,\varphi_1,\ldots,\varphi_n=\varphi'$ each materially equivalent to the next; and so the first of these sentences, φ, is materially equivalent to the last, φ'. (R. K. Meyer has pointed out that the appeal to the disjunctive syllogism in this argument can be avoided by making use of the substitution scheme $(\boldsymbol{t}\wedge(p\rightleftarrows q))\rightarrow(A(p)\rightleftarrows A(q))$, for $\boldsymbol{t}$ a suitable conjunction of formulas of the form $C\rightarrow C$, to derive a chain of relevant equivalents. We do not find the appeal to disjunctive syllogism problematic in this context. But it should be noted that the proof of the alternative substitution scheme requires a heavier commitment to the principles of $\mathbf{R}^{\forall\exists x}$.)

We now appeal to the specific formula (K) used in §52 to prove the incompleteness of $\mathbf{R}^{\forall\exists x}$. Any concrete instance of (K) is true; for the validity of (K) under the standard semantics implies that the corresponding quantifier-free sentence φ' is true, and so, given the material equivalence of φ and φ', φ itself is true. Since every concrete instance φ of (K) is true, it would appear to follow that (K) should be valid under any reasonable semantics. However, (K) is not a theorem of $\mathbf{R}^{\forall\exists x}$, and therefore $\mathbf{R}^{\forall\exists x}$ is not complete for the semantics.

There would appear to be only two plausible ways in which the reasoning to this conclusion might be challenged. The first is to argue that something more than the truth of all its concrete instances is required for the validity of a formula. This might, in general, be a reasonable point to make; but it is hard to see, in the present context, how it is capable of providing the basis for an alternative semantics for $\mathbf{R}^{\forall\exists x}$. The other possibility, which is the one to be pursued here, is to dispute the assumption that a universal sentence is materially equivalent to the conjunction of its (canonical) instances. Since $\forall xFx \supset Fy$ is a theorem of $\mathbf{R}^{\forall\exists x}$, it will be conceded that the truth of a universal sentence yields the truth of its instances. But it will be denied that the converse holds.

The basis for our approach will be a new interpretation of the quantifier. We shall say that a universal sentence $\forall x\psi(x)$ is true just in case $\psi(x)$ is true of an arbitrary or generic individual. But let us not be misunderstood. Our saying that $\psi(x)$ is true of an arbitrary individual is not a fancy way of saying that $\psi(x)$ is true of every individual. We mean to be taken literally; for the universal sentence $\forall x\psi(x)$ to be true, there must actually be an arbitrary individual of which the condition $\psi(x)$ is true.

If $\psi(x)$ is true of an arbitrary individual then it is true of any specific individual. But what of the converse? It seems to us that there are conditions such that it is quite plausible to suppose that they are not true for any arbitrary individual even though they are true for every specific individual. Let the condition $\psi(x)$ be that $\chi(x)$ is provable, in either a formal or an informal sense; and let us suppose that $\psi(x)$ is true of an arbitrary individual if the formula $\chi(x)$, with free variable x, is provable and that $\psi(x)$ is true of a specific individual if $\chi(n)$ is true for n the canonical name of the individual. Then it is perfectly conceivable that $\psi(x)$ be true of every specific individual, i.e., that $\chi(n)$ be provable for each canonical name n, even though $\psi(x)$ is not true of an arbitrary individual; i.e. $\chi(x)$ itself is not provable.

Relevance contexts are similar. The formula $\chi_1 \rightarrow \chi_2(x)$ may be true of every individual, i.e., $\chi_2(n)$ may always be relevantly implied by χ_1, even though $\chi_1 \rightarrow \chi_2(x)$ is not true of an arbitrary individual, i.e., $\chi_2(x)$ is not relevantly implied by χ_1. Indeed, the example in which χ_1 is the conjunction of all the instances $\chi_2(n)$ would appear to be just such a case: χ_1 relevantly implies each $\chi_2(n)$, but not $\chi_2(x)$ itself.

In general, there will be a notion of necessity for which it is the case that the condition $\psi(x)$ is true of an arbitrary individual if it is necessary that $\psi(x)$ is true of every individual. It is not enough that every individual satisfies the condition; it must be in the nature of an individual to satisfy the condition.

The present semantics for $\mathbf{R}^{\forall\exists x}$ and its cousins is based upon this possibility of double interpretation for the quantifier, both in terms of a generic individual and in terms of necessity. However, it turns out that there are certain technical difficulties in guaranteeing that the generic individual is genuinely arbitrary; so the actual formal elaboration of the simple underlying idea is rather complicated.

In some ways, the present interpretation is close to the generic semantics developed elsewhere (Fine 1983, 1985, 1985a). But there are differences. In the earlier work, there exists an explanation of the truth of statements about arbitrary objects in terms of individuals; a statement about an arbitrary object is true just in case it is true for all individuals. On the present account, there is no such explanation. Truths about arbitrary objects are just given; they are about those objects as objects in their own right. Another difference is that the notion of dependence is integral to the earlier work, but in the present account it does not even make an appearance.

It is perhaps worth remarking, in the light of these differences, that the present semantics received no impetus from the earlier work; the connections between the two were a happy accident.

This section is divided into five subsections. The first two lay out the semantics, the third presents the logics, and the last two establish soundness and completeness. A basic knowledge of the semantics for propositional relevance logic is presupposed (see §51). It is conceivable that the methods of the present section might be used to simplify the proofs of incompleteness for the standard semantics; but this is not here investigated.

§53.1. Models. This subsection introduces the models with respect to which the various systems of quantified relevance logic are proved complete.

It is supposed that the formulas of the systems are constructed from a nonempty set of predicates; to each predicate is assigned a degree $n \geq 0$.

A *possible model* $\mathfrak{A}$ is then an 11-tuple $(T, S, D, l, \circ, -, \geq, \uparrow, \downarrow, \rightarrow, \varphi)$ where:

(i) T (theories) is a set.

(ii) S (saturated theories) is a subset of T.

(iii) D (relative domain) is a function from T into sets (we use $\mathfrak{D}$ (domains) for Range(D), I (individuals) for $\bigcup \mathfrak{D}$, and $\approx$ (domain equivalence) for $\{\langle t, u\rangle \in T \times T\colon D_t = D_u\}$).

(iv) l (logics) is a function from $\mathfrak{D}$ into T, with $D_{l(\alpha)} = \alpha$ for all $\alpha \in \mathfrak{D}$.

(v) $\circ$ (fusion) is a partial function from $T \times T$ into T. It is defined, for arguments t and u only when $t \approx u$; then it takes a value $v \approx t$.

(vi) $-$ (complementation) is a unary operation on S for which $-a \approx a$ whenever $a \in S$.

(vii) $\geq$ is a binary relation on T, holding between t and u from T only when $t \approx u$.

(viii) $\uparrow$ (the up operator) is a partial function from $T \times \mathfrak{D}$ into T. It is defined, for given arguments t and α, only when $D_t \subseteq \alpha$; then its value is a u for which $D_u = \alpha$.

(ix) $\downarrow$ (the down operator) is a partial function from $T \times \mathfrak{D}$ into T. It is defined, for given arguments t and α, only when $D_t \supseteq \alpha$; then its value is a u for which $D_u = \alpha$.

(x) $\rightarrow$ (the across operator) is a partial function from $T \times \{\{i, j\} \subseteq I\colon i \neq j\}$ into T. It is defined on t and i, j just in case the distinct $i, j \in D_t$; then its value is a $u \approx t$.

(xi) φ (valuation) is a relation that holds between a theory t from T and an $(n+1)$–tuple $\langle R, i_1, \ldots, i_n \rangle$ consisting of an n-place predicate R and individuals $i_1, \ldots, i_n$ from D_t.

Let us briefly explain the intuitive significance of each of the components. The significance of the components T, S, $\circ$, $-$, $\geq$, and φ should already be clear from the study of propositional relevance logic (see §51). We follow §51 in using $t, s, u, \ldots$ and $a, b, c, \ldots$ for members of T and S, respectively, and in writing $(t \circ u)$ as tu.

The function D takes each theory t into its ontology, or domain of individuals D_t. It will also be useful to think of the members of I as constants. The members of $\mathfrak{D}$ then represent various levels of language which differ only in which constants they contain. For the levels, i.e., the members of $\mathfrak{D}$, we shall use $\alpha, \beta, \gamma, \ldots$; and for the individuals, i.e., the members of I, we shall use $i, j, k, \ldots$.

For each level α, $l(\alpha)$ is the logic appropriate to α. Often the (α) will be omitted.

Given a theory t with language α and given an extension β of α, $\uparrow(t, \beta)$ is the expansion of t to β. The theorems of $\uparrow(t, \beta)$ are the consequences, in the language of β, of the theorems of t. The expansion $\uparrow(t, \beta)$ of a theory t should be contrasted with an extension $u \geq t$ of a theory t: in an expansion, the axioms can remain the same although the language will change; in an extension, the language will remain the same although the axioms can change. This isolation of a purely linguistic expansion of a theory is a distinctive feature of the present semantics. We shall sometimes write the expansion as $t{\uparrow}^{\beta}$, or as t^{β}, or even as $t{\uparrow}$ if no ambiguity can result.

Given a theory t with language α and a refinement β of α, $\downarrow(t, \beta)$ is the contraction of t to β. The theorems of $\downarrow(t, \beta)$ are those theorems of t which belong to the language of β. We shall sometimes write the contraction as $t{\downarrow}_{\beta}$, or t_{β}, or even as $t{\downarrow}$ if no ambiguity can result.

If t is a theory with language α and if i and j are two individuals from α, then $\rightarrow(t, \{i, j\})$ is the minimal extension of t in which the individuals i and j are identified. So if, for example, the proposition pii is a theorem of t, then each of the propositions pii, pij, pji, and pjj will be a theorem of $\rightarrow(t, \{i, j\})$. We shall sometimes write this symmetric extension as ${}^{i,j}\vec{t}$ or as $\vec{t}$ if no ambiguity can result. Points of the form ${}^{i,j}\vec{t}$ are said to be *symmetric in i and j*.

For a possible model $\mathfrak{A}$, as above, to be an *actual* model, it must satisfy certain further conditions, which are detailed below. The conditions are divided into five groups: the first consists of conditions already familiar from the study of propositional relevance logic; the second concerns the structure of levels as given by the inclusion relation; the third concerns the behavior of the up and down operators $\uparrow$ and $\downarrow$; the fourth concerns the interaction of ↑ and ↓ with each of the standard components S, $-$, and $\circ$ of a model; and the fifth concerns the behavior of the across operator $\rightarrow$.

The number of conditions is rather large. It is possible to provide a more compact presentation, especially in the presence of conditions corresponding to some of the stronger logics. But the present account aims for an illuminating analysis of the conditions rather than an economical synthesis.

I. *Standard*

(i) $\geq$ is a partial ordering;
(ii) $t \geq u \Rightarrow (tv \geq uv)$ and $(vt \geq u)$
(iii) $a \geq tu \Rightarrow (\exists b,c)(b \geq t$ and $c \geq u$ and $a \geq bu$ and $a \geq tc)$
(iv) $lt \geq t$
(v) $t \geq lt$
(vi) $a \geq l \Rightarrow a \geq -a$
(vii) $--a = a$
(viii) $a \geq b \Rightarrow -b \geq -a$
(ix) $\varphi t\langle \mathrm{R}, i_1, \ldots, i_\mathrm{n}\rangle \Leftrightarrow (\forall a)(a \geq t \Rightarrow \varphi a\langle \mathrm{R}, i_1, \ldots, i_\mathrm{n}\rangle)$, for R an n-place predicate and $i_1, \ldots, i_\mathrm{n} \in D_t$

II. *Levels*

(i) (Extendability) $\forall\alpha\exists\beta(\beta \supset \alpha)$
(ii) (Upper Bound) $(\forall\alpha,\beta)(\exists\alpha)(\alpha\cup\beta \subseteq \gamma)$
(iii) (Reversibility) $(\forall\alpha,\beta,\gamma)(\alpha \subseteq \beta \subseteq \gamma \Rightarrow \alpha\cup(\gamma-\beta) \in \mathfrak{D})$

III. *Behavior of ↑ and ↓*

(i) (Monotonicity)
  (a) $t \leq u \Rightarrow t\uparrow \leq u\uparrow$
  (b) $t \leq u \Rightarrow t\downarrow \leq u\downarrow$
(ii) (Transitivity)
  (a) $t\uparrow\uparrow = t\uparrow$
  (b) $t\downarrow\downarrow = t\downarrow$
(iii) (Up-down Principles)
  (a) $t\uparrow\downarrow = t$
  (b) $t\downarrow\uparrow \leq t$
  (c) $t\downarrow\uparrow = t\uparrow\downarrow$, as long as $D_t \cap D_{t\downarrow\uparrow} = D_{t\downarrow}$
(iv) (Valuation)
  $\varphi t\downarrow\langle \mathrm{R}, i_1, \ldots, i_\mathrm{n}\rangle \Rightarrow \varphi t\langle \mathrm{R}, i_1, \ldots, i_\mathrm{n}\rangle$, for R an n-place predicate and $i_1, \ldots, i_\mathrm{n} \in D_{t\downarrow}$

IV. *Interaction of* $\uparrow$ *and* $\downarrow$
  (i) (With S)
    (a) $a\downarrow \in S$
    (b) $a \leq b\downarrow \Rightarrow (\exists a^+)(a^+ \leq b$ and $a^+\downarrow = a)$
    (c) $a \geq t\downarrow \Rightarrow (\exists a^+)(a^+ \geq t$ and $a^+\downarrow = a)$
  (ii) (With $-$)
    (a) $(-a)\downarrow = -(a\downarrow)$
  (iii) (With $\circ$)
    (a) $(tu)\uparrow = t\uparrow u\uparrow$
    (b) $(tu\uparrow)\downarrow \leq t\downarrow u$
  (iv) (With l)
    (a) $l\uparrow = l$; i.e., for $\alpha,\beta \in \mathfrak{D}$ and $\alpha \subseteq \beta$, $l(\alpha)\uparrow^{\beta} = l(\beta)$

V. *Behavior of* $\rightarrow$
  (i) $t \leq \vec{t}$
  (ii) $\underline{t} \leq u \Rightarrow \vec{t} \leq \vec{u}$
  (iii) $\vec{\vec{t}} \leq \vec{t}$
  (iv) (t symmetric in i, j and $t \leq a) \Rightarrow (\exists a')(a'$ symmetric in i, j and $\underline{t \leq} a' \leq a)$
  (v) $\overrightarrow{(-\vec{a})} \leq -\vec{a}$
  (vi) $\overrightarrow{(tu)} \leq \vec{t}\vec{u} \leq \vec{t}u$
  (vii) $\overrightarrow{(t\uparrow)} = (\vec{t})\uparrow$
  (viii) $\overrightarrow{(t\uparrow)}\downarrow \leq t$, where $\rightarrow$ is $i, j\!\rightarrow$, $i \in (D_{t\uparrow} - D_t)$, and $j \in D_t$
  (ix) $\varphi t\langle \mathrm{R}, i_1, \ldots, i_{\mathrm{n}}\rangle \Rightarrow \varphi\vec{t}\langle \mathrm{R}, i'_1, \ldots, i'_{\mathrm{n}}\rangle$, where $\rightarrow$ is $j,k\!\rightarrow$ for distinct $j, k \in D_t$, and, for $\mathrm{p} = 1, 2, \ldots, \mathrm{n}$, $i'_{\mathrm{p}} = i_{\mathrm{p}}$ if $i_{\mathrm{p}} \neq j$ or k, and $i'_{\mathrm{p}} = j$ or k, if $i_{\mathrm{p}} = j$ or k.

Some comment on the formulation and the content of the conditions is in order. In the formulation, level and index have rarely been explicitly indicated and should be supposed present in the most natural way possible. So, in I(ii), t, u, and v are to be of the same level; in I(iv), l and t are to be of the same level; in III(iii)(a), $t\uparrow\downarrow$ and t are to be of the same level; in V(i)–(vii), each occurrence of $\rightarrow$ is to be indexed with superscripts for the same pair of individuals from either D_t or D_u or D_a; the full form of III(ii)(a) is: $t\uparrow^{\alpha}\uparrow^{\beta} = t\uparrow^{\beta}$, for $\beta \supseteq \alpha$; and similarly for III(ii)(b).

The conditions under I correspond closely to the conditions on an orthodox model (§51.1). As a consequence, each of our models can be broken up into a system of orthodox models. To be more exact, given one of our models $\mathfrak{A} = (T, S, D, l, \circ, -, \geq, \uparrow, \downarrow, \rightarrow, \varphi)$ and given a level α from $\mathfrak{D}$, let $\mathfrak{A}^{\alpha}$ be the model $(T^{\alpha}, S^{\alpha}, l^{\alpha}, \circ^{\alpha}, -^{\alpha}, \geq^{\alpha}, \varphi^{\alpha})$, where $T^{\alpha} = \{t\colon D_t = \alpha\}$, $S^{\alpha} = \{a\colon D_a = \alpha\}$, $l^{\alpha} = l(\alpha)$, $\circ^{\alpha} = \circ\upharpoonright T^{\alpha}$, $-^{\alpha} = -\upharpoonright S^{\alpha}$, $\geq^{\alpha} = \geq\upharpoonright T^{\alpha}$, and

$$\varphi^{\alpha} = \{\langle t, \langle \mathrm{R}, i_1, \ldots, i_{\mathrm{n}}\rangle\rangle \in \varphi\colon t \in T^{\alpha}\}.$$

Then it follows by the conditions under I that each $\mathfrak{A}^{\alpha}$, for $\alpha \in \mathfrak{D}$, is an orthodox model (with atomic sentences $\mathrm{R}i_1 \ldots i_n$ taking the place of sentence letters). The operators $\uparrow$ and $\downarrow$ show how the different models $\mathfrak{A}^{\alpha}$ interconnect, and the operator $\rightarrow$ provides some additional internal structure. The models $\mathfrak{A}$ and the semantics that results from them might on this account be called *stratified.*

The conditions under II can be understood in terms of an iterable process whereby an arbitrary language α can be extended to any one of a permitted range of extensions. So (i) says that the process can always be performed; (ii) says that no step in the process is irrevocable, that even if one is at α but might have been at β, still there will be an extension of α that also includes β; and (iii) says that the steps in the process may be reversed, that if β is obtained from α by adding the individuals $\beta - \alpha$ and γ from β by adding the individuals $\gamma - \beta$, then the individuals from $\gamma - \beta$ may first be added to α to obtain $\alpha \cup (\gamma - \beta)$ and the individuals from $\beta - \alpha$ may then be added to $\alpha \cup (\gamma - \beta)$ to obtain γ.

The conditions under III are very natural and plausible for the intended meaning of $\uparrow$ and $\downarrow$. Indeed, conditions (i)(a), (b), (ii)(a), (b), and (iii)(a), (b) are plausible whenever $\uparrow$ and $\downarrow$ are so understood that $t\uparrow$ is the representation of t at a higher level, where information may be retained, and $t\downarrow$ is the representation of t at a lower level, where information may be lost (see the comparable functions in Scott's construction of models for the λ-calculus in Scott 1973a).

From the conditions (ii)(b) and (iii)(a) of III it follows that $t\downarrow = t = t\uparrow$ whenever $t\downarrow$, t, and $t\uparrow$ all have the same level. For $t\downarrow = t\uparrow\downarrow\downarrow$ (by (iii)(a)) $= t\uparrow\downarrow$ (by (ii)(b)) $= t$ (by (iii)(a)). But, given $t = t\downarrow$, $t\uparrow = t\uparrow\downarrow = t$ (by (iii)(a)).

From the conditions (ii)(a), (b), (iii)(a) of III and II(ii), it follows that a counterpart $t\beta$ to a theory t can be found at any level β. For suppose α is the level of t. By II(ii), take a level $\gamma \supseteq \alpha \cup \beta$. Then let $t\beta = t^{\gamma}\beta$. The choice of γ does not matter; if $\gamma' \supseteq \alpha \cup \beta$ then $t^{\gamma'}\beta = t^{\gamma}\beta$. For take $\delta \supseteq \gamma' \cup \gamma$. Then it suffices to show that $t^{\delta}\beta = t^{\gamma}\beta$, since then also $t^{\delta}\beta = t^{\gamma'}\beta$. But $t^{\delta}\beta = t^{\gamma\delta}\beta$ (by (ii)(a) and (b)) $= t^{\gamma}\beta$ (by (iii)(a)).

Condition III(iii)(c) says when the result of going down-up is the same as going up-down, as illustrated below:

$$\begin{tikzcd}
 & t\uparrow \arrow[dr, no head] & \\
t \arrow[ur, no head] \arrow[dr, no head] & & t\uparrow\downarrow = t\downarrow\uparrow \\
 & t\downarrow \arrow[ur, no head] &
\end{tikzcd}$$

In going down to $t\downarrow$, certain information may be lost, viz., that embodied in the individuals of $D_t - D_{t\downarrow}$. We must therefore be sure that none of that information can be present in $t\uparrow\downarrow$; so $D_{t\uparrow\downarrow} \cap D_t$ must be the same as $D_{t\downarrow}$. Note that, in contrast to (iii)(a), the level of $t\uparrow\downarrow$ need not be the same as that of t.

The conditions under IV indicate to what extent the operations $\downarrow$ and $\uparrow$ respect the structure of each internal model, as given by the components S, $-$, $\circ$, and l. Condition (i)(a) says that saturation is preserved downwards. It is important to note that saturation is not preserved upwards; if a is saturated, then $a\uparrow$ will in general not be.

Conditions (i)(b) and (i)(c) of IV may be understood in terms of the concept of conservative extension. One theory u at level β is a conservative extension of another theory t at level $\alpha \subset \beta$ if $u\downarrow = t$. The conditions (i)(b) and (i)(c) then tell us when we can get a saturated theory a^+ that is a conservative extension of another saturated theory and is subject to some other constraint. In the first case, the constraint is that a^+ should be included in a saturated theory b, and what is required for the constraint to be satisfied is that $a \le b\downarrow$. In the other case, the constraint is that a^+ should contain a theory t, and what is then required is that $a \ge t\downarrow$.

Condition IV(ii)(a) says that complementation is preserved downwards. The question of upward closure does not properly arise, since $-(a\uparrow)$ is not in general defined.

Condition IV(iii)(a) says that fusion is preserved upwards. Fusion is not in general preserved downwards. One direction, viz., $t\downarrow u\downarrow \le (tu)\downarrow$, does hold and is, indeed, derivable. For $(t\downarrow u\downarrow)\uparrow \le t\downarrow\uparrow u\downarrow\uparrow$ (by IV(iii)(a)) $\le tu$ (by III(iii)(a)). So $(t\downarrow u\downarrow)\uparrow\downarrow \le (tu)\downarrow$ (by III(i)(b)). Hence $t\downarrow u\downarrow \le (tu)\downarrow$ (III(iii)(a)). However, the other direction, viz., $(tu)\downarrow \le t\downarrow u\downarrow$, does not in general hold. Condition (iii)(b) states a special circumstance in which it does hold, viz., when the second term is of the form $u\uparrow$.

Condition IV(iv)(a) states that the status of being a "logic" is preserved upwards. It also appears correct to assume that it is preserved downwards; but this is not an assumption we shall need.

The final conditions, under V, give the intrinsic and interactive behavior of the across operator $\rightarrow$. Conditions (i)–(iii) state that $\rightarrow$ is a closure operator. Condition (iv) states that any saturated cover for a symmetric theory has a saturated and symmetric subcover. The converse of (v) follows from (i); the two together state that the complement of a symmetric point is also symmetric.

In condition V(vi), the second inequality, $\vec{t}\vec{u} \le \vec{t}u$, is reversible, by V(i) and I(ii). The first inequality, $\overrightarrow{tu} \le \vec{t}\vec{u}$ is not reversible. The condition (vi) as a whole is equivalent to (vi′) $\overrightarrow{(\vec{t}\vec{u})} \le \vec{t}u$. For, given (vi′), we obtain (vi) as follows: $\overrightarrow{(tu)} \le \overrightarrow{(\vec{t}\vec{u})}$ (V(i) and I(ii)) $\le \vec{t}u$ and $\vec{t}\vec{u} \le \overrightarrow{(\vec{t}\vec{u})}$ (V(i)) $\le \vec{\vec{t}}u \le \vec{t}u$ (V(iii)). The argument from (vi) to (vi′) is this: $\overrightarrow{(\vec{t}\vec{u})} \le \vec{\vec{t}}\vec{\vec{u}} \le \vec{t}\vec{u}$ (V(iii)) $\le \vec{t}u$.

Condition (vii) says that the operation of going across is preserved upwards.

Condition V(viii) is most significant. It says: suppose a new individual i is added to t in forming the purely linguistic extension $t\uparrow$; and suppose that an old individual j from t is assumed to have all the properties of i (and vice versa); then the resulting theory $\overrightarrow{(t\uparrow)}$ will not contain any new theses concerning the individual i, i.e., $\overrightarrow{(t\uparrow)}\downarrow \leq t$. The condition therefore states that the individuals introduced in the purely linguistic extensions $t\uparrow$ are genuinely arbitrary; nothing is true of them that was not already true of the pre-existing individuals in t. The converse of the condition is derivable. For $t\uparrow \leq \vec{t}\uparrow$ (by V(i)); and so, $t \leq t\uparrow\downarrow$ (III(iii)(a)) $\leq \vec{t}\uparrow\downarrow$ (V(i) and III(i)(b)).

We cull some facts from the preceding remarks, for future reference.

LEMMA 1. The following hold in any stratified model:

(i) $t\downarrow u\downarrow \leq (tu)\downarrow$
(ii) $\overrightarrow{(-a)} = -\vec{a}$
(iii) $\overrightarrow{(t\vec{u})} \leq \vec{t}u$.

§53.2. Truth. In this section we introduce the concept of truth at a point and establish its fundamental properties. We also sketch various alternative versions of the semantics.

We shall be interested in a certain language $\mathfrak{L}$ of first-order relevance logic. The formulas of $\mathfrak{L}$ are constructed in the usual way from the given stock of predicates, using the truth-functional connectives $\wedge$ and $\sim$, the connective $\rightarrow$ for relevant implication, and the universal quantifier $\forall$. For convenience in the completeness proof and also for more doctrinaire reasons, it is supposed that there is a typographic distinction between free and bound variables. We use x, y, z, ... as metalinguistic symbols for the bound variables, and u, v, w, ... as metalinguistic symbols for the free variables. A variable is taken to be free unless it is explicitly indicated to be bound.

Given a stratified model $\mathfrak{A}$, we may add its individuals $I = \bigcup \mathrm{Range}(D)$ as self-designating names to the language $\mathfrak{L}$ to obtain the enlarged language $\mathfrak{L}^+$. Given a formula A of $\mathfrak{L}^+$, let $\mathrm{I(A)} = \{i \in I\colon i \text{ occurs (as a name) in A}\}$. We say that A is *defined at* the point t of $\mathfrak{A}$ if $\mathrm{I(A)} \subseteq D_t$, and we let $\mathrm{S}(t) = \{\mathrm{A}\colon \mathrm{A}$ is a sentence of $\mathfrak{L}^+$ that is defined at $t\}$.

Relative to a model $\mathfrak{A}$, the relation $\vDash$ of truth is to hold between a theory t of $\mathfrak{A}$ and a sentence A of S(t). It is defined by the following clauses:

(i) $t \vDash \mathrm{R}i_1 \ldots i_\mathrm{n}$ iff $\varphi t\langle \mathrm{R}, i_1, \ldots, i_\mathrm{n}\rangle$, for R an n-place predicate;
(ii) $t \vDash \mathrm{B} \wedge \mathrm{C}$ iff $t \vDash \mathrm{B}$ and $t \vDash \mathrm{C}$;
(iii) $t \vDash {\sim}\mathrm{B}$ iff $(\forall a \geq t)(\text{not } -a \vDash \mathrm{B})$;
(iv) $t \vDash \mathrm{B} \rightarrow \mathrm{C}$ iff $(\forall u \approx t)(u \vDash \mathrm{B} \Rightarrow tu \vDash \mathrm{C})$;
(v) $t \vDash (\forall \mathrm{x})\mathrm{B(x)}$ iff $(\exists t\uparrow)(\exists i \in D_{t\uparrow} - D_t)(t\uparrow \vDash \mathrm{B}(i))$.

The clauses (i)–(iv) should be familiar from the study of propositional relevance logic (§51.1). The clause for atomic sentences is modified in the obvious way to take care of their predicational structure. The clause for implication imposes the requirement on the point u verifying the antecedent that it should be at the same level as the point t that is meant to verify the conditional. This has the consequence that, where A is a quantifier-free sentence of a point t of level α, the evaluation of A at t in the stratified model will proceed in exactly the same way as the evaluation of A at t in the corresponding orthodox model $\mathfrak{A}$.

The clause for the universal quantifier represents the most distinctive aspect of our semantics. A universal sentence $\forall \mathrm{x} \mathrm{B}(\mathrm{x})$ is taken to be true iff the condition $\mathrm{B}(\mathrm{x})$ is true for an arbitrary individual i. The requirement that $i \in (D_{t\uparrow} - D_t)$ guarantees that the individual i is indeed arbitrary. It should be noted that we require that $\mathrm{B}(\mathrm{x})$ be true only for one such individual i; for if the individual is genuinely arbitrary, $\mathrm{B}(\mathrm{x})$ should then be true of any individual.

We shall later prove that the universal quantifier also conforms to the more orthodox intuitionistic-type condition:

(v′) $\quad t \vDash \forall \mathrm{x}\mathrm{B}(\mathrm{x})$ iff $(\forall t\uparrow)(\forall u \geq t\uparrow)(\forall i \in D_u)(u \vDash \mathrm{B}(i))$.

This clause could have been used in place of our own in the truth definition. But it would then have been necessary, for the purpose of establishing soundness, to derive condition (v); and so there would have been no essential gain.

We now establish several fundamental results on models. They show that the various components of a model behave in conformity with their informal explanations. The first such result reveals that $\geq$ is indeed a relation of extension.

Lemma 2. For $\mathrm{A} \in \mathrm{S}(t)$,

$t \vDash \mathrm{A}$ and $t \leq u \Rightarrow u \vDash \mathrm{A}$.

Proof. By induction on the construction of A. The proofs for the cases in which A is atomic or has one of the forms $\mathrm{B} \wedge \mathrm{C}$, $\sim \mathrm{B}$, and $(\mathrm{B} \rightarrow \mathrm{C})$ are just as for propositional relevance logic (§51.1).

This leaves the case in which $\mathrm{A} = \forall \mathrm{x}\mathrm{B}(\mathrm{x})$. Assume $t \vDash \forall \mathrm{x}\mathrm{B}(\mathrm{x})$. Then $\exists t\uparrow$, $\exists i \in D_{t\uparrow} - D_t$: $t\uparrow \vDash \mathrm{B}(i)$. Take any $u \geq t$. By III(i)(a), $u\uparrow \geq t\uparrow$. Since $t\uparrow \vDash \mathrm{B}(i)$, it follows by inductive hypothesis (IH) that $u\uparrow \vDash \mathrm{B}(i)$. But $D_u = D_t$ and $D_{u\uparrow} - D_{t\uparrow}$; so $i \in (D_{u\uparrow} - D_u)$; and so $u \vDash \forall \mathrm{x}\mathrm{B}(\mathrm{x})$, as required.

The second result shows that the across operator → behaves as expected. Call the sentence A′ an *i,j-variant* of A if it is obtained from A by substituting i for any occurrences of j or j for any occurrences of i. So, for example, the i,j-variants of Rii are Rii, Rij, and Rji, and Rjj.

Lemma 3. (Truth Across). Suppose t is an i,j-symmetric point, $i \neq j$, and that A′ is an i,j-variant of A. Then:

$t \vDash \mathrm{A} \Rightarrow t \vDash \mathrm{A}'$.

PROOF. By induction on the construction of A. It should be noted that in the respective cases in which A is of the form $B \wedge C$, $\sim B$, $B \rightarrow C$, or $\forall x B(x)$, A′ will be of form $B' \vee C'$, $\sim B'$, $B' \rightarrow C'$, or $\forall x B'(x)$, A′ will be of form $B' \vee C'$, $\sim B' \rightarrow C'$, or $\forall x B'(x)$, where B′ and C′ are *i*,*j*-variants of B and C. Let us now go through the cases in turn.

A *atomic*. By V(ix).

$A = B \wedge C$. Suppose $t \vDash B \wedge C$. Then $t \vDash B$ and $t \vDash C$. By IH, $t \vDash B'$ and $t \vDash C'$. So $t \vDash B' \wedge C'$.

$A = \sim B$. Suppose $t \nvDash A'$. Then $\exists a \geq t$: $-a \vDash B'$. By V(iv), $\exists a'$: a' is symmetric in i, j, and $t \leq a' \leq a$. Since $a' \leq a$, $-a' \geq -a$ (by I(viii)). Since $-a \vDash B'$, it follows by Lemma 2 that $-a' \vDash B'$. Since a' is symmetric, $-a'$ is symmetric (by Lemma 1(ii)). So, by IH and the fact that B is an *i*,*j*-variant of B′, $-a' \vDash B$. But then, given that $t \leq a'$, $t \vDash \sim B$.

$A = B \rightarrow C$. Suppose $t \nvDash B' \rightarrow C'$. Then $\exists u \approx t$: $u \vDash B'$ and $tu \nvDash C'$. By V(i), $u \leq \vec{u}$; and so, by the previous lemma, $\vec{u} \vDash B'$. From V(i) and (iii) it follows that $\vec{u}$ is symmetric. So, by IH, $\vec{u} \vDash B$. Now suppose, for reductio, that $t \vDash B \rightarrow C$. Then $t\vec{u} \vDash C$. Since $t\vec{u} \leq \overrightarrow{(t\vec{u})}$ (by V(i)), it follows by Lemma 2 that $\overrightarrow{(t\vec{u})} \vDash C$; and since $\overrightarrow{(t\vec{u})}$ is symmetric, it follows by IH that $\overrightarrow{(t\vec{u})} \vDash C'$. But $\overrightarrow{(t\vec{u})} \leq \vec{t}u$ (Lemma 1(iii)) $= tu$ (since t is symmetric). Therefore $tu \vDash C'$, which is a contradiction.

$A = \forall x B(x)$. Suppose $t \vDash \forall x B(x)$. Then $\exists t\uparrow$, $\exists k \in (D_{t\uparrow} - D_t)$: $t\uparrow \vDash B(k)$. Since $t\uparrow \leq \overrightarrow{t\uparrow}$, it follows by Lemma 2 that $\overrightarrow{t\uparrow} \vDash B(k)$; and, since $\overrightarrow{t\uparrow}$ is symmetric, it follows by IH that $\overrightarrow{t\uparrow} \vDash B'(k)$. By V(vii), $\overrightarrow{t\uparrow} \leq (\vec{t})\uparrow$; and so, by Lemma 2 again, $(\vec{t})\uparrow \vDash B'(k)$. But $k \in (D_{\vec{t}\uparrow} - D_{\vec{t}})$; and therefore $\vec{t} = t \vDash \forall x B(x)$.

The third result shows that the down operator $\downarrow$ behaves as expected; t is indeed a conservative extension of $t\downarrow$.

LEMMA 4 (Truth Down). For any sentence A of $S(t\downarrow)$, $t \vdash A$ iff $t\downarrow \vDash A$.

PROOF. By induction on the construction of A.

A *atomic*. By III(iv).

$A = B \wedge C$. Straightforward.

$A = \sim B$. $\Rightarrow$. Suppose $t\downarrow \nvDash \sim B$. Then $\exists a \geq t\downarrow$: $-a \vDash B$. By IV(i)(c), $\exists a^+$: $a^+ \geq t$ and $a^+\downarrow = a$. But $-a = -(a^+\downarrow) = (-a^+)\downarrow$ (by IV(ii)(a)). So by IH, $-a^+ \vDash B$; and therefore $t \nvDash \sim B$, as required.

$\Leftarrow$. Suppose $t \nvDash \sim B$. Then $\exists a \geq t$: $-a \vDash B$. By IH, $(-a)\downarrow \vDash B$; and so, by IV(ii)(a), $-(a\downarrow) \vDash B$. Since $a \geq t$, $a\downarrow \geq t\downarrow$ (by III(i)(b)); and, by IV(i)(a), $a\downarrow \in S$. It therefore follows that $t\downarrow \nvDash \sim B$.

$A = (B \rightarrow C)$. $\Rightarrow$. Suppose $t\downarrow \nvDash B \rightarrow C$. Then $\exists u \approx t\downarrow$: $u \vDash B$ and $t\downarrow u \nvDash C$. Since $u \vDash B$ and since $u = u\uparrow\downarrow$ (by III(iii)(a)), it follows by IH that $u\uparrow \vDash B$. Suppose, for reductio, that $tu{\uparrow} \vDash \mathrm{C}$. Then $(tu{\uparrow}){\downarrow} \vDash \mathrm{C}$ by IH. But $(tu{\uparrow}){\downarrow} \leq t{\downarrow}u$ (by IV(iii)(b)). So, by Lemma 2, $t{\downarrow}u \vDash \mathrm{C}$. A contradiction.

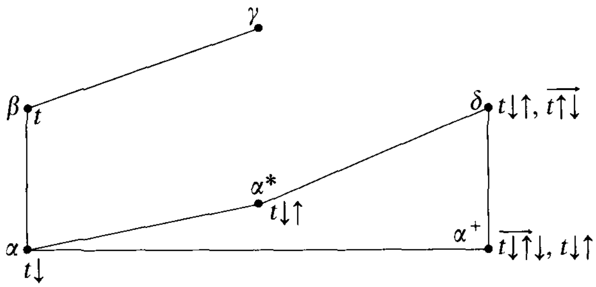
$\Leftarrow$. Suppose $t \nvDash \mathrm{B} \rightarrow \mathrm{C}$. Then $\exists u \approx t$: $u \vDash \mathrm{B}$ and $tu \nvDash \mathrm{C}$. By IH, $u{\downarrow} \vDash \mathrm{B}$ and $(tu){\downarrow} \nvDash \mathrm{C}$. But $t{\downarrow}u{\downarrow} \leq (tu){\downarrow}$ (Lemma 1(i)). So, by Lemma 2, $t{\downarrow}u{\downarrow} \nvDash \mathrm{C}$; and therefore $t{\downarrow} \nvDash \mathrm{B} \rightarrow \mathrm{C}$, as required.

$\mathrm{A} = \forall x\mathrm{B}(x)$. $\Rightarrow$. Suppose $t \vDash \forall x\mathrm{B}(x)$. Then $\exists t{\uparrow}$, $\exists i \in (D_{t\uparrow} - D_t)$: $t{\uparrow} \vDash \mathrm{B}(i)$. Let $D_{t\downarrow} = \alpha$, $D_t = \beta$, and $D_{t\uparrow} = \gamma$. Then $\alpha \subseteq \beta \subseteq \gamma$. So, by II(iii), $\delta = \alpha \cup (\gamma - \beta) \in \mathfrak{D} = \mathrm{Range(D)}$. Since $i \in \gamma - \beta$, $i \in \delta$. So, by IH, $t{\uparrow}{\downarrow} \vDash \mathrm{B}(i)$ (for ${\downarrow} = {\downarrow}_\delta$). Now $\beta \cap \delta = \beta \cap (\alpha \cup (\gamma - \beta)) = (\beta \cap \alpha) \cup (\beta \cap (\gamma - \beta)) = \alpha$. So, by III(iii)(c), $t{\uparrow}{\downarrow} = t{\downarrow}{\uparrow}$ *(where the second* ${\uparrow} = {\uparrow}^\delta$*). Therefore,* $t{\downarrow}{\uparrow} \vDash \mathrm{B}(i)$*; and, since* $i \in (\delta - \alpha)$, $t{\downarrow} \vDash \forall x\mathrm{B}(x)$, as required.

$\Leftarrow$. Suppose $t{\downarrow} \vDash \forall x\mathrm{B}(x)$. Then $\exists t{\downarrow}{\uparrow}$, $\exists i \in (D_{t\downarrow\uparrow} - D_{t\downarrow})$: $t{\downarrow}{\uparrow} \vDash \mathrm{B}(i)$. Let us assume $D_{t\downarrow} = \alpha$, $D_t = \beta$, and $D_{t\downarrow\uparrow} = \alpha^+$. There are two cases.

CASE 1. $i \notin D_t$. By II(ii), $\exists \gamma$: $\gamma \supseteq \beta \cup \alpha^+$. Consider the expansion at level γ of t to $t{\uparrow}$ and of $t{\downarrow}{\uparrow}$ to $t{\downarrow}{\uparrow}{\uparrow}$. Since $t{\downarrow}{\uparrow} \vDash \mathrm{B}(i)$, it follows by IH that $t{\downarrow}{\uparrow}{\uparrow} \vDash \mathrm{B}(i)$. By III(ii)(a), $t{\downarrow}{\uparrow}{\uparrow}$ is of the form $t{\downarrow}{\uparrow}$; and so, for $t{\downarrow}{\uparrow}$ of level γ, $t{\downarrow}{\uparrow} \vDash \mathrm{B}(i)$. Now $t{\downarrow}{\uparrow} \leq t{\uparrow}$ for $t{\downarrow}{\uparrow}^\beta \leq t$ (III(iii)(b)); and so $t{\downarrow}{\uparrow}^\beta{\uparrow}^\gamma = t{\downarrow}{\uparrow}$ (III(ii)(a)) $\leq t{\uparrow}^\gamma$ (III(i)(a)) $= t{\uparrow}$. It then follows by Lemma 2 that $t{\uparrow} \vDash \mathrm{B}(i)$. But $i \in \gamma - \beta$; and therefore $t \vDash \forall x\mathrm{B}(x)$.

CASE 2. $i \in D_t$. We reduce this case to the first case by finding a $t{\downarrow}{\uparrow}$ and a j in $\alpha^+ - \alpha$ that is not also in β. By II(i), $\exists \gamma \supset \beta$. (At this point the reader may find it helpful to consult the diagram below.) Choose a j from $\gamma - \beta$. By II(iii), $\alpha^* = \alpha \cup (\gamma - \beta) \in \mathfrak{D}$; and, by II(ii), $\exists \delta$: $\delta \supseteq \alpha^+ \cup \alpha^*$. Let $t{\downarrow}{\uparrow}$ now be the expansion of $t{\downarrow}$ to δ. Then, by the same reasoning as before, $t{\downarrow}{\uparrow} \vDash \mathrm{B}(i)$. Consider $\overrightarrow{t{\downarrow}{\uparrow}}$ for $\rightarrow = i,j{\rightarrow}$. Since $\mathrm{B}(j)$ is an i,j-variant of $\mathrm{B}(i)$, it follows by Lemma 3 that $\overrightarrow{t{\downarrow}{\uparrow}} \vDash \mathrm{B}(j)$. Now $t{\downarrow}{\uparrow} = t{\downarrow}{\uparrow}^{\alpha^*}{\uparrow}^\delta$ (by III(ii)(a)), which we shall write simply as $t{\downarrow}{\uparrow}{\uparrow}$. So $\overrightarrow{t{\downarrow}{\uparrow}{\uparrow}} \vDash \mathrm{B}(j)$. By IH, $\overrightarrow{t{\downarrow}{\uparrow}{\uparrow}}{\downarrow} \vDash \mathrm{B}(j)$, where the last ${\downarrow} = {\downarrow}_{\alpha^*}$. But $i \in (\delta - \alpha^*)$. So, by V(viii), $\overrightarrow{t{\downarrow}{\uparrow}{\uparrow}}{\downarrow} \leq t{\downarrow}{\uparrow}$ (i.e., $t{\downarrow}{\uparrow}^{\alpha^*}$); and, therefore, by Lemma 2, $t{\downarrow}{\uparrow} \vDash \mathrm{B}(j)$, as required.


γ
β
t
δ
$t{\downarrow}{\uparrow}$, $\overrightarrow{t{\uparrow}{\downarrow}}$
α^*
$t{\downarrow}{\uparrow}$
α^+
α
$\overrightarrow{t{\downarrow}{\uparrow}}{\downarrow}$, $t{\downarrow}{\uparrow}$
$t{\downarrow}$

Three simple-consequences of this result should be noted. The first states that truth is exactly preserved upwards.

COROLLARY 5 (Truth Up). For any sentence A of $S(t)$,

$t \vDash A$ iff $t\uparrow \vDash A$.

PROOF. Let the t in Lemma 4 be $t\uparrow$. The corollary then follows by III(iii)(a).

The other two consequences show that a more orthodox clause for the universal quantifier can be given.

COROLLARY 6. Suppose that $t \vDash \forall x B(x)$. Then, for any $j \in D_t$, $t \vDash B(j)$.

PROOF. Under the supposition, $\exists t\uparrow, \exists i \in (D_{t\uparrow} - D_t)$: $t\uparrow \vDash B(i)$. Take a $j \in D_t$. Consider $\overrightarrow{t\uparrow}$, for $\rightarrow\ = i,j\rightarrow$. By Lemma 3, $\overrightarrow{t\uparrow} \vDash B(j)$; and, by Lemma 4, $\overrightarrow{t\uparrow}\downarrow \vDash B(j)$. But $\overrightarrow{t\uparrow}\downarrow \leq t$ (by V(viii)); and so, by Lemma 2, $t \vDash B(j)$.

COROLLARY 7. Suppose that $t \vDash \forall x B(x)$. Then, for any $t\uparrow$ and for any $j \in D_{t\uparrow}$, $t\uparrow \vDash B(j)$.

PROOF. Given that $t \vDash \forall x B(x)$, it follows from Corollary 5 that $t\uparrow \vDash \forall x B(x)$. The result then follows from Corollary 6.

The next result complements Lemma 2. Its proof depends upon having an orthodox clause for the universal quantifier, and it is for this reason that its proof has had to be postponed until now.

LEMMA 8. For $A \in S(t)$,

$(\forall a \geq t)(a \vDash A) \Rightarrow t \vDash A$.

PROOF. By induction on the construction of A. The proofs for the cases in which A is atomic or has one of the forms $B \wedge C$, $\sim B$, and $(B \rightarrow C)$ are just as for propositional relevance logic (§51.1).

This leaves the case in which $A = \forall x B(x)$. So suppose $t \nvDash \forall x B(x)$. By II(i), $\exists \beta$: $\beta \supset D_t$. Choose an $i \in (\beta - D_t)$, and take a $t\uparrow$ of level β. Then $t\uparrow \nvDash B(i)$. By IH, $\exists a \geq t\uparrow$: $a \nvDash B(i)$. Given $a \geq t\uparrow$, $a\downarrow \geq t\uparrow\downarrow$ (III(i)(b)) $= t$ (III(iii)(a)). By IV(i)(a), $a\downarrow \in S$. So it remains to show that $a\downarrow \nvDash \forall x B(x)$. Now $a\downarrow\uparrow \leq a$ (III(iii)(b)); and so, since $a \nvDash B(i)$, it follows by Lemma 2 that $a\downarrow\uparrow \nvDash B(i)$. But then, by Corollary 7, $a\downarrow \nvDash \forall x B(x)$.

From Lemmas 2 and 8 we obtain the following results in the same way as for propositional relevance logic (§51.1):

COROLLARY 9.

(i) $a \vDash {\sim}\mathrm{B} \Leftrightarrow -a \nvDash \mathrm{B}$, $\mathrm{B} \in \mathrm{S}(a)$,
(ii) $a \vDash \mathrm{B} \vee \mathrm{C} \Leftrightarrow a \vDash \mathrm{B}$ or $a \vDash \mathrm{C}$, for $\mathrm{B,C} \in \mathrm{S}(a)$,
(iii) $(\forall a \approx t)(a \vDash \mathrm{B} \Rightarrow ta \vDash \mathrm{C}) \Rightarrow t \vDash \mathrm{B} \rightarrow \mathrm{C}$, for $\mathrm{B,C} \in \mathrm{S}(t)$,
(iv) $(\forall a \approx l)(a \vDash \mathrm{B} \Rightarrow a \vDash \mathrm{C}) \Leftrightarrow l \vDash \mathrm{B} \rightarrow \mathrm{C}$, for $\mathrm{B,C} \in \mathrm{S}(l)$.

Although the result is not required for later proofs, it may be of interest to work out the derived clause for the existential quantifier. $\exists \mathrm{xB(x)}$ is used as an abbreviation of ${\sim}\forall \mathrm{x} {\sim} \mathrm{B(x)}$.

LEMMA 10. For any formula $\exists \mathrm{xA(x)}$ that is defined at a,

$a \vDash \exists \mathrm{xB(x)}$ iff $(\exists b)(\exists i \in D_b)(b{\downarrow} \leq a$ and $b \vDash \mathrm{B}(i))$.

PROOF. $a \vDash \exists \mathrm{xB(x)} = {\sim}\forall \mathrm{x} {\sim} \mathrm{B(x)}$ iff $-a \nvDash \forall \mathrm{x} {\sim} \mathrm{B(x)}$ iff $(\exists(-a)\uparrow)(\exists i \in D_{-(a\uparrow)})$ $((-a)\uparrow \nvDash {\sim}\mathrm{B}(i))$. Now $(-a)\uparrow \nvDash {\sim}\mathrm{B}(i)$ iff $\exists b(-b \geq (-a)\uparrow$ and $b \vDash \mathrm{B}(i))$ iff $\exists b(b{\downarrow} \leq a$ and $b \vDash \mathrm{B}(i))$. Therefore $a \vDash \exists \mathrm{xB(x)}$ iff $(\exists b)(\exists i \in D_b)(b{\downarrow} \leq a$ and $b \vDash \mathrm{B}(i))$.

Since $t \vDash \exists \mathrm{xB(x)}$ iff $(\forall a \geq t)(a \vDash \mathrm{B(x)})$, a clause for $t \vDash \exists \mathrm{xB(x)}$ is indirectly determined. The clause above can be made to hold with t in place of a, but only when certain additional structural conditions are imposed on a model.

Various alternative versions of the semantics may be given. To some extent, they enable us to simplify the conditions on a model. Their disadvantage, from our own point of view, is that they all involve some departure from the stratified conception of a model.

First of all, we can drop the requirement that the extension relation be defined only on theories t and u whose domain is the same. We can allow, instead, that it also be defined on theories t and u for which $D_t \subseteq D_u$. Using $\precsim$ for the new relation of extension and retaining $\leq$ for the old one, $\precsim$ may be defined in terms of $\leq$ by $t \leq u{\downarrow}$, and $\leq$ may be defined in terms of $\precsim$ by ($t \precsim u$ and $t \approx u$).

If the new relation is used in place of the old, then we may declare that $t^\beta \succsim t$ for $\beta \supseteq D_t$, that $t \succsim t_\beta$ for $\beta \subseteq D_t$, and that $t^\beta \precsim t \precsim t_\beta$ for $\beta = D_t$. Condition III(ii)(a) can then be replaced with the direction $t{\uparrow\uparrow} \precsim t$, III(ii)(b) with the direction $t{\downarrow\downarrow} \succsim t$, III(iii)(a) with $t{\uparrow\downarrow} \precsim t$, IV(iii)(a) with $(tu){\uparrow} \succsim t{\uparrow}u{\uparrow}$, and V(vii) with $(\overrightarrow{t\uparrow}) \succsim (\vec{t}){\uparrow}$; condition III(iii)(b) can be dropped altogether. The proofs are straightforward. For example, to derive $t{\downarrow\uparrow} \leq t$, we note that $t_\beta \precsim t$ and hence that $t_\beta^\alpha \precsim t^\alpha = t$.

Given the liberalized extension relation, the operators ↑ and ↓ can be defined. We may take t^β to be the smallest theory to contain t at level β;

and we may take t_β to be the greatest theory to be included in t at level β. The various structural conditions on $\uparrow$ and $\downarrow$ can then be stated in terms of $\succsim$; and some of them may be dropped.

A more radical reformulation of the semantics can be obtained by no longer requiring, for a sentence to be evaluated at a point, that its individuals must belong to the domain of the point. A domain of individuals D_t will still be associated with each point t. But the individuals in this domain will now be thought to be those pertaining to the axioms of the theory; the consequences of the theory can concern any individuals whatever.

On this approach, the clause for the universal quantifier can simply take the form:

$$t \models \forall x B(x) \text{ iff } (\exists i \notin D_t)(t \models B(i)).$$

There will therefore be no need for the up operator $\uparrow$. However, there will still be a need for the down operator $\downarrow$, since this is required for the proof of soundness.

The fusion operator $\circ$ is now naturally taken to apply to any two theories t and u; the domain of tu is taken to be identical with the domain of t. With this treatment of $\circ$, with the use of $\succeq$ in place of $\geq$, and with the removal of $\uparrow$, various further conditions can be simplified or dropped.

Another radical alternative is to work exclusively in terms of saturated theories. (See §51.5, and §§48.3–9.) On this approach there is no room for the upward theory $a\uparrow$, since $a\uparrow$ will not in general be saturated when a is. Instead, a possible model $\mathfrak{A}$ will be a 9-tuple $(S, D, N, R, -, \geq, \downarrow, \rightarrow, \varphi)$, where $S, D, \circ, -, \geq, \downarrow, \rightarrow$, and φ are as before, N (normal points) is a subset of S, and R is a ternary relation on S that relates only points of the same level. To become an actual model, a possible model must be subject to certain additional conditions. In stating these, we shall use a^+ for an arbitrary conservative extension of a of given level β, i.e., for an arbitrary point b for which $D_b = \beta$ and $b\downarrow = a$. So $\exists a^+$ will mean $\exists b(D_b = \beta$ and $b\downarrow = a$ and $\ldots)$; and similarly for $\forall a^+$.

I. *Standard*

(i) $Rabc$ and $a' \leq a \Rightarrow Ra'bc$

(ii) $Rabc$ and $a \in N \Rightarrow b \leq c$

(iii) $a \in N \Rightarrow Rabb$

(iv) $a \in N \Rightarrow a \geq -a$

(v) $a = --a$

(vi) $a \geq b \Rightarrow -b \geq -a$

(vii) $\varphi a\langle R, i_1, \ldots, i_n\rangle$ and $a \leq b \Rightarrow \varphi b\langle R, i_1, \ldots, i_n\rangle$, for R an n-place predicate and $i_1, \ldots, i_n \in D_a$

II. *Levels.* (i)–(iii). As before.

III. *Basic Behavior of* ↓

(i) $a \le b \Rightarrow a{\downarrow} \le b{\downarrow}$
(ii) $a{\downarrow}{\downarrow} = a{\downarrow}$
(iii) $\varphi a{\downarrow}\langle \mathrm{R}, i_1, \ldots, i_{\mathrm{n}}\rangle \Leftrightarrow \varphi a\langle \mathrm{R}, i_1, \ldots, i_{\mathrm{n}}\rangle$, for R an n-place predicate and $i_1, \ldots, i_{\mathrm{n}} \in D_{a\downarrow}$

IV. *Interactive Behavior of* ↓

(i) (a) $a \le b{\downarrow} \Rightarrow \exists a^+(a^+ \le b)$
(b) for $D_a = \alpha$, $D_b = \beta$, $\gamma(=\alpha \cap \beta)$, $\delta(=\alpha \cup \beta) \in D$: $a{\downarrow}_\gamma = b{\downarrow}_\gamma \Rightarrow \exists b^+(D_{b+} = \delta$ and $b^+{\downarrow}_\alpha \ge a)$
(ii) $(-a){\downarrow} = -(a){\downarrow}$
(iii) (a) $Rabc \Rightarrow Ra{\downarrow}b{\downarrow}c{\downarrow}$
(b) $Rabc \Rightarrow \forall a^+ \exists b^+ \exists c^+(Ra^+b^+c^+)$
(c) $Rabc \Rightarrow \forall c^+ \exists a^+ \exists b^+(Ra^+b^+c^+)$
(iv) for $D_a = \alpha$, $\alpha^+ \in D$, $\alpha^+ \ge \alpha$: $a \in N \Rightarrow \exists a^+$ of level $\alpha^+(a^+ \in N)$

V. *Behavior of* →

(i) $a \le \vec{a}$
(ii) $a \le b \Rightarrow \vec{a} \le \vec{b}$
(iii) $\vec{\vec{a}} \le \vec{a}$
(iv) $(\overrightarrow{-\vec{a}}) \le -\vec{a}$
(v) $a{\downarrow}$ symmetric in $i; j \Rightarrow a$ symmetric in i, j
(vi) $R\vec{a}bc \Rightarrow R\vec{a}\vec{b}\vec{c}$
(vii) $Rabc$ and a symmetric in $i, j \Rightarrow \exists c'(Rabc', c' \le c$, and c' symmetric in $i, j)$
(viii) for $D_a = \alpha$, $\beta \supseteq \alpha$, $\beta \in D$: $j \in \alpha$ and $i \in \beta - \alpha \Rightarrow \exists a^+(D_{a+} = \beta$ and $\mathrm{ij}_{\overrightarrow{a+}} = a)$
(ix) $\varphi a\langle \mathrm{R}, i_1, \ldots, i_{\mathrm{n}}\rangle \Rightarrow \varphi\vec{a}\langle \mathrm{R}, i'_1, \ldots, i'_{\mathrm{n}}\rangle$, where → is j,k→ for distinct $j, k \in D_{\mathrm{a}}$ and, for $\mathrm{p} = 1, 2, \ldots, \mathrm{n}$, $i'_{\mathrm{p}} = i_{\mathrm{p}}$ if $i \ne j$ or k and $i'_{\mathrm{p}} = j$ or k if $i_{\mathrm{p}} = j$ or k.

The generic clause for the quantifier now takes the form:

$$a \vDash \forall x\mathrm{B}(x) \text{ iff } (\exists \alpha^+ \supseteq D_a)(\exists i \in (\alpha^+ - D_a))(\forall a^+)(D_{a^+} = \alpha^+ \Rightarrow a^+ \vDash \mathrm{B}(i)).$$

Thus the single theory $t{\uparrow}$ in the earlier condition gives way to a "cover" of saturated theories a^+. The orthodox clause, on the other hand, takes the more acceptable form:

$$a \vDash \forall x\mathrm{B}(x) \text{ iff } (\forall a^+)(\forall b \ge a)(\forall i \in D_b)(b \vDash \mathrm{B}(i)).$$

In some ways the presentation of the semantics is simpler in terms of saturated theories *a* than in terms of arbitrary theories *t*. But the great advantage is lost of being able to appeal to the upward theory $t\uparrow$ in the formulation of the generic clause for the quantifier and in the formulation of some of the conditions on a model.

It should be possible to apply the present semantics for the quantifier to other intensional logics, to intuitionistic logic, and even to classical logic itself. The application to classical logic is of special interest, since it allows us to give a finite representation of quantification over an infinite domain. However, this is not an idea we shall pursue here.

§53.3. The logics.

We here set out the logics that will be of interest to us. Some of their basic deductive properties are established.

The minimal logic $\mathbf{B}^{\forall\exists x}$ is given by the following axioms and rules:

Axioms

1 $A\wedge B \to A$
2 $A\wedge B \to B$
3 $(A\to B)\wedge(A\to C) \to (A\to B\wedge C)$
4 $(A\to C)\wedge(B\to C) \to (A\vee B\to C)$
5 $A\wedge(B\vee C) \to (A\wedge B)\vee(A\wedge C)$
6 $\sim\sim A \to A$
7 $\sim A\vee A$
8 $\forall xA(x) \to A(v)$
9 $\forall x(A\to B) \to (A\to\forall xB)$
10 $\forall x(A\vee B) \to (A\vee\forall xB)$

Rules

1 $A, B \,/\, A\wedge B$
2 $A, A\to B \,/\, B$
3 $A\to B \,/\, (B\to C)\to(A\to C)$
4 $B\to C \,/\, (A\to B)\to(A\to C)$
5 $A\to\sim B \,/\, B\to\sim A$
6 $A(v) \,/\, \forall xA(x)$.

In axiom schemes 9 and 10, x may not be free in A; in rule 6, v must be free for x in A(v).

Note that axiom schemes 1–7 and rules 1–5 are the same as for propositional relevance logic as in §51.2 (except that there we used single axioms instead of axiom schemes). Axiom scheme 8 is known as *Specification*, and rules 2 and 6 as *Modus Ponens* and *Generalization*, respectively. We call scheme 9 *Relevant Distribution* and scheme 10 *Classical Distribution.*

A *logic* is identified with its set of theorems. It is taken to be any set of formulas that contains the minimal logic $\mathbf{B}^{\forall\exists x}$ and is closed under rules 1–6 and a rule of substitution for predicate letters. We shall sometimes find it convenient to work with the concept of a *quasi-logic*; this is like a logic, but need not be closed under the rule of generalization (6 above) or the rule of substitution.

If **L** is a propositional relevance logic in the sense of §51.2, then $\mathbf{L}^{\forall\exists x}$ is the smallest logic in our sense to contain all substitution instances of the theorems of **L**. Given an axiomatization of **L** with rules 1–5 above, an axiomatization of $\mathbf{L}^{\forall\exists x}$ can be obtained by adding axiom schemes 8–10 and rule 6. In the case of many propositional logics, such as **R** and **E**, the axiomatization of the corresponding quantificational logics can be considerably simplified; but this is not a question we shall go into. See §38 for a few formulations.

The notion of *deduction* is defined in the usual way (§51.2). It should be recalled that theorems cannot figure as "free" assumptions in deductions, but only as "free" major premisses in the application of *modus ponens*; so $\vdash_{\mathbf{L}}$ here, as in §51, is a relevant rather than an Official (§22.2.1) deductive relation. We then have:

LEMMA 11 (Deduction theorem). For any logic (indeed, quasi-logic) **L**, $A \vdash_{\mathbf{L}} B \Rightarrow A \to B \in \mathbf{L}$.

Basic propositional facts about logics will be taken for granted. We also require the following quantificational facts:

LEMMA 12. For any logic **L**,

(i) $\Delta \vdash_{\mathbf{L}} B(v) \Rightarrow \Delta \vdash_{\mathbf{L}} \forall x B(x)$, for v not in Δ and free for x in B(v);

(ii) $\forall x_1 \ldots \forall x_n(A \to B(x_1, \ldots, x_n)) \vdash_{\mathbf{L}} (A \to \forall x_1 \ldots \forall x_n B(x_1, \ldots, x_n)$, for $x_1, \ldots, x_n$ not free in A.

PROOF. (i) Suppose $\Delta \vdash_{\mathbf{L}} B(v)$. Then, for some $A_1, \ldots, A_m \in \Delta$, $A = A_1 \wedge \ldots \wedge A_m \vdash_{\mathbf{L}} B(v)$. By the Deduction theorem, $(A \to B(v)) \in \mathbf{L}$. By Generalization, $\forall x(A \to B(x)) \in \mathbf{L}$; and by Relevant Distribution and *Modus Ponens*, $(A \to \forall x B(x)) \in \mathbf{L}$. But then $\Delta \vdash_{\mathbf{L}} \forall x B(x)$.

(ii) Let $C = \forall x_1 \ldots \forall x_n(A \to B(x_1, \ldots, x_n))$. Then $C \vdash_{\mathbf{L}} A \to B(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are new variables, by repeated applications of Specification. By a single application of (i) above, $C \vdash_{\mathbf{L}} \forall x_n(A \to B(v_1, \ldots, v_{n-1}, x_n))$; and so, by Relevant Distribution, $C \vdash_{\mathbf{L}} A \to \forall x_n B(v_1, \ldots, v_{n-1}, x_n)$. Repeating this process $(n-1)$ more times, we then obtain $C \vdash_{\mathbf{L}} A \to \forall x_1 \ldots \forall x_n B(x_1, \ldots, x_n)$.

§53.4. Soundness. Let $\mathfrak{A}$ be a stratified model. A sentence A of the corresponding language $\mathfrak{L}^+$ is said to be *true in* $\mathfrak{A}$—in symbols, $\mathfrak{A} \models A$—if $l \models A$ for any l at which A is defined. On the other hand, a formula $A(v_1, \ldots, v_n)$

with free variables $v_1, \dots, v_n$ is said to be *true in* $\mathfrak{A}$ if, for any individuals $i_1, \dots, i_n \in I$ and for any l at which $A(i_1, \dots, i_n)$ is defined, $l \models A(i_1, \dots, i_n)$. A formula of the given language $\mathfrak{L}$ is then said to be *valid*—$\models$ A—if it is true in every model.

We now show:

THEOREM 13. Each theorem of $\mathbf{B}^{\forall\exists x}$ is valid.

PROOF. By induction on the proof of a formula A in $\mathbf{B}^{\forall\exists x}$. The non-quantificational cases are taken care of in the same way as for propositional relevance logic (§51.3). This leaves the quantificational axioms and rules. Let us deal with each in turn.

AXIOM 8. Take a designated point l of a model $\mathfrak{A}$ and a sentence $\forall xA(x) \to A(i)$ of $S(l)$. We must show that $l \models (\forall xA(x) \to A(i))$. So suppose $t \models \forall xA(x)$, for $t \approx l$. Then, by Corollary 6, $t \models A(i)$, as required.

AXIOM 9. Take a designated point l of a model $\mathfrak{A}$ and a sentence $\forall x(A \to B(x)) \to (A \to \forall xB(x))$ of $S(l)$, x not free in A. Suppose $t \models \forall x(A \to B(x))$, for $t \approx l$. To show $t \models (A \to \forall xB(x))$, suppose $u \models A$, for $u \approx t$. We then wish to show $tu \models \forall xB(x)$. Since $t \models \forall x(A \to B(x))$, $\exists t{\uparrow}\exists i \in (D_{t\uparrow} - D_t)$: $t{\uparrow} \models (A \to B(i))$. Since $u \models A$, $u{\uparrow} \models A$ (by Corollary 5). So $t{\uparrow}u{\uparrow} \models B(i)$. But $t{\uparrow}u{\uparrow} = (tu){\uparrow}$; so $(tu){\uparrow} \models B(i)$; and so $tu \models \forall xB(x)$.

AXIOM 10. Take a designated point l of a model $\mathfrak{A}$ and a sentence $\forall x(A \vee B(x)) \to (A \vee \forall xB(x))$ of $S(l)$, x not free in A. Suppose $a \models \forall x(A \vee B(x))$ and $a \not\models A$. To show $a \models \forall xB(x)$. Since $a \models \forall x(A \vee B(x))$, $(\exists a{\uparrow})(\exists i \in D_{a\uparrow} - D_a))$: $a{\uparrow} \models A \vee B(i)$. Take any $b \geq a{\uparrow}$. Now $b{\downarrow} \geq a{\uparrow}{\downarrow} = a$. So, by IV(i)(b), $\exists a^+$: $a^+ \leq b$ and $a^+{\downarrow} = a$. Since $a \not\models A$, it follows by Lemma 4 that $a^+ \not\models A$. Since $a{\uparrow} = a^+{\downarrow}{\uparrow} \leq a^+$ and $a{\uparrow} \models (A \vee B(i))$, it follows by Lemma 2 that $a^+ \models (A \vee B(i))$. Given $a^+ \not\models A$ and $a^+ \models (A \vee B(i))$, $a^+ \models B(i)$; and so, given $a^+ \leq b$, $b \models B(i)$. But the $b \geq a{\uparrow}$ was arbitrary. Therefore $a{\uparrow} \models B(i)$, by Lemma 8; and so $a \models \forall xB(x)$.

RULE 6. Take any model $\mathfrak{A}$. It suffices to show, given that each sentence $A(i)$ is true in the model $\mathfrak{A}$, that $\forall xA(x)$ is true in $\mathfrak{A}$. So take a designated point l at which $\forall xA(x)$ is defined. Pick an $l{\uparrow}$ and an $i \in (D_{l\uparrow} - D_l)$. Then $l{\uparrow}$ is a designated point, by IV(iv)(a); so $l \models A(i)$; and so $l \models \forall xA(x)$.

§53.5. Completeness. We prove completeness for the minimal logic $\mathbf{B}^{\forall\exists x}$ by means of a canonical model. This result is then extended to various other logics, including $\mathbf{R}^{\forall\exists x}$ and $\mathbf{E}^{\forall\exists x}$.

The construction of the canonical model requires certain preliminary notions, and the verification of its fundamental properties requires certain preliminary results. Recall that, for $\mathbf{L}$ a logic, a set of formulas Δ is an $\mathbf{L}$-*theory* if it is closed under **L**-deduction; i.e., $(\forall A)(\Delta \vdash_{\mathbf{L}} A \Rightarrow A \in \Delta)$; and that Δ is **L**-*prime* if all disjunctions are decided; i.e., $(\forall A,B)(\Delta \vdash_{\mathbf{L}} A \vee B \Rightarrow A \in \Delta$ or $B \in \Delta)$. These definitions apply equally well to a quasi-logic **L**. We then have:

LEMMA 14 (Lindenbaum's). Let **L** be a quasi-logic, Γ a set of formulas closed under disjunction, and Δ an **L**-theory that does not intersect Γ. Then there is an **L**-prime $\Delta' \supseteq \Delta$ that also does not intersect Γ.

PROOF. As for propositional relevance logic (§51.3). Note that the proof does not require that **L** be closed under the rule of substitution.

For any formula A, let Var(A) be the set of variables that occur in A; and, for any set of variables V, let Fml(V) be the set of formulas A for which $\mathrm{Var}(A) \subseteq V$. The above notions of being an **L**-theory and being **L**-prime can then be relativized to the formulas Fml(V). We say that Δ is an **L**-*theory in* V if:

$$(\forall A \in \Delta)(\mathrm{Var}(A) \subseteq V) \text{ and } (\forall A)(\mathrm{Var}(A) \subseteq V \text{ and } \Delta \vdash_{\mathbf{L}} A \Rightarrow A \in \Delta),$$

and that Δ is **L**-*prime in* V if:

$$(\forall A \in \Delta)(\mathrm{Var}(A) \subseteq V) \text{ and } (\forall A,B)(\mathrm{Var}(A), \mathrm{Var}(B) \subseteq V, \text{ and}$$
$$\Delta \vdash_{\mathbf{L}} A \vee B \Rightarrow A \in \Delta \text{ or } B \in \Delta)$$

It should be noted that Lindenbaum's lemma obtains with the relativized notions in place of the absolute ones.

The syntactic account of the across operator in the canonical model will require the use of a somewhat kinky notion of deduction. Given distinct variables v and w and formulas A and B, say that B is a v,w-*variant of* A if it is obtained from A by replacing any of the occurrences of w with v and any of the occurrences of v with w. The special v,w-variant of A in which all occurrences of w are replaced with v will be denoted by $A^{v}\!/_{w}$.

Say that B is **L**-*deducible from* Δ *under the identification of* v *and* w—in symbols, $\Delta \vdash^{v,w}_{\mathbf{L}} B$—if there is a sequence of formulas $A_0, A_1, \ldots, A_n$ such that $A_n = B$ and $(\forall i \le n)[A_i \in \Delta$ or $(\exists j,k < i)(A_i = A_j \wedge A_k)$ or $(\exists j < i)(A_j \to A_i \in \mathbf{L})$ or $(\exists j < i)$ (A_i is a v,w-variant of A_j)]. Deducibility under identification of the two variables v, w is obtained from orthodox deducibility by adding the one new rule: from A infer any v,w-variant of A. This may appear a somewhat odd rule to add. But imagine that v and w are merely typographic variants of each other; one would then want to treat A and its variants as if they were the same formula.

The new notion of deduction can be very simply related to the orthodox notion. Given a logic **L**, let $\mathbf{L}^{v,w}$ be the smallest quasi-logic to contain **L** and all theorems of the form $A \to A'$ for A' a v,w-variant of A. It is to be noted that $\mathbf{L}^{v,w}$ will not in general be a logic; for $Fv \to Fw$, let us say, will belong to it, but presumably not $\forall x \forall y(Fx \to Fy)$. We have:

LEMMA 15. For distinct variables v and w, and formulas A and B, the following are equivalent:

(i) $A \vdash_{\mathbf{L}}^{v,w} B$
(ii) $A \vdash_{\mathbf{L}^{v,w}} B$
(iii) $\vdash_{\mathbf{L}^{v,w}} A \to B$
(iv) $\vdash_{\mathbf{L}} A\,{}^{v}\!/_{w} \to B\,{}^{v}\!/_{w}$.

PROOF. By a chain of implications.

(i) ⇒ (ii). Obvious from the definitions.

(ii) ⇒ (iii). By the Deduction theorem.

(iii) ⇒ (iv). By a straightforward induction on proofs, we can show that $\vdash_{\mathbf{L}^{v,w}} C$ implies $\vdash_{\mathbf{L}} C\,{}^{v}\!/_{w}$ for any formula C.

(iv) ⇒ (i). Given that $\vdash_{\mathbf{L}} A\,{}^{v}\!/_{w} \to B\,{}^{v}\!/_{w}$, the following sequence of formulas constitutes an **L**-deduction of B from A under the identification of v and w: A; $A\,{}^{v}\!/_{w}$; $B\,{}^{v}\!/_{w}$; B.

In the light of this result we shall use the notations $\vdash_{\mathbf{L}}^{v,w}$ and $\vdash_{\mathbf{L}^{v,w}}$ interchangeably. A similar result can be proved with an arbitrary set V of variables in place of the doubleton set $\{v, w\}$; but it will not be required.

In defining the canonical model, it will be convenient to indicate explicitly the language of each theory. Accordingly, for **L** a logic and V a set of variables, say that t is an **L**,*V-theory* if it is an ordered pair $\langle \Delta, V \rangle$, where Δ is an **L**-theory in V. We say that a language-specific theory $\langle \Delta, V \rangle$ is **L**,*V-prime* if Δ is **L**-prime in V. If t is the theory $\langle \Delta, V \rangle$, we use $\mathrm{Thm}(t)$ for Δ and $\mathrm{Var}(t)$ for V.

If $\Delta = \mathrm{Thm}(t)$ is nonempty, then $V = \mathrm{Var}(t)$ is determinable from t itself; it is simply $V' = \bigcup\{\mathrm{Var}(A)\colon A \in \Delta\}$. For $V' \subseteq V$, by definition; and $V \subseteq V'$, since, if $A \in \Delta$, $v \in V$, and R is an n-place predicate of **L**, then $(A \vee Rvv \ldots v) \in \Delta$. However, Var(t) is not determinable from $\mathrm{Thm}(t)$ in case $\mathrm{Thm}(t)$ is empty; and it is because of this one annoying fact that we have had to work with the more complicated definition of theory.

If $t = \langle \Delta, V \rangle$ is a language-specific theory and U is any set of variables, then we let $t|U$ be the pair $\langle \Gamma, U \rangle$, where $\Gamma = \{A\colon \mathrm{Var}(A) \subseteq U$ and $\mathrm{Thm}(t) \vdash_{\mathbf{L}} A\}$. It is readily seen that $t|U$ is also a theory. In case $U \supseteq V$, $t|U$ is called the *expansion of t to U*; and in case $U \subseteq V$, $t|U$ is called the *contraction of t to U*. Note that $\mathrm{Thm}(t|U)$, for $t|U$ a contraction, is simply $\{A \in t\colon \mathrm{Var}(A) \subseteq U\}$.

We are now ready to define the canonical model. Given a logic **L**, let the *canonical model* $\mathfrak{A}_{\mathbf{L}}$ for **L** be $(T, S, D, l, \circ, -, \geq, \uparrow, \downarrow, \to, \varphi)$, where:

(i) $T = \{t\colon t$ is an **L**,V-theory for a finite set of variables $V\}$;

(ii) $S = \{t \in T\colon t \text{ is } \mathbf{L}, V\text{-prime}\}$;
(iii) $D = \{\langle t, \mathrm{Var}(t)\rangle\colon t \in T\}$;
(iv) $I = \{\langle V, t\rangle\colon t \in T, \mathrm{Var}(t) = V, \text{ and } \mathrm{Thm}(t) = \mathbf{L} \cap \mathrm{Fml}(V)\}$;
(v) $\circ = \{\langle t, u, v\rangle\colon t, u, v \in T, \mathrm{Var}(t) = \mathrm{Var}(u) = \mathrm{Var}(v), \text{ and } \mathrm{Thm}(v) = \{\mathrm{B}\colon \mathrm{A} \rightarrow \mathrm{B} \in \mathrm{Thm}(t) \text{ and } \mathrm{A} \in \mathrm{Thm}(u)\}\}$;
(vi) $- = \{\langle a, b\rangle\colon a, b \in S, \mathrm{Var}(b) = \mathrm{Var}(a), \text{ and } \mathrm{Thm}(b) = \{\mathrm{B}\colon \sim\mathrm{B} \notin a\}\}$;
(vii) $\geq = \{\langle t, u\rangle \in T \times T\colon \mathrm{Thm}(t) \supseteq \mathrm{Thm}(u) \text{ and } \mathrm{Var}(t) = \mathrm{Var}(u)\}$;
(viii) $\uparrow = \{\langle t, \alpha, u\rangle\colon t, u \in T, \alpha \supseteq \mathrm{Var}(t), \text{ and } u \text{ is the expansion of } t \text{ to } \alpha\}$;
(ix) $\downarrow = \{\langle t, \alpha, u\rangle\colon t, u \in T, \alpha \subseteq \mathrm{Var}(t), \text{ and } u \text{ is the contraction of } t \text{ to } \alpha\}$;
(x) $\rightarrow = \{\langle t, \mathrm{v}, \mathrm{w}, u\rangle\colon t, u \in T$, v and w are distinct variables of $\mathrm{Var}(t)$, $\mathrm{Var}(u) = \mathrm{Var}(t)$, and $\mathrm{Thm}(u)$ is the smallest $\mathbf{L}^{\mathrm{v},\mathrm{w}}$-theory in the language of $\mathrm{Var}(u)$ to contain $\mathrm{Thm}(t)\}$;
(xi) $\varphi = \{\langle t, \langle \mathrm{R}, \mathrm{v}_1, \ldots, \mathrm{v}_n\rangle\rangle\colon t \in T$, R is an n-place predicate, and $\mathrm{Rv}_1 \ldots \mathrm{v}_n \in \mathrm{Thm}(t)$.

It is readily checked that any canonical model is indeed a possible model. We now show:

LEMMA 16 (Modelhood). Any canonical model $\mathfrak{A}_{\mathrm{L}}$ satisfies the conditions for being a stratified model.

PROOF. Since there are so many conditions, we shall suppress uninteresting detail.

I. The verification is as for propositional relevance logic (§51.3).

II. (i). Since $\mathrm{Var}(t)$, for $t \in T$, is always finite. (ii) and (iii). Trivial.

III. By the basic properties of deduction, with the single exception of one half of (iii)(c), viz., $t{\uparrow}{\downarrow} \leq t{\downarrow}{\uparrow}$. Let $\mathrm{Var}(t) = U$, $\mathrm{Var}(t{\downarrow}) = V_1$, $\mathrm{Var}(t{\uparrow}) = V_2$, and $\mathrm{Var}(t{\uparrow}{\downarrow}) = \mathrm{Var}(t{\downarrow}{\uparrow}) = W$. Now suppose $\mathrm{A} \in \mathrm{Thm}(t{\uparrow}{\downarrow})$. Then $\mathrm{A} \in \mathrm{Fml}(W)$. Write A in the form $\mathrm{A}(\mathrm{w}_1, \ldots, \mathrm{w}_n)$, where $\mathrm{w}_1, \ldots, \mathrm{w}_n$ are all the variables of A that are not in V_1. By the basic properties of deduction, $\mathrm{Thm}(t) \vdash \mathrm{A}(\mathrm{w}_1, \ldots, \mathrm{w}_n)$. By the restriction on the condition, $\mathrm{w}_1, \ldots, \mathrm{w}_n \notin U$. So, by Lemma 12(i), $\mathrm{Thm}(t) \vdash \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n)$, for new bound variables $\mathrm{x}_1, \ldots, \mathrm{x}_n$. But then $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathrm{Thm}(t{\downarrow})$; and so, by Specification, $\mathrm{A}(\mathrm{w}_1, \ldots, \mathrm{w}_n) \in \mathrm{Thm}(t{\downarrow}{\uparrow})$, as required.

IV. (i)(a). Trivial.

(i)(b). Suppose $a \leq b{\downarrow}$. Let $\mathrm{Thm}(a) = \Delta$, $\mathrm{Thm}(b) = \Gamma$, $\mathrm{Var}(a) = V$, and $\mathrm{Var}(b) = V^+$. Then we have: Δ **L**-prime in V, $V \subseteq V^+$, Γ **L**-prime in V^+, and $\Delta \subseteq \Gamma$. We wish to find a Δ^+ such that: Δ^+ is **L**-prime in V^+, $\Delta^+ \subseteq \Gamma$, and $\Delta^+ \cap \mathrm{Fml}(V) = \Delta$. For then we can set $a^+ = \langle \Delta^+, V^+\rangle$.

Let $\bar{\Delta} = \{\mathrm{A} \in \mathrm{Fml}(V)\colon \mathrm{A} \notin \Delta\}$, $\bar{\Gamma} = \{\mathrm{B} \in \mathrm{Fml}(V^+)\colon \mathrm{B} \notin \Gamma\}$, $\Sigma = \bar{\Delta} \cup \bar{\Gamma}$, $\Delta' = \{\mathrm{A} \in \mathrm{Fml}(V^+)\colon \Delta \vdash_{\mathrm{L}} \mathrm{A}\}$, and Σ' = the closure of Σ under disjunction. By

Lindenbaum's lemma (confined to the case in which the formulas come from V^+), it suffices to show that Δ' and Σ' do not intersect. For then the $\Delta^+ \supseteq \Delta$ that is **L**-prime in V^+ and does not intersect Σ' will satisfy the required conditions.

Since Δ is **L**-prime in V, $\bar{\Delta}$ is already closed under disjunction; and since Γ is **L**-prime in V^+, $\bar{\Gamma}$ is already closed under disjunction. It therefore suffices to show that $\bar{\Delta}$, $\bar{\Gamma}$, and $\Sigma^* = \{\mathrm{A} \vee \mathrm{B}\colon \mathrm{A} \in \bar{\Delta} \text{ and } \mathrm{B} \in \bar{\Gamma}\}$ do not intersect Δ'. Now it is clear, since Δ is an **L**-theory in V, that $\bar{\Delta}$ does not intersect Δ'; and it is clear, since Γ is an **L**-theory in V^+, that $\bar{\Gamma}$ does not intersect Δ'. So this leaves the case ot the set Σ^*.

Suppose, for reductio, that Δ' and Σ^* intersect. Then $\exists \mathrm{A} \in \bar{\Delta}$, $\mathrm{B} \in \bar{\Gamma}$: $\Delta \vdash_{\mathbf{L}} \mathrm{A} \vee \mathrm{B}$. Write B in the form $\mathrm{B}(\mathrm{v}_1, \ldots, \mathrm{v}_n)$, where $\mathrm{v}_1, \ldots, \mathrm{v}_n$ are all the free variables of B not belonging to V. By Lemma 12(i), $\Delta \vdash_{\mathbf{L}} \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n(\mathrm{A} \vee \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n))$. By Classical Distribution, $\Delta \vdash_{\mathbf{L}} \mathrm{A} \vee \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n)$. Since Δ is **L**-prime in V, $\mathrm{A} \in \Delta$ or $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \Delta$. We have already seen that $\mathrm{A} \notin \Delta$. But if $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \Delta$, then it follows, given $\Delta \subseteq \Gamma$, that $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \Gamma$; and so, by Specification, $\mathrm{B}(\mathrm{v}_1, \ldots, \mathrm{v}_n) \in \Gamma$, which is impossible.

(i)(c). Suppose $a \geq t{\downarrow}$. Let $\mathrm{Thm}(a) = \Delta$, $\mathrm{Thm}(t) = \Gamma$, $\mathrm{Var}(a) = V$, and $\mathrm{Var}(t) = V^+$. Then we have: Δ **L**-prime in V, $V \subseteq V^+$, Γ an **L**-theory in V^+, and $\Delta \supseteq \Gamma \cap \mathrm{Fml}(V)$. We want a Δ^+ such that: Δ^+ is **L**-prime in V^+, $\Delta^+ \supseteq \Gamma$, and $\Delta^+ \cap \mathrm{Fml}(V) = \Delta$. For then we can set $a^+ = \langle \Delta^+, V^+ \rangle$.

Since Δ is **L**-prime in V, $\bar{\Delta} = \{\mathrm{A} \in \mathrm{Fml}(V)\colon \mathrm{A} \notin \Delta\}$ is closed under disjunction. Since $\Delta \supseteq \Gamma \cap \mathrm{Fml}(V)$, $\bar{\Delta}$ and Γ do not intersect. So, by Lindenbaum's lemma, $\exists \Delta^+\colon \Delta^+$ is **L**-prime in V^+, $\Delta^+ \supseteq \Gamma$, and Δ^+ does not intersect $\bar{\Delta}$. This Δ^+, then, has the required properties.

(ii)(a). Straightforward.

(iii)(a). We break up the equation into two inclusions. Let $\mathrm{Thm}(t) = \Delta$, $\mathrm{Thm}(u) = \Gamma$, $\mathrm{Var}(t) = \mathrm{Var}(u) = V$, $\mathrm{Thm}(t{\uparrow}) = \Delta^+$, $\mathrm{Thm}(u{\uparrow}) = \Gamma^+$, and $\mathrm{Var}(\Delta^+) = \mathrm{Var}(\Gamma^+) = V^+ \supseteq V$. For the rightward inclusion, we suppose that $\mathrm{B} \in \mathrm{Thm}((tu){\uparrow})$. Then $\exists \mathrm{B}' \in \mathrm{Thm}(tu)\colon \mathrm{B}' \vdash_{\mathbf{L}} \mathrm{B}$. So $\exists \mathrm{A} \in \Gamma\colon \mathrm{A} \rightarrow \mathrm{B}' \in \Delta$. Since $\mathrm{A} \rightarrow \mathrm{B}' \in \Delta$ and $\mathrm{B}' \vdash_{\mathbf{L}} \mathrm{B}$, $\mathrm{A} \rightarrow \mathrm{B} \in \Delta^+$; and so $\mathrm{B} \in \mathrm{Thm}(t{\uparrow}u{\uparrow})$.

For the leftward inclusion, suppose that $\mathrm{B} \in \mathrm{Thm}(t{\uparrow}u{\uparrow})$. Then $\exists \mathrm{A}^+ \in \Gamma^+\colon \mathrm{A}^+ \rightarrow \mathrm{B} \in \Delta^+$. Since $u{\uparrow}$ is an expansion of u, $\exists \mathrm{A} \in \Gamma\colon \mathrm{A} \vdash_{\mathbf{L}} \mathrm{A}^+$; and, since $t{\uparrow}$ is an expansion of t, $\Delta \vdash_{\mathbf{L}} \mathrm{A} \rightarrow \mathrm{B}$. Write B in the form $\mathrm{B}(\mathrm{v}_1, \ldots, \mathrm{v}_n)$, where $\mathrm{v}_1, \ldots, \mathrm{v}_n$ are all the free variables of B not in V. Since $\Delta \vdash_{\mathbf{L}} \mathrm{A} \rightarrow \mathrm{B}(\mathrm{v}_1, \ldots, \mathrm{v}_n)$, it follows by Lemma 12(ii) that $\Delta \vdash_{\mathbf{L}} \mathrm{A} \rightarrow \forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n)$. Since $\mathrm{A} \in \Gamma$, $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_n \mathrm{B}(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathrm{Thm}(tu)$; and so, by Specification, $\mathrm{B}(\mathrm{v}_1, \ldots, \mathrm{v}_n) \in \mathrm{Thm}((tu){\uparrow})$, as required.

(iii)(b). Let us set $\mathrm{Thm}(t{\downarrow}) = \Delta$, $\mathrm{Thm}(u) = \Gamma$, $\mathrm{Var}(t{\downarrow}) = \mathrm{Var}(u) = V$, $\mathrm{Thm}(t) = \Delta^+$, $\mathrm{Thm}(u{\uparrow}) = \Gamma^+$, and $\mathrm{Var}(t) = \mathrm{Var}(u{\uparrow}) = V^+ \supseteq V$. Suppose $\mathrm{B} \in \mathrm{Thm}(tu{\uparrow}){\downarrow}$. Then $\mathrm{B} \in \mathrm{Fml}(V)$ and $\exists \mathrm{A}^+ \in \Gamma^+\colon \mathrm{A}^+ \rightarrow \mathrm{B} \in \Delta^+$. So $\exists \mathrm{A} \in \Gamma\colon \mathrm{A} \vdash_{\mathbf{L}} \mathrm{A}^+$ and $\mathrm{A} \rightarrow \mathrm{B} \in \Delta$. But then $\mathrm{B} \in \mathrm{Thm}(t{\downarrow}u)$, as required.

V. (i)–(iii). Straightforward.

(iv). Suppose that t is v,w-symmetric, for v, w distinct variables of $V = \mathrm{Var}(t)$, and that $t \leq a$. Let $\Delta = \mathrm{Thm}(t)$ and $\Gamma = \mathrm{Thm}(a)$. We wish to show that $\exists\Delta'$: $\Delta \subseteq \Delta' \subseteq \Gamma$ and that Δ' is $\mathbf{L}^{v,w}$-prime in V. For then we can set $a' = \{\Delta', v\}$.

Since Γ is **L**-prime, $\bar{\Gamma} = \{A \in \mathrm{Fml}(V)\colon A \notin \Gamma\}$ is closed under disjunction; and, since t is v,w-symmetric, Δ is an $\mathbf{L}^{v,w}$-theory. But then, by Lindenbaum's lemma, applied now to the quasi-logic $\mathbf{L}^{v,w}$ under the restriction to the formulas of $\mathrm{Fml}(V)$, there is an $\mathbf{L}^{v,w}$-prime theory $\Delta' \supseteq \Delta$ in V that does not intersect Γ.

(v). Use the fact that if $\mathrm{Thm}(\vec{a})$ is an $\mathbf{L}^{v,w}$-theory in V then so is $\mathrm{Thm}(-(\vec{a})) = \{A \in \mathrm{Fml}(V)\colon {\sim}A \notin \mathrm{Thm}(\vec{a})\}$ (see §51.3).

(vi). Take $\rightarrow$ to be v,w$\rightarrow$ for v, $w \in \mathrm{Var}(t)$. First, let us suppose $B \in \mathrm{Thm}(\overrightarrow{tu})$. Then $\exists B' \in \mathrm{Thm}(tu)$: $B' \vdash_{\mathbf{L}^{v,w}} B$. So $\exists A$: $A \in \mathrm{Thm}(u)$ and $A \rightarrow B' \in \mathrm{Thm}(t)$. But then $A \rightarrow B \in \mathrm{Thm}(\vec{t})$ and $B \in \mathrm{Thm}(\vec{t}\vec{u})$, as required.

Now suppose $B \in \mathrm{Thm}(t\vec{u})$. Then $\exists A$: $A \in \mathrm{Thm}(\vec{u})$ and $A \rightarrow B \in \mathrm{Thm}(\vec{t})$. So $\exists A' \in \mathrm{Thm}(u)$: $A' \vdash_{\mathbf{L}^{v,w}} A$. But then $A' \rightarrow B \in \mathrm{Thm}(\vec{t})$ and $B \in \mathrm{Thm}(\vec{t}u)$, as required.

(vii). Again, take $\rightarrow\, =$ v,w$\rightarrow$ for v, $w \in \mathrm{Var}(t)$. Suppose $C \in \mathrm{Thm}(\overrightarrow{t\uparrow})$. Then $\exists B \in \mathrm{Thm}(t\uparrow)$: $B \vdash_{\mathbf{L}^{v,w}} C$. So $\exists A \in \mathrm{Thm}(t)$: $A \vdash_{\mathbf{L}} B$. From $A \vdash_{\mathbf{L}} B$ and $B \vdash_{\mathbf{L}^{v,w}} C$ it follows that $A \vdash_{\mathbf{L}^{v,w}} C$. By Lemma 15, $\vdash_{\mathbf{L}} A{}^{v}\!/_{w} \rightarrow C{}^{v}\!/_{w}$. Write C in the form $C(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are all the variables of C not in $\mathrm{Var}(t)$. By Lemma 12(i) and the Deduction theorem, $\vdash_{\mathbf{L}} A{}^{v}\!/_{w} \rightarrow \forall x_1 \ldots \forall x_n C(x_1, \ldots, x_n){}^{v}\!/_{w}$. So, by Lemma 15 again, $A \vdash_{\mathbf{L}^{v,w}} \forall x_1 \ldots \forall x_n C(x_1, \ldots, x_n)$. But then $\forall x_1 \ldots \forall x_n C(x_1, \ldots, x_n) \in \mathrm{Thm}(\vec{t})$; and so $C(w_1, \ldots, w_n) \in \mathrm{Thm}(\vec{t}\uparrow)$, as required.

The other direction is straightforward.

(viii). Let $\mathrm{Var}(t) = V$ and $\mathrm{Var}(t\uparrow) = V^+$; and take $\rightarrow\, =$ v,w$\rightarrow$, where $v \in (V^+ - V)$ and $w \in V$. Suppose $B \in \mathrm{Thm}(t\uparrow\downarrow)$. Then $\exists A \in \mathrm{Thm}(t)$: $A \vdash_{\mathbf{L}^{v,w}} B$. Since v does not occur in either A or B, it follows from Lemma 15 that $A \vdash_{\mathbf{L}} B$. But then $B \in \mathrm{Thm}(t)$, as required.

(ix). By definition.

The Truth lemma for the canonical model takes the following form:

LEMMA 17. For **L** a logic, t a point of the canonical model $\mathfrak{A}_{\mathbf{L}}$, and A a formula with variables from $\mathrm{Var}(t)$,

$t \models A$ iff $A \in \mathrm{Thm}(t)$.

PROOF. By induction on the construction of the formula A. The cases in which A is atomic or of the form ${\sim}B$, $B \wedge C$, or $B \rightarrow C$ are much the same as for propositional relevance logic (§51.3). This leaves the case in which A is of the form $\forall x B(x)$.

Suppose first that $t \vDash \forall x B(x)$. Then $\exists t{\uparrow}\exists v \in (D_{t\uparrow} - D_t)$: $t{\uparrow} \vDash B(v)$. By IH, $B(v) \in \mathrm{Thm}(t{\uparrow})$. By definition of $\mathrm{Thm}(t{\uparrow})$, $\mathrm{Thm}(t) \vdash_{\mathbf{L}} B(v)$. By Lemma 12(i), $\mathrm{Thm}(t) \vdash_{\mathbf{L}} \forall x B(x)$; and so $\forall x B(x) \in \mathrm{Thm}(t)$.

Now suppose that $\forall x B(x) \in \mathrm{Thm}(t)$. Choose a variable $v \notin \mathrm{Var}(t)$. Let $t{\uparrow}$ be the expansion of t to $\mathrm{Var}(t) \cup \{v\}$. By Specification, $B(v) \in \mathrm{Thm}(t{\uparrow})$. By IH, $t{\uparrow} \vDash B(v)$; and so $t \vDash \forall x B(x)$.

Completeness for $\mathbf{B}^{\forall\exists x}$ can now be established.

THEOREM 18. For any formula A,

$$A \in \mathbf{B}^{\forall\exists x} \text{ iff } \vDash A.$$

PROOF. The direction $\Rightarrow$ (Soundness) follows from Theorem 13.

For the other direction, suppose that $A \notin \mathbf{B}^{\forall\exists x}$. By the Truth lemma 17, $(\mathfrak{A}_{\mathbf{L}}, l) \nvDash A$ for appropriate l. But then, by Lemma 16, A is not true in all models and, therefore, is not valid.

Compactness and the Skolem-Löwenheim theorem can be extracted from the proofs in the usual way. These results can also be extended to a wide range of further logics. Let a *frame* $\mathfrak{F}$ be a model $\mathfrak{A}$ shorn of its valuation; the model $\mathfrak{A}$ is then said to be *based upon* the frame $\mathfrak{F}$. Say that a formula A is *valid in* the frame $\mathfrak{F}$ (we write $\mathfrak{F} \vDash A$) if it is true in all models based upon $\mathfrak{F}$. Given a class of frames X, say that the logic $\mathbf{L}$ is *sound for* X if $(\forall A)[A \in \mathbf{L} \Rightarrow (\forall \mathfrak{F} \in X)(\mathfrak{F} \vDash A)]$, that $\mathbf{L}$ is *sufficient for* X if $(\forall A)[(\forall \mathfrak{F} \in X)(\mathfrak{F} \vDash A) \Rightarrow A \in \mathbf{L}]$, and that $\mathbf{L}$ is *complete for* X if it is both sound and sufficient for X.

Consider the following list of postulates and conditions (taken from §51.4):

	Postulate	*Condition*
1	$(A \to B) \wedge (B \to C) \to (A \to C)$	$t(tu) \leq tu$
2	$(A \to B) \to ((B \to C) \to (A \to C))$	$t(uv) \leq (ut)v$
3	$(B \to C) \to ((A \to B) \to (A \to C))$	$t(uv) \leq (tu)v$
4	$(A \to {\sim}B) \to (B \to {\sim}A)$	$a \geq tb \Rightarrow -b \geq t - a$
5	$A \wedge (A \to B) \to B$	$tt \leq t$
6	$(A \to B) \to .\ {\sim}A \vee B$	$a - a \leq a$
7	$(A \to (B \to C)) \to (A \wedge B \to C)$	$(tu)u \leq tu$
8	$A\ /\ (A \to B) \to B$	$tl \leq t$
9	$A \to ((A \to B) \to B)$	$tu \leq ut$

THEOREM 19. Let $\mathbf{L}$ be the logic obtained by adding any of the postulates in the above list to $\mathbf{B}^{\forall\exists x}$, and let X be the corresponding class of frames. Then $\mathbf{L}$ is complete for X.

PROOF. It can be shown, much as in §51.4, that $\mathbf{L}$ is sound for X and that the canonical frame $\mathfrak{F}_{\mathbf{L}} \in X$. It follows, in particular, that the logics $\mathbf{R}^{\forall\exists x}$ and $\mathbf{E}^{\forall\exists x}$ (see §38) are complete for the corresponding classes of frames.

The above method of proof generalizes. Say that a propositional logic **L** is *supercanonical* if, for any quasi-logic $\mathbf{L}' \supseteq \mathbf{L}$, the canonical frame $\mathfrak{F}_{\mathbf{L}'}$ (as defined in §51.3) is a frame for **L**. (A logic **L** is canonical if $\mathfrak{F}_{\mathbf{L}}$ is a frame for **L**. I do not know of any logics that are canonical without being supercanonical.)

THEOREM 20. Suppose **L** is a propositional relevance logic that is supercanonical. Then the corresponding quantificational relevance logic $\mathbf{L}^{\forall\exists x}$ is complete.

PROOF. $\mathbf{L}^{\forall\exists x}$ is complete for its canonical frame $\mathfrak{F}_{\mathbf{L}^{\forall\exists x}}$; for, by identifying atomic and universal formulas with sentence letters, we can see that each "stratum" of the canonical frame is isomorphic to the canonical frame $\mathfrak{F}_{\mathbf{L}'}$ for some quasi-logic $\mathbf{L}' \supseteq \mathbf{L}$.

§54. $\mathbf{KR}_{\rightarrow\&}$: A conjunction-arrow fragment corrupted by Boolean structure. **KR**, developed by A. Abraham and R. Meyer and R. Routley, is described in §65.1.2 as the result of adding to **R** (as in §R2) a postulate sufficient to add a Boolean twist to its previously straight-as-a-cue relevant negation:

$$A\&{\sim}A \rightarrow B.$$

The reader of these volumes will not require of us a philosophical justification for such a disagreeable postulate, and we do not propose to provide one; instead, we wish only to make some specialized remarks of slight but not void philosophical interest.

It is reported in §65.1.2 that one obtains an appropriate semantics for **KR** in the style of §48.5 by postulating that * is an identity operator, $x^* = x$, and that accordingly **KR** is not just another formulation for two-valued logic. Taking the star operator as identity has the obvious consequence of making the three-termed relation R six ways symmetric. We cannot find an intuitive path from the informal readings of R (say, in terms of relative commitment as in §51.5) to this symmetry, but the fact remains that there is no plunge into two-valued logic where one might have been expected. The calculus **KR** plays a central role in the undecidability inquiries of Urquhart in §65, which is, perhaps, its chief technical importance, whereas its chief importance in the logical dimension is doubtless the mere datum of no collapse. Intuitive connections to projective geometry are made manifest in §65.

We can, however, add just a little more revealing information. One might have thought that the absence of degeneration into two-valued logic signaled that the aforementioned Boolean sporting with negation possessed only isolated consequences and, in particular, that the *positive*, or negation-free fragment of **KR** did not itself outrun the healthful relevance principles of **R**. What we highlight in this section is how false such a thought would have been, and indeed one has no need of disjunction to reveal the decay: the Boolean infection curses even the arrow-conjunction fragment of **KR**, as can be seen by examining the peculiar postulates KR1 and KR2 below, with which we axiomatize that fragment.

§54.1. Axioms for $\mathbf{KR}_{\to\&\circ t}$ and their consistency. Although we are philosophically more interested in the arrow-conjunction fragment $\mathbf{KR}_{\to\&}$ of **KR**, it is mentally and visually easier to process postulates stated with the help of both the fusion operation, $\circ$, and the constant $\boldsymbol{t}$. Accordingly, even though references to $\circ$ and to $\boldsymbol{t}$ could have been avoided at the cost of several arrows, we shall state our technical result for the arrow-conjunction-fusion-$\boldsymbol{t}$ fragment of **KR**, which we call $\mathbf{KR}_{\to\&\circ t}$. We put up with the long name because of its welcome assistance to an overloaded memory when it is important to keep in mind just which connectives are licensed, and we henceforth take for granted that $\mathbf{KR}_{\to\&\circ t}$ is a conservative extension of $\mathbf{KR}_{\to\&}$. Just to keep things in perspective, we note from §65.2.5 that $\mathbf{KR}_{\to\&}$ and, accordingly, $\mathbf{KR}_{\to\&\circ t}$ are undecidable.

We take over the semantics for $\mathbf{KR}_{\to\&\circ t}$ from §48.3, adding to the list of conditions there stated only the one new entry:

6. (Commutation in the second two places) If Rabc then Racb.

Evidently this acts with the other conditions of §48.3 to yield six-ways symmetry. We use "$\mathbf{KR}_{\to\&\circ t}$-frame" and "$\mathbf{KR}_{\to\&\circ t}$-model" in analogy to the concepts of $\mathbf{R}_{\to}$-frame and -model of §48.3.

We state two additional axioms KR1 and KR2—that is, axioms to be added to the **R**-family axioms and rules for these connectives as given in §R2—for $\mathbf{KR}_{\to\&\circ t}$.

KR1. $[A\&\boldsymbol{t}\&(((B\to C)\&B)\to E)\to F]\&[A\&\boldsymbol{t}\&C\to F] \to.\ A\&\boldsymbol{t}\to F$.

The "$\boldsymbol{t}$" here can be replaced, just as in §45.1, by a conjunction of identity axioms in some of the local variables. Semantic verification of this axiom depends only on Rxx0, to show that at least one of $0\models((B\to C)\&B)\to E$ and $0\models C$, and hence that at least one of those formulas is true wherever $\boldsymbol{t}$ is true.

The postulate KR1 may be considered the counterpart in $\mathbf{KR}_{\to\&\circ t\vee}$ of the following, which employs a connective, disjunction, not present in the vocabulary of $\mathbf{KR}_{\to\&\circ t}$:

KR1′. $(A\circ A)\vee(A\to B)$.

It is perhaps preeminently the presence of this postulate in $\mathbf{KR}_{\to\&\circ t\vee}$ that exhibits the peculiarly widespread consequences of making Boolean assumptions about negation, for it is easy to see that KR1′ is a cousin too close to kiss of the two-valued oddity, $A\vee(A\supset B)$, and of the modal curiosity, $\Diamond A\vee(A ⥽ B)$.

The second and last additional postulate for $\mathbf{KR}_{\to\&\circ t}$ is

KR2. $A\&(B\circ C\circ D) \to.\ [(A\circ C)\&(B\circ D)]\circ C$.

Semantic verification of KR2 uses both Pasch and commutation of §48.3, as follows. Suppose that the antecedent of KR2 is true at a, so that $A \vDash$ a and also, for some x, Rxda, Rbcx, d $\vDash D$, b $\vDash B$, and c $\vDash C$. By Pasch and commutation, for some y, Racy and Rbdy; whence y $\vDash (A\circ C)\&(B\circ D)$. But also Ryca, by six-ways symmetry; so the consequent is true at a, as required.

A simpler Pasch-free substitute for KR2 is this, where "X" more or less plays the role of "$B\circ D$": $(A\&(C\circ X))\to.[(A\circ C)\&X]\circ C$. This substitute for KR2 is formally a form of "modularity" (a limited or conditional version of distributivity) if we write conjunction as a lattice meet and fusion as a lattice join:

$$\mathrm{A(C\vee X) \le [(A\vee C)X]\vee C.}$$

The reader should consult §§65.1.3–4 to see how and why this observation not only makes sense but is of some interest.

§54.2. Completeness. This section shows that $\mathbf{KR}_{\to\&\circ t}$ is indeed the appropriate fragment of **KR** by proving that $\mathbf{KR}_{\to\&\circ t}$ is complete with respect to the semantics conferred by the addition of (6) above; that is to say, KR1 and KR2 are enough postulates.

The strategy consists in building a canonical $\mathbf{KR}_{\to\&\circ t}$-model out of $\mathbf{KR}_{\to\&\circ t}$-theories, where by a T-theory (for T a set of formulas) we mean a set closed under adjunction and *modus ponens*—for implications in T. This is the strategy common to §48.3 and §51, and our exposition, which will be more in the style of §48.3, will assume familiarity with those arguments.

Let F be some nontheorem of $\mathbf{KR}_{\to\&\circ t}$. Using "0" as in §48.3, let 0 be a maximal F-free $\mathbf{KR}_{\to\&\circ t}$-containing $\mathbf{KR}_{\to\&\circ t}$-theory (it is unexpected that even for $\mathbf{KR}_{\to\&\circ t}$—no disjunction—we seem to require a *maximal* 0).

Let K be defined as the set of all nontrivial 0-theories. "Nontrivial" means: neither empty nor universal.

We need (for convenience) two canonical three-place relations. Let R′ be defined on K as in §48.3:

R′abc iff for all A, B, if $A\to B\in$ a and $A\in$b then $B\in$c.

The equivalent "fusion" version would make it that R′abc iff, for every A, B, if $A\in$a and $B\in$b, then $(A\circ B)\in$ c.

Then let R* build in 3/6 of six-ways symmetry:

R*abc iff R′abc and R′acb and R′cba.

The rest of the six ways are guaranteed by commutativity in the first two places, which comes from the theorem of **R**, $A\to.A\to B\to B$.

The "canonical $\mathbf{KR}_{\to\&\circ t}$-frame" is then $\langle$K, R*, 0$\rangle$, and the "canonical $\mathbf{KR}_{\to\&\circ t}$-model" comes canonically by the identification of truth and membership,

a $\vDash A$ iff $A\in$a.

Since obviously the nontheorem *F* does not belong to 0, what we need to show is that the canonical model is in fact a $\mathbf{KR}_{\rightarrow\&\circ t}$-model.

FACT 1. R* is six ways symmetric.

FACT 2. If R*0ab then a = b.

We verify the conditions 1–5 on an **R**-frame as listed in §48.3, plus 6 from above, §54.1; and the Atomic Hereditary condition and Valuation clauses of §48.3.

1. *Identity.* We need to show R*0aa, that is (1) R′0aa, (2) R′0aa (again), and (3) R′aa0. The first two come as in §48.3, solely because a is a 0-theory. The third comes from KR1, as we will show in a moment, but the argument would be simpler if we had disjunction and KR1′, as follows. 0 for the construction based on $\mathbf{KR}_{\rightarrow\&\circ t\vee}$ would certainly be a prime extension of $\mathbf{KR}_{\rightarrow\&\circ t\vee}$. Now think of the fusion version of the canonical definition of R′, and suppose $A\in$a. Since a is nontrival, one cannot have $A\rightarrow B$ in 0 for every *B*, and so, given KR1′, $A\circ A$ must be in 0; which is just what is required for R′aa0, fusion version.

The argument with KR1 is only more tortuous than that for KR1′, relying on the *maximal* *F*-freeness of 0. Suppose $B\rightarrow C$ and *B* are both in a; for R′aa0 we need *C* in 0. If it is not, then $(A\&t\&C)\rightarrow F$ is provable in $\mathbf{KR}_{\rightarrow\&\circ t}$, for some *A* in 0, by maximality. Also because a is nontrivial, we cannot have $X = ((B\rightarrow C)\&B)\rightarrow E$ in 0 for *every E*; so choose an *E* where *X* is not in 0. By maximality, for suitable *A* (we might as well say, the same *A*), $(A\&t\&X)\rightarrow F$ is in 0. But then KR1 would put $(A\&t)\rightarrow F$ in 0, which cannot be.

2. *Commutation* (in 1–2 places). We need to show that R*abc gives R*bac. Use Fact 1.

3′. *Pasch.* We verify this instead of the equivalent (3) Associativity. Suppose R*abx and R*xcd. We need to find a y such that R*acy and R*ybd. Given a and b, let a&b as an operation on theories be the obvious one: the smallest 0-theory containing *A*&*B* for each $A\in$a and $B\in$b; and let a∘b be the smallest 0-theory, c, such that R′abc. The desired y is then given by

$$y = (a\circ c)\&(b\circ d).$$

We need to show that this y has the right R* relations to a, b, c, d. (These arguments will also show that y is nontrivial.)

The premisses give us six facts (up to commutations of ∘):

1. $a\circ b \subseteq x$
2. $x\circ a \subseteq b$
3. $x\circ b \subseteq a$
4. $c\circ d \subseteq x$
5. $x\circ c \subseteq d$
6. $x\circ d \subseteq c$.

We need to establish the following six:

1. $a \circ c \subseteq y$. Obvious from definition of y.
2. $b \circ d \subseteq y$. Ditto.
3. $y \circ c \subseteq a$. Suppose A, B, C, C', D belong to their alphabetically mated theories; we must show that $((A \circ C)\&(B \circ D) \circ C' \in a$. Choose $C'' = (C\&C') \in c$. By 3 and 4 of the premisses above, we know that $B \circ C'' \circ D \in a$, and so that $A\&(B \circ C'' \circ D) \in a$ as well. KR2 clearly suffices to put $((A \circ C'')\&(B \circ D)) \circ C''$ and hence $((A \circ C)\&(B \circ D)) \circ C'$ in a, as required.
4. $y \circ a \subseteq c$. Symmetrical to 3.
5. $y \circ b \subseteq d$. Ditto.
6. $y \circ d \subseteq b$. Ditto.

4. *Idempotence.* R^*aaa comes by definition from $R'aaa$. Observe that six ways symmetry verifies $B \rightarrow . B \circ B \circ B$ *without* idempotence.

5. *Monotony.* We need that R^*0aa' and $R^*a'bc$ yield R^*abc. By Fact 2 above, the first premiss gives that $a = a'$.

6. *Commutation* (in 2–3 places). Use Fact 1.

Atomic Hereditary condition. Use Fact 2.

Valuation clauses (for connectives). Given the axioms and rules of §R2, the clauses for & and $\boldsymbol{t}$ are trivial, and the clause for $\circ$ is easily derived from that for $\rightarrow$, to which we now attend.

Since R^*abc implies $R'abc$, it is obvious that if $(A \rightarrow B) \in a$ and if $A \in b$ then $B \in c$.

Suppose now that $A \rightarrow B \notin a$, and choose $X \in a$, given that a is nontrival; we need to find b and c such that R^*abc, $A \in b$, and $B \notin c$. Let b be the principal 0-theory generated by A; this will be nontrivial, for if $A \rightarrow . X \rightarrow B$ were in 0, so would be $X \rightarrow . A \rightarrow B$, forcing $(A \rightarrow B) \in a$. Let c be $a \circ b$. (Clearly $a \circ b$ will contain $X \circ A$ if fusion is present, and at least $(X \rightarrow . A \rightarrow Y) \rightarrow Y$—for all Y—even if fusion is not.)

Obviously $A \in b$ and $B \notin c = a \circ b$. We need to be sure that R^*abc; i.e., in addition to the already secured $R'ab(a \circ b)$, we need $R'a(a \circ b)b$ and $R'(a \circ b)ba$. Thinking of the fusion definition of R', these would be secured by

$$Y \rightarrow . X \circ (X \circ Y) \text{ and}$$
$$X \rightarrow . (X \circ Y) \circ Y,$$

each being in 0 for every $X \in a$ and $Y \in b$. We have already established that $R'aa0$ and $R'bb0$, and so that $X \circ X$ and $Y \circ Y$ are in 0; hence, quite in general, $Z \rightarrow . X \circ X \circ Z$ and $Z \rightarrow . Y \circ Y \circ Z$ are in 0, and so fooling around with commutativity and associativity gives what is required: $R^*ab(a \circ b)$.

This completes the proof.

PROBLEM. $\mathbf{KR}_{\rightarrow\&\circ t\vee}$ is certainly undecidable, by §65, and certainly recursively enumerable as a subsystem of **KR**; so we ask, Is there a finite axiomatization of $\mathbf{KR}_{\rightarrow\&\circ t\vee}$ with *modus ponens* and adjunction as its only rules?

CHAPTER X

PROOF THEORY AND DECIDABILITY

§60. **Relevant analytic tableaux** (with Michael A. McRobbie). "*Logic is at root all about trees.*"

In this section we present a tableau-style analysis of the implication-negation fragments of the principal relevance logics **E**, **R**, and **RM**, as well as the classical propositional calculus **TV**. This analysis, although similar in certain respects to that given by Smullyan 1968 for **TV**, differs substantially in that it is purely *proof-theoretic* in character, as opposed to that of Smullyan, which is *semantical*. Henceforth the kinds of structures studied in a *semantically* based tableau-style analysis will be called *analytic semantic tableaux*, while the kinds of structures studied in a *proof-theoretically* based tableau-style analysis will be called simply *analytic tableaux*. Further, logical systems in which the theoremhood {validity} of formulas is determined by whether or not analytic tableaux {analytic semantic tableaux} constructed in a prescribed way from these formulas meet certain necessary and sufficient conditions will be called *analytic tableau systems* {*analytic semantic tableau systems*}. Hence the *analytic tableau formulation* {*analytic semantic tableau formulation*} of some arbitary logic **S** is simply an analytic tableau system {analytic semantic tableau system} that is provably equivalent to **S**. In what follows, by "tableau" we shall be referring to the proof-theoretic species and not to the semantic unless otherwise indicated.

The calculuses **E**, **R**, and **RM** are defined in §27 and §R1, and their implication-negation fragments are defined in §14. The calculus $\mathbf{TV}_{\overset{\sim}{\to}}$ is the implication-negation fragment of the two-valued calculus **TV**. For a discussion of how these results fit into the context of other tableau investigations, see note 2 of McRobbie and Belnap 1979, from which this section was taken. For an example of a semantic tableau system for the first degree entailments, see §50.

§60.1. **The tableau systems.** Here we present the definitions of the various structures, and the concepts defined on these structures, which we shall be using. These definitions are essentially based on, and are extensions of, those given in Smullyan 1968 and Toledo 1975.

A *tableau* τ is a finite *tree* to whose *nodes* formulas have been assigned. Some of the nodes may also be *annotated* in a sense to be specified. There may be *barriers* between certain adjacent nodes. We use $<_0$, $<$, and $\leq$ respectively, for *immediate predecessor*, *predecessor*, and *predecessor-identity*.

We let F be the function that assigns formulas to nodes, so that $F(i)$ is the formula assigned to node i. An *end node* is one that is not the predecessor of any node. A *branch* of a tableau is a sequence of nodes commencing with the *origin* such that each node that is not an end node of the tableau has as its immediate successor in the branch one of its immediate successors in the tableau.

With each logical system S subject to our investigation we will be associating a tableau system TS. TS will be defined by (a) listing some *rules* that TS admits in the construction of its tableaux, and (b) listing some *global requirements* that the tableaux of TS must satisfy. We pause to characterize the two sorts of rules with which we shall be dealing. It is important to note that the prepositions "*at*" and "*to*" when italicized are being used technically and that special attention needs to be paid when they are used in this way.

A *connective rule* is a rule applied *at* a given node n in some tableau *to* some node $m \leq n$, and, in a derivative sense, *to* the formula (of a prescribed form) assigned to m. The application of a connective rule *to* this formula generates (1) less complex formulas of a prescribed kind and (2) successor nodes to n. The generated formulas are assigned to the generated nodes in accordance with some prescribed pattern. The node n is annotated with "$R(m)$," where "R" stands in for the name of the connective rule applied *at* n. (We note that ordinarily (e.g., in Jeffrey 1967) it is the node{s} generated by application of a rule that is {are} annotated, but it will become clear why it is important for us to annotate the node *at* which the rule is applied rather than either the generated node{s} or the node *to* which the rule is applied.) A connective rule may also generate a *barrier* between n and its successor node{s}; as is explained below, in certain systems only certain rules are allowed to "cross a barrier."

A *branch closure rule* is a rule applied *at* a given end node n in some tableau *to* each member of a sequence of nodes $m_1, \ldots, m_k \leq n$, and derivatively *to* each member of a sequence of formulas (each one of which is of a prescribed form) assigned to the nodes $m_1, \ldots, m_k$. There are no new nodes or formulas generated by an application of a branch closure rule *at* n. The node n is annotated with "$R(m_1, \ldots, m_k)$," where "R" stands in for the name of the particular branch closure rule applied *at* n.

Recall that each TS has associated with it both a family of rules and some global requirements. The former permits us to define, abstractly and inductively, a TS-*tableau*: a finite unannotated chain of nodes to which formulas have been assigned is a TS-tableau, and if n is an end node of a TS-tableau *at* which no rule has been applied, then a new tableau resulting from the application *at* n of one of the rules admitted by TS is also a TS-tableau.

Note that, although a formula may have a rule applied *to* it more than once, no more than one rule can be applied *at* a node. Hence, if the end node of some branch of a tableau has a branch closure rule applied *at* it, then no further construction can take place in that branch. Such a branch is said to be *closed.*

For each TS-tableau τ there is a uniquely determined sequence of nodes commencing with the origin of τ and continuing to (and including) the first annotated node of τ—or continuing to the end of τ if no node is annotated. We say that τ *begins with* those nodes and also that τ *begins with* the sequence α of formulas assigned to those nodes. When thinking of building tableaux, we will also speak of *beginning* a tableau with a sequence of formulas α; i.e., constructing a sequence of nodes to which the members of α are assigned (in order).

Global requirements are brought in to characterize those tableaux which really do some work for us. We say that τ is a TS-*refutation of* a sequence of formulas α if (1) τ is a TS-tableau, (2) τ begins with α, and (3) τ satisfies the global requirements of TS. In these circumstances we say that τ is a TS-refuting tableau, and that α is TS-*refutable.* We write

$$\alpha \vdash_{\mathrm{TS}}$$

when α is TS-refutable.

Note that the difference between TS-refuting tableaux and (mere) TS-tableaux is satisfaction of the global requirements.

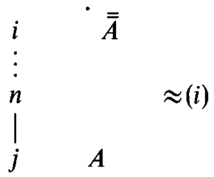
Naturally, refutability is the dual of provability; so we say that A is a theorem of TS just in case $\bar{A} \vdash_{\mathrm{TS}}$, and we write $\vdash_{\mathrm{TS}} A$ for theoremhood.

In order to flesh out this abstract characterization, we next give the rules and the global requirements on which we shall be drawing. In our pictures, a solid [dotted] line from i down to j indicates $i <_0 j$ [$i \leq j$]—i.e., our trees grow downwards.

Connective rules

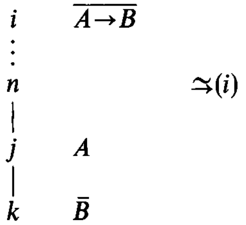
Double Negation ($\approx$)

Let $\bar{\bar{A}}$ and A be assigned to i and j, respectively, and let n be annotated with $\approx(i)$, where $i \leq n <_0 j$. Then we say that $\approx$ has been applied *at n to* $\bar{\bar{A}}$ at i, generating A at j.



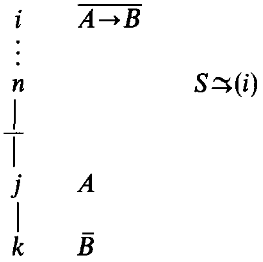
Negated Arrow ($\overline{\rightarrow}$)

Let $\overline{A\rightarrow B}$, *A and* $\bar{B}$ be assigned to i, j, and k, respectively, and let n be annotated with $\overline{\rightarrow}(i)$, where $i \leq n <_0 j <_0 k$. Then we say that $\overline{\rightarrow}$ is applied *at n to* $\overline{A\rightarrow B}$ at i, generating A at j and $\bar{B}$ at k.


i $\overline{A\rightarrow B}$
n $\overline{\rightarrow}(i)$
j A
k $\bar{B}$

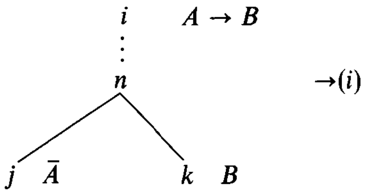
Strict Negated Arrow (S$\overline{\rightarrow}$)

This rule is exactly like $\overline{\rightarrow}$, except that a barrier is generated between the node n *at* which the rule is applied and its immediate successor node. We may indicate barriers by horizontal lines. Hence an application of the rule S$\overline{\rightarrow}$ has the following picture:


i $\overline{A\rightarrow B}$
n $S\overline{\rightarrow}(i)$
j A
k $\bar{B}$

Arrow ($\rightarrow$)

Let $A \rightarrow B$, $\bar{A}$, and B be assigned to i, j, and k, respectively, and let n be annotated with $\rightarrow(i)$, where $i \leq n <_0 j$ and $i \leq n <_0 k$ and $j \neq k$. Then we say that $\rightarrow$ is applied *at n to* $A\rightarrow B$ at i, generating $\bar{A}$ at j and B at k.


i $A \rightarrow B$
n $\rightarrow(i)$
j $\bar{A}$
k B

Branch closure rules

Closure (Cl)

Let A and $\bar{A}$ be assigned to i and j, respectively, and let n be annotated with $\mathrm{Cl}(i, j)$, where $i, j \leq n$. Then we say that Cl is applied *at* n *to* A at i and $\bar{A}$ at j.

$$\begin{array}{llllll} i & A & & j & \bar{A} & \\ \vdots & & & \vdots & & \\ j & \bar{A} & & i & A & \\ \vdots & & & \vdots & & \\ n & & \mathrm{Cl}(i, j) & n & & \mathrm{Cl}(i, j) \end{array}$$

Mingle Closure (MCL)

Let $A_1, \ldots, A_m, \bar{A}_1, \ldots, \bar{A}_m$ be assigned to (not necessarily distinct) nodes $i_1, \ldots, i_m, i_{m+1}, \ldots, i_{2m}$ $(m \geq 1)$ respectively, and let n be annotated with $\mathrm{MCl}(i_1, \ldots, i_{2m})$, where, for any j between 1 and $2m$, $i_j \leq n$. Then we say that MCl is applied *at* n *to* $A_1, \ldots, \bar{A}_m$ *at* $i_1, \ldots, i_{2m}$, respectively.

The following illustrates one possibility:

$$\begin{array}{lll} i_1 & A & \\ \vdots & & \\ i_2 & \bar{B} & \\ \vdots & & \\ i_3 & \bar{A} & \\ \vdots & & \\ i_4 & B & \\ \vdots & & \\ i_5 & A & \\ \vdots & & \\ n & & \mathrm{MCl}(i_1, i_4, i_5, i_3, i_2, i_3) \end{array}$$

Next we list the global requirements on which we shall draw. (Not all will apply to all systems.)

Global requirements

Closure requirement. Every branch of τ is closed, i.e., has a branch closure rule applied *at* its end node.

Use requirement. If a formula at some node in τ has a rule applied *to* it, then both the formula and the node will be said to be *used*. The requirement is that *each* node in τ (and hence the formula assigned to it) must be used at least once.

This is the requirement with which we catch the concept of relevance in a tableau. If τ satisfies the Use requirement, then it has no inessential ingredients, no loose pieces, no irrelevant or extraneous bits.

Barrier requirement. A TS-tableau τ satisfies the Barrier requirement iff the only rule that *crosses a barrier* is $\rightarrow$. That is, if there is a barrier between j and k and if any rule is applied *at k to j*, then the rule must be $\rightarrow$.

This requirement turns out to be modal in character, answering to the necessitive character of entailments. The concept of a barrier was suggested by Meyer as a simplification of a more complex earlier device. By varying the conditions on what rules can cross barriers and certain other conditions, it is possible to provide analytic tableau formulations of many strict implicational calculuses and of the same calculuses alternatively formulated using $\Box$ and $\Diamond$. See McRobbie 197+a.

Finally we may define the four tableau systems by stating for each (a) which rules it admits and (b) which global requirements it imposes on its tableaux. This may be summed up in the following table:

	Rules	Global requirements
$\mathbf{TTV}_{\overset{\sim}{\rightarrow}}$	$\approx$, $\rightarrow$, $\overset{\sim}{\rightarrow}$, Cl	Closure
$\mathbf{TRM}_{\overset{\sim}{\rightarrow}}$	$\approx$, $\rightarrow$, $\overset{\sim}{\rightarrow}$, MCl	Closure, Use
$\mathbf{TR}_{\overset{\sim}{\rightarrow}}$	$\approx$, $\rightarrow$, $\overset{\sim}{\rightarrow}$, Cl	Closure, Use
$\mathbf{TE}_{\overset{\sim}{\rightarrow}}$	$\approx$, $\rightarrow$, S$\overset{\sim}{\rightarrow}$, Cl	Closure, Use, Barrier

We illustrate as follows:

EXAMPLE 1. The following is available in TTV.

1 $\overline{A\rightarrow.B\rightarrow A}$ $\overset{\sim}{\rightarrow}(1)$
|
2 A
|
3 $\overline{B\rightarrow A}$ $\overset{\sim}{\rightarrow}(3)$
|
4 B
|
5 $\bar{A}$ Cl(2, 5)

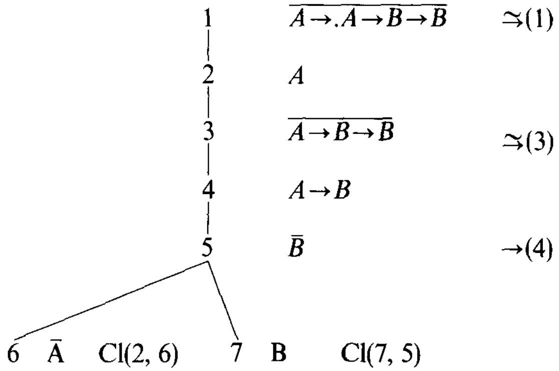
Observe that this tableau does *not* satisfy the Use requirement: no rule is ever applied *to* 4, as a quick inspection of the nodes mentioned in the annotations reveals.

Example 2. The following tableau is available in **TRM**—or, with a different annotation, in **TTV**.

1	$\overline{A \to . A \to A}$	$\overset{\sim}{\to}(1)$
2	A	
3	$\overline{A \to A}$	$\overset{\sim}{\to}(3)$
4	A	
5	$\bar{A}$	MCl(2, 4, 5, 5)

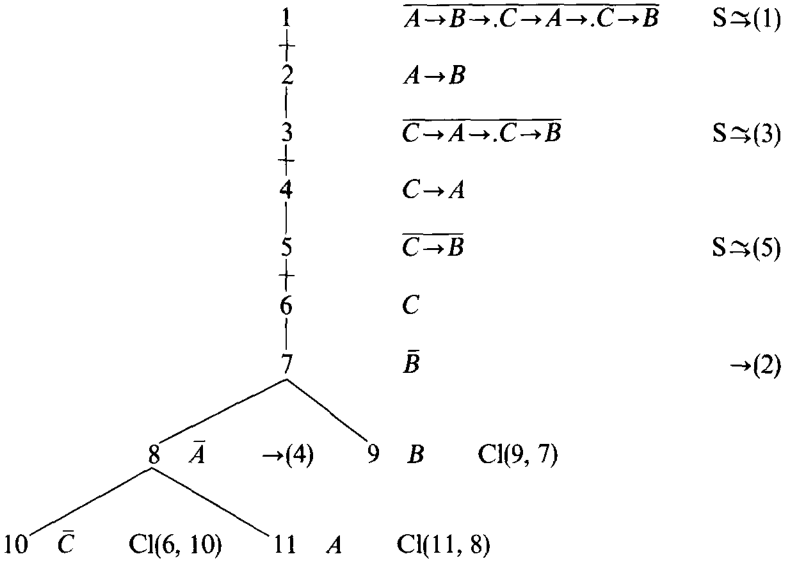
Observe that, although this tableau satisfies the Use requirement, as it must for **TRM**, it does so by using the rule MCl (allowed in **TRM**) instead of plain Cl.

Example 3. The following tableau is available also in **TR**:


1 $\overline{A \to . A \to B \to B}$ $\overset{\sim}{\to}(1)$
2 A
3 $\overline{A \to B \to B}$ $\overset{\sim}{\to}(3)$
4 $A \to B$
5 $\bar{B}$ $\to(4)$
6 $\bar{A}$ Cl(2, 6)
7 B Cl(7, 5)

Observe that the Use requirement is satisfied. On the other hand, note that, if S$\overset{\sim}{\to}$ had been used to create a barrier between 3 and 4, then the application of Cl at 6 would have violated the Barrier requirement—so even if its annotation were changed, this tableau would not be available in **TE**.

Example 4. The following tableau is available in **TE**, and, with alternative annotations, in all four systems.


1 $\overline{A\to B\to.C\to A\to.C\to B}$ $\mathrm{S}\overset{\sim}{\to}(1)$
2 $A\to B$
3 $\overline{C\to A\to.C\to B}$ $\mathrm{S}\overset{\sim}{\to}(3)$
4 $C\to A$
5 $\overline{C\to B}$ $\mathrm{S}\overset{\sim}{\to}(5)$
6 C
7 $\bar{B}$ $\to(2)$
8 $\bar{A}$ $\to(4)$
9 B Cl(9, 7)
10 $\bar{C}$ Cl(6, 10)
11 A Cl(11, 8)

Note how the uses of → *at* 7 and 8 cross barriers, but that no other rules do so; hence, the Barrier requirement of TE is satisfied. One can further read off from the annotations that every node is used; i.e., every node has a rule applied *to* it.

§60.2. Equivalence via left-handed consecution calculuses. Each of the four tableau systems we have defined is equivalent in the appropriate sense to its corresponding Hilbert system:

Theorem 1. Let S be $\mathbf{E}_{\overset{\sim}{\to}}$, $\mathbf{R}_{\overset{\sim}{\to}}$, $\mathbf{RM}_{\overset{\sim}{\to}}$ or $\mathbf{TV}_{\overset{\sim}{\to}}$. For all formulas A, $\vdash_{\mathrm{TS}} A$ *iff* $\vdash_{\mathrm{S}} \mathrm{A}$, i.e., iff there is a proof of A in the corresponding Hilbert system S.

In order to expedite the proof of this theorem, we present "left-handed" Gentzen consecution formulations of the four systems of interest as intermediaries between the Hilbert calculuses and the tableau systems. Notation is from §13.1: Greek letters stand for sequences of formulas; all members of $\check{\alpha}$ have the form $A\to B$; $\bar{\alpha}$ is the sequence of negations of members of α.

We give a set of axioms and rules from which the various left-handed Gentzen formulations of $\mathbf{E}_{\overset{\sim}{\to}}$, $\mathbf{R}_{\overset{\sim}{\to}}$, $\mathbf{RM}_{\overset{\sim}{\to}}$ and $\mathbf{TV}_{\overset{\sim}{\to}}$ are defined.

Axioms

$A, \bar{A}\vdash$ (Ax⊢) $\alpha, \bar{\alpha}\vdash$ (MAx⊢)

Structural rules

Permutation (C$\vdash$)

$$\frac{\alpha, A, B, \beta \vdash}{\alpha, B, A, \beta \vdash}$$

Contraction (W$\vdash$)

$$\frac{\alpha, A, A \vdash}{\alpha, A \vdash}$$

Weakening (K$\vdash$)

$$\frac{\alpha \vdash}{\alpha, A \vdash}$$

Connective rules

Double Negation ($\approx\vdash$)

$$\frac{\alpha, A \vdash}{\alpha, \bar{\bar{A}} \vdash}$$

Arrow ($\rightarrow\vdash$)

$$\frac{\alpha, \bar{A} \vdash \qquad \beta, B \vdash}{\alpha, \beta, A \rightarrow B \vdash}$$

Negated Arrow ($\overset{\sim}{\rightarrow}\vdash$)

$$\frac{\alpha, A, \bar{B} \vdash}{\alpha, \overline{A \rightarrow B} \vdash}$$

Strict Negated Arrow (S$\overset{\sim}{\rightarrow}\vdash$)

$$\frac{\vec{\alpha}, A, \bar{B} \vdash}{\vec{\alpha}, \overline{A \rightarrow B} \vdash}$$

The left-handed Gentzen systems for $\mathbf{E}_{\overset{\sim}{\rightarrow}}$, $\mathbf{R}_{\overset{\sim}{\rightarrow}}$, $\mathbf{RM}_{\overset{\sim}{\rightarrow}}$ and $\mathbf{TV}_{\overset{\sim}{\rightarrow}}$ are defined from these as follows.

$\mathrm{L_l}\mathbf{E}_{\overset{\sim}{\rightarrow}}$ = Ax, C$\vdash$, W$\vdash$, $\approx\vdash$, $\rightarrow\vdash$, S$\overset{\sim}{\rightarrow}\vdash$
$\mathrm{L_l}\mathbf{R}_{\overset{\sim}{\rightarrow}}$ = Ax, C$\vdash$, W$\vdash$, $\approx\vdash$, $\rightarrow\vdash$, $\overset{\sim}{\rightarrow}\vdash$
$\mathrm{L_l}\mathbf{RM}_{\overset{\sim}{\rightarrow}}$ = MAx, C$\vdash$, W$\vdash$, $\approx\vdash$, $\rightarrow\vdash$, $\overset{\sim}{\rightarrow}\vdash$
$\mathrm{L_l}\mathbf{TV}_{\overset{\sim}{\rightarrow}}$ = Ax, C$\vdash$, W$\vdash$, K$\vdash$, $\approx\vdash$, $\rightarrow\vdash$, $\overset{\sim}{\rightarrow}\vdash$

We write $\alpha \vdash_{\mathrm{L_lS}}$ when $\alpha \vdash$ is a theorem of $\mathrm{L_l}$S.

THEOREM 2. Let **S** be $\mathbf{E}_{\overset{\sim}{\rightarrow}}$, $\mathbf{R}_{\overset{\sim}{\rightarrow}}$, $\mathbf{RM}_{\overset{\sim}{\rightarrow}}$ or $\mathbf{TV}_{\overset{\sim}{\rightarrow}}$. For all formulas A, $\vdash_{\mathbf{S}} A$ iff $A \vdash_{\mathrm{L_lS}}$. That is, the left-handed systems exactly correspond to the respective Hilbert systems.

PROOF. The cases for $\mathbf{E}_{\overset{\sim}{\rightarrow}}$ and $\mathbf{R}_{\overset{\sim}{\rightarrow}}$ can be recovered from §13, since the systems $\mathrm{L_l}$S are the duals of the right handed systems $\mathrm{L_r}$S treated there. The result for $\mathbf{RM}_{\overset{\sim}{\rightarrow}}$ is new, but straightforward. The case for **TV** can be extracted from Gentzen 1935.

The next theorem gives us half of the equivalence between the tableau systems and the left-handed systems.

THEOREM 3. For each considered S, if $\alpha \vdash_{\mathrm{L_lS}}$ then $\alpha \vdash_{\mathrm{TS}}$.

PROOF is by straightforward induction on the length of the proof of $\alpha \vdash$ in $\mathrm{L_l}$S.

If $\alpha \vdash$ is an axiom of $\mathrm{L_1S}$, then beginning a tableau with α and applying the appropriate branch closure rule *at* its last node *to* all its nodes constitutes a refutation, in the appropriate system, of α.

For the remaining rules of the $\mathrm{L_1S}$-calculuses, we assume we have TS-refutations of their premiss or premisses and then show how to construct a TS-tableau that will refute its conclusion because it satisfies the various global requirements of TS.

Permutation. Begin a tableau with its conclusion, and continue it in exactly the same way as for the tableau refuting its premiss, except for switching annotations referring to the permuted items. Because only the last node of the beginning of the tableau is annotated, all *to*-nodes will remain above their *at*-nodes, and there will be no problem about barriers. Nor is there any problem about any of the global requirements.

Contraction. Begin a tableau with its conclusion, and continue it in exactly the same way as the tableau refuting its premiss, except change references to one of the A's to become references to the other. Note in particular that MCl permits repetition of a node in its annotation. There is no problem about any of the global requirements.

Weakening. Begin a tableau with its conclusion, and continue it in exactly the same way as the tableau refuting its premiss. The inserted step will not be used, but this is not a problem since if $\mathrm{L_1S}$ has Weakening as a rule, then TS does not have the Use requirement.

Of the connective rules, we go through only Arrow and Strict Negated Arrow.

Arrow. By hypothesis we have refutations τ_1 of α, $\bar{A}$ and τ_2 of β, B. Begin a tableau with the conclusion of Arrow, and apply $\rightarrow$ at its last step. Now continue down the left as in τ_1, and down the right as in τ_2, changing annotations to suit. Clearly the Closure and Barrier requirements will be satisfied by the constructed tableau if they are satisfied by the given one, and so will the Use requirement, since members of α and $\bar{A}$ are used (just) down the left side, members of β and B are used (just) down the right side, and $A \rightarrow B$ is used *at* itself.

Strict Negated Arrow. By hypothesis we have a refutation τ of the premiss. Begin a tableau with the conclusion of Strict Negated Arrow, and apply $\mathrm{S}\overline{\rightarrow}$ at its last step. Now continue with τ, changing annotations to suit. Because of the restriction on Strict Negated Arrow, the Barrier requirement will be satisfied by the constructed tableau if it is satisfied by τ; and so will the Use and Closure requirements.

We leave verification of the other rules to the reader. (Theorem 3 can be established *directly* in the fashion of McRobbie 197+a; i.e., a detour via L_1S is not necessary. What is primarily involved in this proof is proving a tableau-theoretic equivalent of Gentzen's *Hauptsatz* (see Gentzen 1935) for TS′, where TS′ is just TS plus a tableau-theoretic equivalent of Gentzen's rule *cut*.)

THEOREM 4. For each considered S, if $\alpha \vdash_{TS}$ then $\alpha \vdash_{L_1S}$.

PROOF. We shall see that a TS-refuting tableau can be looked at as a sort of Gentzen proof turned upside down; the global restrictions will come heavily into play.

From this point onward we follow Curry 1963 by "identifying" sequences that are permutes of each other (all our L_1-systems have permutation)—but we shall have to keep track of which formulas occur in our sequences, and how many times each occurs (some of our L_1-systems do not have Weakening).

Given any TS-refuting tableau, we define a function *Seq* from its nodes into sequences of formulas as follows: for each node n, A is to have n_A occurrences in Seq(n) just in case there are n_A distinct applications of rules *at* nodes $\geq n$ *to* nodes $\leq n$ to which A is assigned (counting separately for multiple mentions of the same node in MCl); that is, just in case n is caught n_A times between the *at* and the *to* of an application of a rule *to* A; that is, just in case there are n_A triples $\langle i, j, m\rangle$ where i and j are nodes and m is an integer, such that $i \leq n \leq j$, $F(i) = A$, and the mth member of the annotation *at* j is i.

LEMMA. If τ is a TS-refuting tableau, then, for each annotated node n of τ, i.e., for each node n *at* which a rule is applied, Seq(n)$\vdash_{L_1S}$.

PROOF is by induction on the number of annotated nodes succeeding a given annotated node.

Suppose first that n is an end node. By the Closure requirement, n must be annotated by an appropriate branch closure rule Cl or MCl; since there will be in Seq(n) an occurrence of a formula corresponding to each reference *to* it in the annotation of n, Seq(n)$\vdash$ will be an axiom of L_1S.

Second, suppose n is annotated with $\rightarrow$(i), where $F(i) = A \rightarrow B$. Let j and k be the immediate successors of n, so that $F(j) = \bar{A}$ and $F(k) = B$. Let Seq(j) $= \alpha, \bar{A}, \ldots, \bar{A}$, with $j_{\bar{A}}$ $\bar{A}$s corresponding to references *to* j, and let Seq(k) $= \beta, B, \ldots, B$, with k_B Bs corresponding to references *to* k, where $j_{\bar{A}}$, $k_B \geq 0$, and where α and β contain formulas corresponding to references *to* nodes $\leq n$. Clearly Seq(n) $= \alpha, \beta, A \rightarrow B$. Now, by inductive hypothesis, Seq(j)$\vdash_{L_1S}$ and Seq(k)$\vdash_{L_1S}$. If TS requires Use, then $j_{\bar{A}}$, $k_B > 1$. If not, L_1S

has Weakening, so we may anyhow suppose $j_{\bar{A}}$, $k_B > 1$. By Contraction and $\to\vdash$, Seq(n) $\vdash_{\mathrm{L_1S}}$.

Third, suppose n is annotated with ⥽(i) or S⥽(i), where $F(i) = A\to B$. Let j be the successor of n, and k the successor of j, so that $F(j) = A$ and $F(k) = \bar{B}$. Let Seq(k) $= \alpha, A, \ldots, A, \bar{B}, \ldots, \bar{B}$, with k_A As corresponding to references to j, and $k_{\bar{B}}$ $\bar{B}$s, corresponding to references to k, each > 0. Since j is not annotated, Seq(n) $= \alpha, \overline{A\to B}$. By the hypothesis of the induction, k being annotated, Seq(k) $\vdash_{\mathrm{L_1S}}$. As before, if TS requires Use, k_A, $k_{\bar{B}} \geq 1$; if not, $\mathrm{L_1S}$ has Weakening; so we may anyhow suppose k_A, $k_{\bar{B}} \geq 1$. By contraction, α, A, $\bar{B} \vdash_{\mathrm{L_1S}}$. If TS requires Barrier, every member of α is an implication, since all the references it represents will cross the barrier generated by the application of *S⥽ at n*; so that Seq(n) $\vdash_{\mathrm{L_1S}}$ by S⥽$\vdash$. Otherwise ⥽$\vdash$ produces the same result.

The case when n is annotated with $\approx$ is left to the reader.

Returning to the proof of Theorem 4, we suppose $\alpha \vdash_{\mathrm{TS}}$, and note by the Lemma that Seq(n) $\vdash_{\mathrm{L_1S}}$, where n is the first annotated node in a tableau TS-refuting α; by Closure there will be such a node. Seq(n) can contain no formula not in α. For each formula A in α, let m_A be its number of occurrences in α, and let n_A be its number of occurrences in Seq(n). If TS imposes the Use requirement, $n_A \geq m_A \geq 1$; and, if not, Weakening is available in $\mathrm{L_1S}$; so it is anyhow harmless to suppose $n_A \geq m_A \geq 1$. So $\alpha \vdash_{\mathrm{L_1S}}$, by Contraction.

Finally, we note that Theorem 1 is an immediate consequence of Theorems 2–4: the tableau and Hilbert systems are in the appropriate sense equivalent.

§60.3. Problems. We conclude with a short list of some of the more interesting problems raised by the results we have presented in this section.

1. The analytic tableau formulation of **R**$_{⥽}$ given in this section has been extended to the system **R**$_{+}$ by McRobbie 1977. Can it be extended further to all of **R**?

2. There is a precise translation between analytic semantic tableau formulations and analytic tableau formulations of **TV** and a large number of modal logics, (e.g., see McRobbie 197+a). Analytic semantic tableau formulations of **E**$_{⥽}$, **R**$_{⥽}$ and **RM**$_{⥽}$ can be quite straightforwardly extracted from the semantics given for their parent systems in Routley and Meyer 1973 (§48). Are the analytic semantic tableau formulations and the analytic tableau formulations of these logics intertranslatable? Put more generally, what do the systems **TE**$_{⥽}$, **TR**$_{⥽}$ and **TRM**$_{⥽}$ mean from the point of view of the Routley/Meyer semantics?

3. By dropping the rule $\approx$ from **TE**$_{⥽}$, **TR**$_{⥽}$, and **TRM**$_{⥽}$ and adjusting Cl and MCl so that closure can take place only on propositional variables and their negates, it can easily be shown that we have the analytic tableau formulations of **E**$_{\to}$, **R**$_{\to}$, and **RM**$_{\to}$. What do these formulations mean from the

point of view of the semilattice semantics given for these logics by Urquhart in §47? What does the system $\mathbf{TR}_+$ mean from the point of view of the theory of Dunn monoids given by Meyer 1973a? The relation between **TTV** and Boolean algebras is discussed by Eytan 1974.

4. Decidability for various analytic tableau formulations of various strict implicational calculuses can be straightforwardly extracted from Davidson, Jackson, and Pargetter 1977, and it is a trivial exercise to show that **TTV** can be used to show **TV** decidable. Can the systems $\mathrm{T}\mathbf{E}_{\strictif}$, $\mathrm{T}\mathbf{R}_{\strictif}$, and $\mathrm{T}\mathbf{RM}_{\strictif}$ be used to show $\mathbf{E}_{\strictif}$, $\mathbf{R}_{\strictif}$, and $\mathbf{RM}_{\strictif}$ decidable directly without translating them into the respective left-handed Gentzen systems whose decidability is known (as in §13)?

5. What are the analytic tableau formulations of at least the implication/negation fragments of the weak relevant systems **T**, **T**$-$**W**, **S**, and **B**?

We close with a final observation. Tableau systems have always been construed semantically; and even given our results, our tableau systems still have a strong semantical flavor. This fact, taken together with the essential simplicity of operation of our tableau systems, leads us to speculate that there may in fact be a simpler semantics for **E**, **R**, and **RM** than those reported in §48, which are the best results to date.

§61. A consecution calculus for positive relevant implication with necessity (with Anil Gupta). **R** (see §§R2 and 28) is one of the principal relevance logics, codifying relations among $\rightarrow$ (relevant but nonmodal implication), &, $\vee$, and $\sim$. $\mathbf{R}^{\Box}$ is its enrichment with an **S4**-ish necessity operator $\Box$ (see §§22 and 27.1.3) so that entailment can be carried by $\Box(A\rightarrow B)$, and $\mathbf{R}^{\Box}$ may be further extended—conservatively—by the addition of postulates for a constant necessarily true proposition $\boldsymbol{t}$ and a cotenability operator $A\circ B$ $[=_{\mathrm{df}}\sim(A\rightarrow\sim B)]$ yielding wha^{t} we might call $\mathbf{R}^{\Box\circ t}$ (see §R2 for postulates). No one yet knows a decent consecution formulation (Gentzen 'sequenzen-kalkül'—see §7.2 for our terminology) of **R**, but in §28.5 the problem is solved for $\mathbf{R}^{\circ t}_{+}$, which is the positive fragment $\mathbf{R}_+$ of **R** conservatively enriched by $\boldsymbol{t}$ and $\circ$. In §29.10 it was announced that this result could be extended to $\mathbf{R}^{\Box\circ t}_{+}$; the purpose of this section is to present the proof. That is, we define a consecution calculus $\mathbf{LR}^{\Box\circ t}_{+}$ which by means of an appropriate Elimination theorem we show equivalent to $\mathbf{R}^{\Box\circ t}_{+}$.

§61.1. History. A word about history is in order. JMD's result for the system *without* necessity was presented in a colloquium at the University of Pittsburgh in the spring of 1968 and by title at a meeting of the Association for Symbolic Logic in December, 1969 (see Dunn 1973). A full treatment appeared in §28.5. The modifications required by the addition of necessity were not quite straightforward; we completed the proof in September, 1972. The final results were written up in the winter of 1973 for circulation and for presentation to the St. Louis Conference on Relevance Logic in 1975. After it developed that the proceedings of that conference would not appear, we finally withdrew the paper in order to offer it as Belnap, Gupta, and Dunn 1980.

In the meantime, Minc 1972 (February 24 is given as the date of earliest presentation) proved essentially the same theorem we report below, i.e., cut for the system *with* necessity. (It is surely needless to say that our work and his have been totally independent.) And there is the well-known work of Prawitz 1965 on normal form theorems for natural deduction forms of relevance logic. We therefore feel called upon to say a few words about what this section adds.

First, Prawitz 1965. Prawitz does in fact prove a normal form theorem for a relevance logic — but not for the system $\mathbf{R}_+$. Minc 1972 even goes so far as to suggest that the totally irrelevant $p \to [(c \& (q \to c)) \to c]$ is provable in Prawitz' $\mathbf{R}_{M,S4}$, but we have not been able to reconstruct his reasoning and do not agree. However, we do agree that in fact Prawitz's system is *not* the same as $\mathbf{R}_+$: the formula $((A \to (B \vee C)) \& (C \to D)) \to (A \to (B \vee D))$ (mentioned in §§27.1.1 and 47.4) distinguishes the two, being provable in Prawitz's system but not in $\mathbf{R}_+$. (Charlwood 1978 shows that Prawitz's system is equivalent to that of Urquhart 1972, using the work of Fine 1976. These systems are also akin to the *constructive* relevance logics of Pottinger 1969.)

Second, Minc 1972. What we offer below is a new proof, which compares with that of Minc 1972 as follows. In the first place, Minc's proof uses the technique of Curry 1963 by which in some cases of the argument one modifies the entire proof-tree in a wholesale manner, substituting, in effect, for what Curry calls "quasi-parametric ancestors." Our proof, in contrast, carries out each case by modifying only the immediately preceding steps in a retail way. The trade-off is this: on the wholesale plan, there are many modifications, but each is rather simple. On the retail plan, there are limited modifications, but each must be more complex. What emerges below is precisely the sort of modification that will permit the retail plan to go through.

In the second place, we provide a detailed analysis of the nature of Gentzen rules in the spirit of Curry, and we offer certain easily verifiable properties of rules under which our sort of argument will succeed. (In §62 we carry out a similar sort of analysis for wholesale-type arguments. We also manage to find a consecution formulation for all of **R** in a sense—but only at the expense of adding Boolean negation, which we take to be foreign to our enterprise.)

We digress momentarily to mention that Minc 1977, Mac Lane 197+, and Szabo 1983 have found strong connections between various consecution calculuses for fragments of relevance logics and various categories. Thus, e.g., the cut elimination theorem for $\mathbf{LR}^{\circ t}_{\to} - \mathbf{W}$ (besides dropping conjunction, disjunction, and contraction, drop extensional sequences from $\mathbf{LR}_+$) yields the Kelly-Mac Lane Coherence theorem for proper shapes.

§61.2. Postulates for L $(=\mathbf{LR}^{\Box\circ t}_{+})$. This section recapitulates §§28.5.1 and 29.10. Turning to the consecution calculus $\mathbf{LR}^{\Box\circ t}_{+}$ we show equivalent to $\mathbf{R}^{\Box\circ t}_{+}$, let us begin by shortening its name to '**L**'. The formation rules of **L** are a generalization of the usual Gentzen formation rules, inasmuch as (1) there are two kinds of sequences allowed, and (2) we allow sequences of sequences of... sequences. We distinguish the two kinds by prefixes: 'I' stands for 'intensional' and corresponds to cotenability, 'E' stands for 'extensional' and corresponds to conjunction. An *antecedent* then is defined as follows: each formula (in $\&$, $\vee$, $\circ$, $\rightarrow$, $\boldsymbol{t}$, and $\Box$) is an antecedent; and if $\alpha_1, \ldots, \alpha_n$ are antecedents, so are (where $n \geqslant 1$)

$$\mathrm{I}(\alpha_1, \ldots, \alpha_n)$$

and

$$\mathrm{E}(\alpha_1, \ldots, \alpha_n).$$

Then a *consecution* in **L** has the form $\alpha \vdash A$, with α an antecedent and A a formula. (*Note:* α cannot be empty in **L**; it is the role of $\boldsymbol{t}$ to allow us so to manage things.)

We use small Greek letters as ranging over antecedents, and capital Greek letters as ranging over (possibly empty) sequences of symbols drawn from the following: the formulas, the symbols I and E, the two parentheses, and the comma. We shall use 'V' as standing indifferently for I or E, so as to be able to state rules common to both. And we agree that displayed parentheses are always to be taken as paired.

We now state the axioms and rules for **L**.

The *axioms* have the usual form:

$$A \vdash A \quad (\mathrm{Id})$$

The structural rules are manifold. First the familiar ones:

Permutation (CV⊢)

$$\frac{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n)\Gamma_2 \vdash A}{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n)\Gamma_2 \vdash A}\,(\mathrm{CV}\vdash)$$

Contraction (WV⊢)

$$\frac{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \alpha, \alpha, \ldots, \alpha_n)\Gamma_2 \vdash A}{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \alpha, \ldots, \alpha_n)\Gamma_2 \vdash A}\,(\mathrm{WV}\vdash)$$

Weakening (KE⊢)

$$\frac{\Gamma_1\alpha\Gamma_2 \vdash A}{\Gamma_1\mathrm{E}(\alpha, \beta)\Gamma_2 \vdash A}\,(\mathrm{KE}\vdash)$$

Note that the α_i, α, and β must be antecedents (*a fortiori* nonempty), whereas Γ_1 and Γ_2 in general will not be antecedents, and may be empty; and recall that 'V' in the above rules varies over E and I.

Now for some structural rules, peculiar to nested sequences, ensuring that each antecedent is equivalent to an I-sequence of E-sequences of I-sequences . . . (counting formulas as either); or perhaps an alternation starting with an E-sequence.

$$\frac{\Gamma_1\mathrm{V}(\alpha)\Gamma_2 \vdash A}{\Gamma_1\alpha\Gamma_2 \vdash A}\,(\mathrm{V}_1\ \mathrm{elim})$$

$$\frac{\Gamma_1\alpha\Gamma_2 \vdash A}{\Gamma_1\mathrm{V}(\alpha)\Gamma_2 \vdash A}\,(\mathrm{V}_1\ \mathrm{int})$$

$$\frac{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \mathrm{V}(\Sigma), \ldots, \alpha_n)\Gamma_2 \vdash A}{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \Sigma, \ldots, \alpha_n)\Gamma_2 \vdash A}\,(\mathrm{V}_2\ \mathrm{elim})$$

$$\frac{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \Sigma, \ldots, \alpha_n)\Gamma_2 \vdash A}{\Gamma_1\mathrm{V}(\alpha_1, \ldots, \mathrm{V}(\Sigma), \ldots, \alpha_n)\Gamma_2 \vdash A}\,(\mathrm{V}_2\ \mathrm{int})$$

Logical rules

$$\frac{\Gamma_1\mathrm{I}(A, \mathrm{B}\Sigma)\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}((A \circ B)\Sigma)\Gamma_2 \vdash C}\,(\circ\vdash) \qquad \frac{\alpha \vdash A \quad \beta \vdash B}{\mathrm{I}(\alpha, \beta) \vdash (A \circ B)}\,(\vdash\circ)$$

$$\frac{\Gamma_1\mathrm{E}(A, B\Sigma)\Gamma_2 \vdash C}{\Gamma_1\mathrm{E}((A\&B)\Sigma)\Gamma_2 \vdash C}\,(\&\vdash) \qquad \frac{\alpha \vdash A \quad \beta \vdash B}{\mathrm{E}(\alpha, \beta) \vdash (A\&B)}\,(\vdash\&)$$

$$\frac{\Gamma_1A\Gamma_2 \vdash C \quad \Gamma_1B\Gamma_2 \vdash C}{\Gamma_1(A \vee B)\Gamma_2 \vdash C}\,(\vee\vdash) \qquad \frac{\alpha \vdash A}{\alpha \vdash (A \vee B)}\,(\vdash\vee) \qquad \frac{\alpha \vdash B}{\alpha \vdash (A \vee B)}\,(\vdash\vee)$$

$$\frac{\alpha \vdash A \quad \Gamma_1B\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}((A \rightarrow B), \alpha)\Gamma_2 \vdash C}\,(\rightarrow\vdash) \qquad \frac{\mathrm{I}(\Sigma\alpha, A) \vdash B}{\mathrm{I}(\Sigma\alpha) \vdash (A \rightarrow B)}\,(\vdash\rightarrow)$$

$$\frac{\Gamma_1\alpha\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}(t, \alpha)\Gamma_2 \vdash C}\,(t\vdash)$$

$$\frac{\Gamma_1A\Gamma_2 \vdash \mathrm{C}}{\Gamma_1\Box A\Gamma_2 \vdash C}\,(\Box\vdash) \qquad \frac{\alpha \vdash A}{\alpha \vdash \Box A}\,(\vdash\Box)$$

Restriction on $(\vdash\Box)$: every constituent in α must either have the form $\Box B$ or be $\boldsymbol{t}$.

§61.3. Translation and equivalence. I-sequences are to be translated into $\mathbf{R}_{+}^{\Box\circ t}$ via cotenability, and E-sequences via conjunction, as in the following definition of a translation function:

$$\begin{aligned} \mathrm{T}(A) &= A \\ \mathrm{T}(\mathrm{V}(\alpha)) &= \mathrm{T}(\alpha) \\ \mathrm{T}(\mathrm{E}(\alpha, \Delta)) &= (\mathrm{T}(\alpha)\&\mathrm{T}(\mathrm{E}(\Delta))) \\ \mathrm{T}(\mathrm{I}(\alpha, \Delta)) &= (\mathrm{T}(\alpha)\circ\mathrm{T}(\mathrm{I}(\Delta))) \\ \mathrm{T}(\alpha \vdash A) &= \mathrm{T}(\alpha)\rightarrow A \end{aligned}$$

EQUIVALENCE THEOREM. Part 1. If $\alpha \vdash A$ is provable in **L**, then $\mathrm{T}(\alpha \vdash A)$ is provable in $\mathbf{R}^{\Box\circ t}_{+}$. Part 2. If A is provable in $\mathbf{R}^{\Box\circ t}_{+}$, then $\boldsymbol{t} \vdash A$ is provable in **L**. Accordingly, since $\boldsymbol{t}$ is provable in $\mathbf{R}^{\Box\circ t}_{+}$, it follows that A is provable in $\mathbf{R}^{\Box\circ t}_{+}$ just in case $\boldsymbol{t} \vdash A$ is provable in **L**.

Proof of Part 1 is left entirely to the reader.

For Part 2 we must tediously prove $\boldsymbol{t} \vdash A$ in **L** for every axiom A of $\mathbf{R}^{\Box\circ t}_{+}$, a procedure we omit; and we must show the admissibility in **L** of the rules

$$\frac{\boldsymbol{t} \vdash A \qquad \boldsymbol{t} \vdash B}{\boldsymbol{t} \vdash A\&B} \qquad\qquad \frac{\boldsymbol{t} \vdash A \qquad \boldsymbol{t} \vdash A\rightarrow B}{\boldsymbol{t} \vdash B}$$

answering to &I and →E. The former is trivial; for the latter we must, as usual prove an Elimination theorem. Its statement involves multiple simultaneous substitution; we prepare by introducing some notation. In the first place, by a *constituent* we shall, as in Curry 1963, always refer to an occurrence of a formula that does *not* lie in the scope of any logical connective, and by an *M-constituent* we shall mean a constituent that is an occurrence of M. Secondly where $X_1, \ldots, X_{p_0}$ are pairwise disjoint sets of constituents of an antecedent δ, we define

$$\delta(\gamma_1/X_1, \ldots, \gamma_{p_0}/X_{p_0})$$

or often

$$\delta(\gamma_p/X_p)_{p=1}^{p_0}$$

to be the result of simultaneously substituting in δ, for each p $(1 \leqslant p \leqslant p_0)$, the antecedent γ_p for every constituent in X_p. Thirdly, we define Y to be a n_0-*ary sequential partition of* X just in case X is a set and Y is an n_0-tuple of subsets of X (including the possibility that some are empty) which are pair-wise disjoint and whose union is X. Where Y is an n_0-tuple, we uniformly take Y_n as its nth member, so that $Y = \langle Y_1, \ldots, Y_{n_0}\rangle$; hence, Y is an n_0-ary sequential partition of X iff $Y_n \cap Y_{n'} = \varnothing$ when $n \neq n'$, and $Y_1 \cup \ldots \cup Y_{n_0} = X$.

Elimination Theorem. Let Y be a n_0-ary sequential partition of a set X of M-constituents in an antecedent δ. Then the following rule is admissible in **L**:

$$\frac{\gamma_p \vdash M \quad (1 \leqslant p \leqslant p_0) \qquad \delta \vdash D}{\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D}$$

What makes this Elimination theorem different from its cousins is that it countenances not just one but a sequence of left premisses. This generalization of the usual theorem is suggested by the interaction of the rules ($\vee\vdash$) and ($\vdash\Box$). The proof will be by a triple induction (double would do, but the extra level is convenient). The Outer induction is on the length of M, the Middle induction on the "combined rank" of M in the left and right premisses, and the Inner induction on the number of left premisses at maximum "rank". The argument requires an analysis of the concept of "rank," to which we next proceed by way of some auxiliary notions.

§61.4. Some definitions and the normality property. An *inference* is an ordered pair consisting of a finite (non-null) sequence of consecutions—the *premisses*—as left entry and a consecution—the *conclusion*—as right entry. A *rule* is a set of inferences; its members are called *instances*. A *calculus* is a set of consecutions—the *axioms*—together with a set of rules; and an inference *of* a calculus is an instance of one of its rules. A *derivation in* a calculus **S** is, as usual, a tree of consecutions, each branch of which terminates in an axiom of **S**, and each nonterminal node being such that the pair consisting of a sequence of nodes immediately above it as left entry and the given node as right entry constitutes an inference of **S**.

Rank in a derivation has to be defined relative to an "analysis" of the inferences of **S**: hence the following. In the first place, a pair $\langle P, C\rangle$ is an *analysis of* an inference *Inf* if P is a subset of the constituents of *Inf* and C is an equivalence relation on P. Secondly, $\langle P, C\rangle$ is a *normal analysis of* an inference *Inf* just in case the following hold:

1. $\langle P, C\rangle$ is an analysis of *Inf*; i.e., P is a subset of the constituents of *Inf*, and C is an equivalence relation on P.

In the following we call members of P *parameters* and say that constituents related by C are *congruent*.

2. Congruent parameters are occurrences of the same formula.
3. Congruent parameters are on the same side of $\vdash$.
4. Each parameter is congruent to exactly one parameter in the conclusion.

These conditions are adapted from Curry 1963, p. 197. We comment several paragraphs below.

The above gives us the notion of an analysis of a single inference; moving up two levels, we say that f is an *analysis-function* for a calculus S if f is a function defined on all inferences of S and such that $f(Inf)$ is always an analysis of *Inf*. (This disallows the possibilities of analyses relative to rules and of multiple analyses, perhaps wanted in the more general case but avoidable in application to **L** by one *ad hoc* step taken below.) Further, f is a *normal analysis-function* for S if it maps each inference of S into a normal analysis of that inference.

We introduce the following terminology in the spirit of Curry 1963. Let $f(Inf) = \langle P, C\rangle$. Then the members of P are *f-parameters* and, more particularly, *f-premiss-parameters* or *f-conclusion-parameters* according to where they lie. Further, constituents of *Inf* that are *not* in P are called *f-principal* if they are in the conclusion of *Inf* and *f-subalterns* if in one of the premisses.

We choose a normal analysis-function $f_{\mathbf{L}}$ for **L** by specifying $f_{\mathbf{L}}(Inf) = \langle P, C\rangle$ as follows. Turning first to the conclusion of *Inf*, we put in P and thereby make f-conclusion-parameters all conclusion-constituents of *Inf* *except* any constituent newly introduced by one of the logical rules; hence, such a constituent (if there is one) is $f_{\mathbf{L}}$-principal. Further, with respect to any premiss of *Inf*, we put in P and thereby make $f_{\mathbf{L}}$-premiss-parameters just those constituents (if any) matching in a way obvious from the statement of the rules an $f_{\mathbf{L}}$-conclusion-parameter of *Inf*; hence the others (if any) are $f_{\mathbf{L}}$-subalterns. Lastly, we define C by taking the reflexive, symmetrical, and transitive closure of the following relation C_0: each $f_{\mathbf{L}}$-premiss-parameter bears C_0 to the $f_{\mathbf{L}}$-conclusion-parameter that it matches in a way obvious (again) from the statement of the rules.

In the above we have relied on "the statement of the rules." In order to be sure this procedure makes sense, we should first verify that no inference falls under more than one rule of **L** and that "the statement of a rule" provides a unique congruence relation. This is not so for those instances of one of the rules (CV⊢) in which (1) the conclusion is identical with the premiss in virtue of the permutation of adjacent instances of the *same* antecedent (i.e., with reference to the statement of the rule, $\alpha_i = \alpha_{i+1}$), and (2) there is in the premiss more than one case of like adjacent constituents; for these instances of the rules (CV⊢) would allow more than one analysis—and indeed could fall under both (CE⊢) and (CI⊢)—in accordance with "the statement of the rules." To avoid this difficulty we must do something or other *ad hoc*; our choice is to modify the above by declaring that in such cases each premiss-constituent shall bear C_0 to the similarly positioned conclusion-constituent.

Normality Property. $f_{\mathbf{L}}$ is a normal analysis-function for **L**.

Proof by inspection.

We note the following in regard to our definition of normality and our particular analysis of **L**. (a) Condition 2, although sensible and faithful to Curry 1963, is not used below. (b) In virtue of 1 and 4, a conclusion-parameter can be congruent only to itself; more generally, congruence is determined uniquely by specifying for each premiss-parameter the conclusion-parameter to which it is congruent; one then takes the reflexive, symmetric, and transitive closure. (c) Condition 4 is important to our argument, and 3 is used though it could be avoided (but it is sensible; indeed, one might think of strengthening it in our context of two kinds of sequences). (d) Curry requires that every conclusion-parameter be congruent to at least one parameter in at least one premiss; our analysis of (KE⊢) does not satisfy this condition. (e) Curry requires that a parameter be congruent to at most one parameter in any one premiss; our analysis of (WV⊢) does not satisfy this condition. (f) Curry requires that there be at most one principal (nonparametric) constituent in the conclusion; our analysis satisfies this further condition, but we do not add it, since alternative analyses, especially of (KE⊢), not satisfying this condition would still be sensible and permit our argument to go through with hardly any modification.

We now define the concept of f-rank, where f is an analysis-function for a calculus S. Let *Der* be a derivation in S; let S be the final consecution in *Der*; unless S is an axiom, let *Inf* be the inference in *Der* of which S is the conclusion; and let X be a set of constituents in S. Define the *f-rank of X in Der* as follows. If X is empty its f-rank in *Der* is 0. If X is nonempty but contains no f-conclusion-parameters—i.e., if all the constituents in X are f-principal or if S is an axiom of S—then the f-rank of X in *Der* is 1. Otherwise let the inference *Inf* be

$$(1)\qquad \frac{S_n \quad (1 \leqslant n \leqslant n_0)}{S}$$

and, for each n $(1 \leqslant n \leqslant n_0)$, let Der_n be the subderivation of *Der* terminating in S_n, and let X_n be the (possibly empty) set of f-premiss-parameters in S_n which are f-congruent to some member of X. Let r be the maximum among the f-ranks of the various X_n in their respective Der_n $(1 \leqslant n \leqslant n_0)$. Then the f-rank in *Der* is defined as $r + 1$. (The feature of the definition giving inductive control is that, if X is nonempty, its rank in *Der* is always greater than that of any of the X_n in their respective Der_n.)

It is convenient also to define *the consequent f-rank of Der* as the f-rank of X in *Der* for the particular case where X is the unit set of the consequent of the final consecution S of *Der*.

When f is in particular f_{L}, we drop it as a prefix on rank, congruence, parameter, etc.

§61.5. Elimination theorem: Outline of proof. We may use these concepts to rephrase the Elimination theorem in a way convenient for the upcoming proof. Make the following abbreviations.

$\Phi(M, k, j, i)$: $\delta, \gamma_1, \ldots, \gamma_{p_0}$ are antecedents, and M and D are formulas; X is a set of M-constituents of δ, and Y is a p_0-ary sequential partition of X; for $1 \leqslant p \leqslant p_0$, Der_p is a derivation in **L** of $\gamma_p \vdash M$, with k the maximum of the consequent ranks of all the Der_p, and i the number of the Der_p having this maximum consequent rank; and finally, Der is a derivation in **L** of $\delta \vdash D$, with j the rank of X in Der.

$\psi(M, k, j, i)$: for all $p_0, \delta, \gamma_1, \ldots, \gamma_{p_0}, D, X$, and Y; if $\Phi(M, k, j, i)$, then $\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$ is provable in **L**.

With these abbreviations we rephrase the Elimination theorem in a way convenient for the upcoming inductive proof.

Elimination theorem. For every M, k $(k \geqslant 0)$, j $(j \geqslant 0)$, and i $(i \geqslant 1)$: $\psi(M, k, j, i)$.

Proof. We proceed by a nested induction. First choose arbitrary M, and suppose

Outer hypothesis: for all M' shorter than M, for all k, j, i, $\psi(M', k, j, i)$. Next choose arbitrary k and j and suppose

Middle hypothesis: for all k' and j' such that $(k'+j') < (k+j)$, for all i, $\psi(M, k', j', i)$.

Inner hypothesis: for all i' such that $i' < i$, $\psi(M, k, j, i')$. Lastly, choose arbitrary p_0, δ, $\gamma_1, \ldots, \gamma_{p_0}$, D, X, and Y, and suppose

Step hypothesis: $\Phi(M, k, j, i)$.

In order to establish the theorem, it suffices to show that, under these hypotheses, $\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$ is provable in **L**. The argument is by cases; but first for convenience we define

L-*premisses:* $\gamma_p \vdash M \quad (1 \le p \le p_0)$ R-*premiss:* $\delta \vdash D$
Conclusion: $\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$

In the first place, if X is empty then R-premiss = Conclusion, so the Step hypothesis may be used. Suppose now that X is nonempty.

Case 1. Each of the derivations of the L-premisses has a consequent rank of 1, hence is either an axiom or comes by a logical rule with the consequent occurrence of M principal. Two subcases.

Case 1.1. X contains no conclusion-parameters; hence R-premiss is either an axiom or comes by a logical inference the principal constituent of which is the unique member of X. Since X has exactly one member, all but one of the

Y_p ($1 \leq p \leq p_0$) is empty, so that, if $p_0 \geq 2$, the Inner hypothesis can be used with the one remaining needed L-premiss. Suppose then that $p_0 = 1$; i.e., there is a unique L-premiss which by 1 is either an axiom or comes by a logical rule. Three subcases.

Case 1.11. The L-premiss is an axiom. Hence R-premiss = Conclusion; use the Step hypothesis.

Case 1.12. R-premiss is an axiom. Hence the L-premiss = Conclusion; use the Step hypothesis.

Case 1.13. Each of the L-premiss and R-premiss comes by a logical inference, with the consequent occurrence of M principal in one and the unique M-constituent in X principal in the other. Evidently the inferences must be instances of matching logical rules. Five subcases (for $\circ$, &, $\vee$, $\rightarrow$, and $\Box$). In each case the argument, using the Outer hypothesis, is straightforward, with an occasional use of one of the rules (V_2 int).

Case 1.2. X contains at least one conclusion-parameter. See below.

Case 2. At least one of the derivations of the L-premisses has a consequent rank of at least 2; hence $k \geq 2$. See below.

In the sequel we treat only the cases 1.2 and 2; to deal with these we elaborate on the concepts of left and right regularity of §28.5.3. The plan is to state easily verifiable properties of the rules of **L**, and then to see how more complex properties of the rules needed in the treatment of the two cases mentioned are corollaries of the easily verified properties.

§61.6. Closure under substitution and case 1.2. Let f be an analysis-function for **S**. We shall say that a rule *Ru* of **S** is *closed under f-parametric substitution* under the following conditions. Let *Inf* be an instance of *Ru*, and let X be a f-congruence class of constituents of *Inf*; i.e., the set of all constituents of *Inf* that are f-congruent to some constituent of *Inf*. Then, for arbitrary antecedent β, the inference *Inf*(β/X) which results from substituting β for all members of X either has a conclusion identical with one of its premisses, or is itself an instance of *Ru*. (Of course on the *right* only *formulas* β may be substituted.) Furthermore, f-parameter and f-congruence for *Inf*(β/X) are as follows: in the first place, constituents of *Inf*(β/X) lying within any substituted occurrence of β are f-parameters and are congruent to just those constituents of *Inf*(β/X) occupying like positions in substituted occurrences of β. Secondly, note that substitution induces a natural one–one correspondence between (a) constituents of *Inf*(β/X) *not* lying within any substituted occurrence of β and (b) the unsubstituted-for occurrence of *Inf*; this correspondence is an isomorphism with respect to both f-parameterhood and f-congruence. That is, such a constituent of *Inf*(β/X) is an f-parameter iff its correspondent in *Inf* is, and such constituents of *Inf*(β/X) are f-congruent iff their correspondents are. (Compare Curry 1963, p. 198, (r6).)

CLOSURE UNDER PARAMETRIC SUBSTITUTION PROPERTY. Each rule of **L** except ($\vdash\Box$) is closed under $f_{\mathbf{L}}$-parametric substitution.

PROOF by inspection of the rules of **L**. Verification is perhaps easier if we use normality, parts 3 and 4, to restate the property. Let

$$(1)\qquad \frac{\alpha_n \vdash C_n \quad (1 \le n \le n_0)}{\delta \vdash D}$$

be an instance of *Ru* and let y be an f-conclusion-parameter. Let X_n be the set of f-parameters in the premiss $\alpha_n \vdash C_n$ which are congruent to y; then

$$(2)\qquad \frac{\alpha_n(\beta/X_n) \vdash C_n \quad (1 \le n \le n_0)}{\delta(\beta/\{y\}) \vdash D}$$

or

$$\frac{\alpha_n \vdash C_n(\beta/X_n) \quad (1 \le n \le n_0)}{\delta \vdash \beta}$$

(according as y is on the left or right; if on the right, β must be a formula) is an instance of *Ru*, and with f-parameterhood and f-congruence as stated.

One form in which we shall need this property is stated in the following

COROLLARY. Let

$$(3)\qquad \frac{\alpha_n \vdash C_n \quad (1 \le n \le n_0)}{\delta \vdash D}$$

be an instance of a rule *Ru* of **L** other than ($\vdash\Box$), let X be a set of conclusion parameters in δ, and let Y be a p_0-ary sequential partition of X. Let Y_{p_n} $(1 \le n \le n_0, 1 \le p \le p_0)$ be the set of parameters in α_n which are congruent to one of those in Y_p, and let $\gamma_1, \dots, \gamma_{p_0}$ be antecedents. Then the following will also be an instance of *Ru*:

$$(4)\qquad \frac{\alpha_n(\gamma_p/Y_{p_n})_{p=1}^{p_0} \vdash C_n \quad (1 \le n \le n_0)}{\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D}$$

Furthermore, parameterhood and congruence are undisturbed for unsubstituted-for constituents.

PROOF. One needs only to verify that the simultaneous substitution of the Corollary can be reduced to successive single substitutions as authorized by the Closure under Parametric Substitution property; and this is guaranteed by normality, especially part 4, which implies that, for each n $(1 \le n \le n_0)$, all the Y_{p_n} $(1 \le p \le p_0)$ are pairwise disjoint.

We can now treat Case 1.2. By the hypothesis of the case, we know that the derivation *Der* of R-premiss terminates in an inference *Inf* with respect to which at least one M-constituent in X is a conclusion-parameter. Suppose first that *Inf* is an instance of a rule *Ru* other than ($\vdash\Box$), and let *Inf* be (3). Define $\bar{X}$ as the set of conclusion-parameters in X, and let $\bar{Y}$ be that p_0-ary sequential partition of $\bar{X}$ such that $\bar{Y}_p = Y_p \cap \bar{X}$ ($1 \leq p \leq p_0$). For $1 \leq n \leq n_0$, let $\bar{Y}_{p_n}$ be the set of premiss-parameters in $\alpha_n \vdash C_n$ which are congruent to a member of $\bar{Y}_p$. By using the L-premisses and $\alpha_n \vdash C_n$ with the Middle hypothesis, obtain the L-provability of

$$(5) \qquad \alpha_n(\gamma_p/\bar{Y}_{p_n})_{p=1}^{p_0} \vdash C_n \quad (1 \leq n \leq n_0)$$

The inference from the premisses (5) to

$$(6) \qquad \delta(\gamma_p/\bar{Y}_p)_{p=1}^{p_0} \vdash D$$

is, by the Corollary to the Closure under Parametric Substitution property, also an instance of *Ru*; so (6) is provable in L. Now, if $X = \bar{X}$ then (6) = Conclusion, and we are done.

Otherwise, let $\bar{\bar{X}}$ be the set of M-constituents in (6) corresponding to those in $X - \bar{X}$, and let $\bar{\bar{Y}}$ be the p_0-ary sequential partition of $\bar{\bar{X}}$ defined by letting $\bar{\bar{Y}}_p$ be the set of M-constituents in (6) corresponding to those in $Y_p - \bar{X}$. By the "furthermore" part of the cited Corollary, all members of $\bar{\bar{X}}$ (actually there will be exactly one, but we do not use this information) must be non-parametric (principal) in the inference from (5) to (6); so the rank of $\bar{\bar{X}}$ in the derivation of (6) terminating in the inference from (5) to (6) is 1. We may therefore use the L-premisses with (6) and the Middle hypothesis (1 being less than $2 \leq j$) to obtain

$$(\delta(\gamma_p/\bar{Y}_p)_{p=1}^{p_0})(\gamma_p/\bar{\bar{Y}}_p)_{p=1}^{p_0} \vdash D$$

which is just Conclusion.

Suppose, second and last, that *Inf* is an instance of ($\vdash\Box$) and, in particular, is

$$\frac{\delta \vdash C}{\delta \vdash \Box C} \quad \text{(R-premiss)}$$

Apply the Middle hypothesis to the L-premisses and $\delta \vdash C$, obtaining the L-provability of

$$(7) \qquad \delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash C.$$

We wish to show that (7) is a suitable premiss for ($\vdash\Box$), since if it is we may thereby obtain Conclusion. In the first place, we observe that, by the conditions on ($\vdash\Box$), every constituent in δ must either have the form $\Box A$ or be t; in particular this is true for all the M-constituents in X. We now invoke the case hypothesis 1: each L-premiss is either an axiom or has its consequent

M-constituent as principal constituent for a logical rule. Since there is no right rule for $\boldsymbol{t}$, all the nonaxiomatic L-premisses must come by ($\vdash\Box$). Accordingly, by the restriction on this rule, every constituent in each γ_p must either have the form $\Box A$ or be $\boldsymbol{t}$, so the same is true for every constituent in (7); hence (7) is indeed an appropriate premiss for an inference by ($\vdash\Box$) to Conclusion.

This completes our treatment of Case 1.2.

Closure under Embedding and Case 2. Let f be an analysis-function for S. We shall say that a rule Ru of S is *closed under embedding in a larger f-parametric context* if the following holds. Suppose Ru has as an instance the inference

$$\text{(8)}\qquad \frac{\delta_m \vdash A_m \quad (1 \le m \le m_0) \qquad \alpha_n \vdash C \quad (1 \le n \le n_0)}{\gamma \vdash C}$$

where (a) the displayed occurrence of C in $\gamma \vdash C$ is an f-conclusion-parameter, where (b) the $\alpha_n \vdash C$ are all the premisses—we suppose there is at least one—containing an f-premiss-parameter on the right of $\vdash$ congruent to the aforementioned occurrence of C, and where (c) the $\delta_m \vdash A_m$ (if any; there may be none) are the premisses in which the right side of $\vdash$ is *not* an f-premiss-parameter (hence a subaltern). (Subsequent references to (8) are all supposed to include these provisos; we call the $\alpha_n \vdash C$ the *f-parametric premisses* and the $\delta_m \vdash A_m$ the *f-nonparametric premisses.*) Let β be an antecedent and let y be a constituent of β. Then closure under embedding in a larger f-parametric context requires that

$$\text{(9)}\qquad \frac{\delta_m \vdash A_m \quad (1 \le m \le m_0) \qquad \beta(\alpha_n/\{y\}) \vdash C \quad (1 \le n \le n_0)}{\beta(\gamma/\{y\}) \vdash C}$$

also be an instance of Ru. (We note that this property, though related to those of Curry 1963, pp. 197–198, has no quite clear analogue there. Its closest cousin is the part of (r6) that speaks of "inserting" new parameters.)

Closure under Embedding property. All the rules of **L** are closed under embedding in a larger f_{L}-parametric context.

Proof. The right logical rules of **L** satisfy the condition vacuously. For the other rules, write (9) as

$$\text{(9}'\text{)}\qquad \frac{\delta_m \vdash A_m \quad (1 \le m \le m_0) \qquad \Gamma\alpha_n\Delta \vdash C \quad (1 \le n \le n_0)}{\Gamma\gamma\Delta \vdash C}.$$

Now verification of closure under embedding can be obtained by inspection of these rules, noting that, whenever (8) is an instance of a rule Ru of L, so is (9′).

We remark that there is only one rule of **L**, namely ($\rightarrow\vdash$), having instances (8) with any nonparametric premisses at all, and in that case there is only one. And there is only one rule of **L**, namely ($\vee\vdash$), having instances (8) with more than a single parametric premiss; and even in that case, there are but two.

For application we are going to need a corollary of the Closure under Embedding property, which will be an immediate consequence of a certain fact to the effect that if a rule is closed under embedding in a larger f-parametric context in the sense defined above, then it is also closed, in a sense, under a more complex sort of embedding. For statement of the fact, we define $\mathrm{Part}_{n_0}(X)$ as the set of all n_0-ary sequenced partitions of X.

FACT. Let f be an analysis-function for a system **S** of which *Ru* is a rule, and let *Ru* be closed under embedding in a larger f-parametric context. For each antecedent β and nonempty set of occurrences X in β, if (8) is an instance of *Ru*, then (10) below is also in a wider sense; that is, the conclusion of (10) may be obtained from the premisses of (10) by a *series* of one or more applications of *Ru*.

$$\frac{\delta_m \vdash A_m \quad (1 \leq m \leq m_0) \qquad \beta(\alpha_n/Y_n)_{n=1}^{n_0} \vdash C \quad (Y \in \mathrm{Part}_{n_0}(X))}{\beta(\gamma/X) \vdash C}. \tag{10}$$

The notation is intended to suggest that, in addition to the m_0 f-nonparametric premisses that come over unchanged from (8), there is a premiss $\beta(\alpha_n/Y_n)_{n=1}^{n_0} \vdash C$ for each member Y of the set Part_{n_0} of n_0-ary sequential partitions of X.

We note that, in the special case $n_0 = 1$—i.e., when there is only one f-parametric premiss—(8) and (10), respectively, assume the simpler forms

$$\frac{\delta_m \vdash A_m \quad (1 \leq m \leq m_0) \qquad \alpha \vdash C}{\gamma \vdash C} \tag{8'}$$

$$\frac{\delta_m \vdash A_m \quad (1 \leq m \leq m_0) \qquad \Gamma_1\alpha\Gamma_2\alpha\Gamma_3 \ldots \Gamma_{n-1}\alpha\Gamma_n \vdash C}{\Gamma_1\gamma\Gamma_2\gamma\Gamma_3 \ldots \Gamma_{n-1}\gamma\Gamma_n \vdash C}. \tag{10'}$$

Among rules of **L** having instances of the form (8) there is only one, namely, ($\vee\vdash$) not falling under the special case (8′)–(10′) of closure under embedding; and even then n_0 is but 2.

PROOF. By simple induction on the cardinality of X. If there is but one member in X, then (10) = (9) and the hypothesis of the Fact suffices. Suppose that the Fact is true for X with q members; we show that it continues to hold for X with $q+1$ members. Choose $y \in X$, and rewrite the f-parametric premisses of (10) in n_0 batches according to which of the sets $Y_1, \ldots, Y_{n_0}$ contains y:

$$(11) \qquad \beta(\alpha_1/\{y\}, \alpha_1/(Y_1 - \{y\}), (\alpha_i/Y_i)_{i \neq 1}) \vdash C$$
$$(y \in Y_1 \text{ and } Y \in \mathrm{Part}_{n_0}(X))$$
$$\vdots$$
$$\beta(\alpha_{n_0}/\{y\}, \alpha_{n_0}/(Y_{n_0} - \{y\}), (\alpha_i/Y_i)_{i \neq n_0}) \vdash C$$
$$(y \in Y_{n_0} \text{ and } Y \in \mathrm{Part}_{n_0}(X)).$$

For each $n\,(1 \leqslant n \leqslant n_0)$, consider the nth batch of premisses, drawn from (11),

$$(11)_n \qquad \beta(\alpha_n/\{y\}, a_n/Y_n - \{y\}, (\alpha_i/Y_i)_{i \neq n} \vdash C$$
$$(y \in Y_n \text{ and } Y \in \mathrm{Part}_{n_0}(X)).$$

These may be rewritten

$$(11')_n \qquad \beta(\alpha_n/\{y\}, \alpha_n/Z_n, (\alpha_i/Z_i)_{i \neq n} \vdash C \quad (Z \in \mathrm{Part}_{n_0}(X - \{y\})),$$

that is,

$$(11'')_n \qquad \beta(\alpha_n/\{y\}, (\alpha_i/Z_i)_{i=1}^{n_0}) \vdash C \quad (Z \in \mathrm{Part}_{n_0}(X - \{y\})).$$

The cardinality of $X - \{y\}$ is q; so, by the hypothesis of the induction, we may obtain from the premisses $(11'')_n$, together with the m_0 nonparametric premisses of (10), by means of a series of applications of *Ru*, the following:

$$(12)_n \qquad \beta(\alpha_n/\{y\}, \gamma/(X - \{y\})) \vdash C.$$

We do this for each $n\,(1 \leq n \leq n_0)$. Now we know by hypothesis that *Ru* is closed under embedding in a larger *f*-parametric context; consequently, from all of the consecutions $(12)_n\,(1 \leq n \leq n_0)$ together with the nonparametric premisses of (10), we can obtain by one further application of *Ru*

$$\beta(\gamma/\{y\}, \gamma/(X - \{y\})) \vdash C,$$

which is just the conclusion of (10), as desired, and which finishes the proof of the Fact.

COROLLARY. Every rule *Ru* of **L** is closed under embedding in the wider sense that if (8) is an instance of *Ru* then one can obtain the conclusion of (10) from its premisses by a series of zero or more applications of *Ru*.

PROOF. Immediate from the Closure under Embedding property and the Fact.

We are now in a position to deal with Case 2, the hypothesis of which is that at least one of the derivations of the L-premisses has a consequent rank of at least 2. Choose one of these whose consequent rank is the maximum, k, and for notational convenience (only) let us pretend we have chosen the derivation Der_1 of the *first* L-premiss, $\gamma_1 \vdash M$. Let Der_1 terminate in an inference

$$(13) \qquad \frac{\beta_m \vdash A_m \quad (1 \leq m \leq m_0) \qquad \alpha_n \vdash M \quad (1 \leq n \leq n_0)}{\gamma_1 \vdash M}$$

with the nonparametric premisses (if any) collected on the left and the parametric premisses—there must by the case hypothesis be at least one—collected on the right. Consider the $n_0 + p_0 - 1$ premisses

$$\alpha_n \vdash M \quad (1 \leq n \leq n_0) \qquad \gamma_p \vdash M \quad (2 \leq p \leq p_0)$$

and choose an arbitrary n_0-ary sequential partition Z of the set Y_1. When put together with the R-premiss, either the Middle hypothesis (if the number i of left derivations with maximum consequent rank is 1) or the Inner hypothesis (otherwise) will justify our claim that the following are all provable:

$$(14) \qquad \delta((\alpha_n/Z_n)_{n=1}^{n_0}, (\gamma_p/Y_p)_{p=2}^{p_0}) \vdash D \quad (Z \in \mathrm{Part}_{n_0}(Y_1)).$$

Now we argue as follows. In the first place, since (13) is an instance of some rule *Ru* of **L**, so also is the result of substituting D for the exhibited parametric M—by the Parametric Substitution Closure property; i.e., we have

$$\frac{\beta_m \vdash A_m \quad (1 \leq m \leq m_0) \qquad \alpha_n \vdash D \quad (1 \leq n \leq n_0)}{\gamma_1 \vdash D}$$

as an instance of *Ru*, and with the sorting into parametric and nonparametric premisses unchanged (by the "furthermore" clause). Consequently, by the Corollary to the Closure under Embedding property, the following consecution can be obtained from the premisses (14) by a series of one or more applications of *Ru*:

$$\delta(\gamma_1/Y_1, (\gamma_p/Y_p)_{p=2}^{p_0}) \vdash D.$$

But this is just Conclusion; which completes the proof of the Elimination theorem and of the equivalence of **L** to $\mathbf{R}_{+}^{\Box\circ t}$.

§62. Display logic. This section provides a type of Gentzen consecution calculus for various relevance logics. We begin by putting the matter in context.

§62.1 Introduction. In §7.3 we offered "merge-style" consecution calculuses (see §7.2 for this terminology) for a variety of pure implicational relevance logics. Then we found other styles of consecution calculuses for implication-negation fragments in §13, and in §17 for the fragment, common to several relevance logics, consisting of just entailments between truth-functional formulas. Then in §28.5 we provided a consecution calculus for $\mathbf{R}_+$ with the vocabulary $\{\rightarrow, \&, \vee, \circ, t\}$, and in §61 for $\mathbf{R}_+$ with vocabulary $\{\rightarrow, \&, \vee, \circ, t, \Box\}$. But, in order to appreciate what *has* been done, it is best to be clear on what has *not*.

1. We have *not* provided a consecution calculus for $\mathbf{R}_+$ with just the standard vocabulary $\{\rightarrow, \&, \vee\}$ of §27.1.1. This limitation on our results is indeed a limitation the overcoming of which constitutes an open problem, but we think the limitation minimal since (a) $A \circ B$ is definable (not in $\mathbf{R}_+$ but) in **R** as $\sim(A \rightarrow \sim B)$, and (b) adding $\boldsymbol{t}$ to **R** has always seemed to us conceptually innocuous (see §27.1.2), perhaps because $\boldsymbol{t}$ is definable in $\mathbf{R}^{\forall\exists p}$ by $\forall p(p \rightarrow p)$. It helps that §45.1 shows that the addition of $\boldsymbol{t}$ to **R** is conservative, and that the semantic techniques of §48 show that the addition of $\circ$ to **R** is conservative as well—as announced in §28.3.1.

2. We have *not* provided a consecution calculus for **R**, which includes negation, in any vocabulary.

3. We have *not* provided consecution calculuses for $\mathbf{E}_+$ or $\mathbf{T}_+$, *a fortiori* not for **E** or **T**, in any vocabulary.

What we do in this section is provide a consecution calculus for each of **R**, **T**, and **E**, including negation. We provide these calculuses, however, only by going outside the standard vocabularies for **R**, **T**, and **E**: to the standard $\{\rightarrow, \&, \vee, \sim\}$ we are forced to add not only $\circ$ and $\boldsymbol{t}$ (see 1 above), but also $\boldsymbol{T}$, and $\sim_b$, where $\boldsymbol{T}$ is the disjunction of all propositions and $\sim_b$ is Boolean negation.

The addition of $\boldsymbol{T}$, which can certainly be made conservatively, was already contemplated in §27.1.2. As in the case of $\boldsymbol{t}$ (see 1 above), the addition seems to us more or less conceptually innocuous, not just because it is conservative, but because in $\mathbf{R}^{\forall\exists p}$ (and $\mathbf{E}^{\forall\exists p}$ and $\mathbf{T}^{\forall\exists p}$), $\boldsymbol{T}$ is naturally definable as $\exists pp$ (see also §33). In contrast, the proposed addition of Boolean negation $\sim_b$, whether or not conservative, raises philosophical questions; for a discussion of these, see §80.2.

Apart from conceptual questions, the mathematical fact is that the addition of Boolean negation to **R** and to **T** is conservative (see §62.5.2 below); we may therefore describe this section as solving the problem of providing a consecution calculus for these logics—but only by adding $\{\circ, \boldsymbol{t}, \boldsymbol{T}, \sim_b\}$ to the standard vocabulary, $\{\rightarrow, \&, \vee, \sim\}$. The problem of providing consecution calculuses with smaller vocabularies remains open.

With respect to **E**, we are faced with an additional difficulty, since at this time no one knows whether or not the addition of Boolean negation $\sim_b$ to **E** is conservative. For **E**, then, we employ a somewhat different strategy, in order to be sure that it is, as we might say, "well displayed."

The problems outlined above led to the research reported in this section; in fact, however, the techniques developed in the course of their partial solution permit us to formulate a Gentzen consecution calculus for an indefinite number of logics all mixed together, including Boolean (two-valued), intuitionistic, relevance, and (various) modal logics. This is accomplished by augmenting and refining the structural ideas of Gentzen 1934. The key feature of the calculus permitting control in the presence of multiple logics is this: every "positive part" of a consecution can be displayed as the consequent, standing alone, of an equivalent consecution; and every "negative part" can be displayed as the antecedent, standing alone, of an equivalent consecution; such a calculus we call a "Display logic." The first spinoff is a generalized Elimination theorem.

§62.2 outlines the grammatical structures needed for Display logic, emphasizing the ways in which it develops themes in Gentzen. §62.3 explains the system itself in a whosesale way, but without attention to the special features of the various calculuses that Display logic can treat. In this section we divide the rules into (1) display equivalences, (2) structural rules, and (3) connective postulates. §62.4 states and proves a wholesale (cut) Elimination theorem. §62.5 discusses in more detail how various logics appear in the context of Display logic. §62.6 elaborates a series of possibilities.

§62.2 Grammar. The grammar of any consecution calculus is complex (though the complexity is frequently played down by means of reliance on geometrical intuitions; see §62.6.8 for a related point), and, because we are treating so many logics simultaneously, the grammar of Display logic is even more so.

§62.2.1. Indices and families. We need a way of distinguishing the connectives of one logic from those of another; to take a familiar example, there is the implication of Boolean logic, and there is the strict implication of **S4**. To distinguish the similar connectives of these logics, and others, we postulate a set of *indices*—the idea, in first approximation, being that each logic shall be associated with a distinct index. To motivate the second approximation, we should recognize that some well-known logics are "hybrid" in the sense that they involve connectives of more than one kind; the easiest example is a modal logic, which treats both modal connectives and Boolean connectives. For this reason, we shall associate indices not with historically given "logics" *per se*, but with *families* of connectives. For example, to treat **S4** we shall need indices for two families, the modal **S4** family for its modal connectives and the Boolean family for its Boolean connectives.

Because many features of Display logic are independent of which families are considered, we shall be vague about precisely which indices are included in Display logic. We shall, however, assume that the following indices are present, so that we can use them as running examples.

EXAMPLE 2-1. (INDICES.)

Index	Family
b	Boolean (for two-valued logic)
s4	modal family for **S4** (its modal connectives)
r	relevance family (for relevance connectives of **R**)
h	intuitionist family ("h" for Heyting)
e	entailment

Names for other specific indices, or variables ranging over all indices, can be introduced *ad hoc.*

§62.2.2 Formula-connectives and structure-connectives. Apart from the multiplicity engendered by the families, the connectives, as (implicitly) in Gentzen, are of two kinds: *formula-connectives*, which take formulas into formulas, and *structure-connectives*, which take structures (analogues of Gentzen's sequences) into structures. The idea of multiple families of *formula*-connectives is old; the idea of characterizing these connectives by using multiple families of *structure*-connectives is due to Minc 1972 and Dunn 1973.

There are in each family several formula-connectives, which is usual, and several structure-connectives, which is not, or not very. Consider Gentzen's "L" calculuses (§7.1). There structure is carried by commas (their meaning is context-sensitive, since they signify conjunction on the left and disjunction on the right; they are of no fixed polyadicity) and by the empty symbol (its meaning is also context-sensitive, since it signifies truth on the left and falsity on the right). Here there are three ways in which these structural ideas are developed.

1. Positive structuring, carried by a structure-connective $\circ$, is always binary, never, as with Gentzen's commas, of no fixed polyadicity. The meaning of $\circ$ remains, like Gentzen's comma, context-sensitive. See §62.6.8 for further discussion of the matter.
2. There is negative structuring *; it is one-place and is essential in realizing the "display" feature.
3. There is a zero-place item of structure, I, which replaces the empty symbol of Gentzen 1934. This item carries a heavy burden in accommodating multiple families and is specially useful in connection with modal logics. Its meaning, too, is context-sensitive.

To repeat: for each family, there are several formula-connectives and several structure-connectives. We catch this idea by postulating a list of *generic connectives*, each of which is a function that maps each family-index into a specific connective of the family associated with that index. With each of these generic connectives is associated a fixed number of *places*. From among the many possibilities for generic connectives, we choose the following (others are added later).

Generic connective	Type of arguments and values	Places	Approximate reading
t	formula	0	truth
f	formula	0	falsity
$\sim$	formula	1	negation
$\Box$	formula	1	necessity
$\Diamond$	formula	1	possibility
&	formula	2	conjunction
$\vee$	formula	2	disjunction
$\rightarrow$	formula	2	implication
I	structure	0	truth/falsity
*	structure	1	negation
$\circ$	structure	2	conjunction/disjunction

Warning: the given "reading" must be understood in context; for the properties of the connectives differ drastically from family to family.

EXAMPLE 2-2. (FAMILIES AND LOGICS.) Classical two-valued logic is formulated using a single family, the Boolean family. For example, $\sim_b$ is Boolean negation, and $\rightarrow_b$ is material implication. $\Box_b$ in this family is just the identity connective, as is $\Diamond_b$. See §62.5.1 below.

The logic **S4** is hybrid, including both the Boolean and the **S4** families: the extensional connectives belong to the former, the modal connectives to the latter. See §62.5.6 below. For example, a typical distribution law would relate $\Box_{s4}(A\rightarrow_b B)$ and $\Box_{s4}A\rightarrow_b\Box_{s4}B$.

The logic **R** uses both the Boolean and the relevance families. The "standard" vocabulary of the formulation of §27.1.1 (see also §R2) involves negation, implication, conjunction, and disjunction; in Display logic, these come through as $\sim_r$ and $\rightarrow_r$ from the relevance family, and $\&_b$ and $\vee_b$ from the Boolean family; see §62.5.2. below. For example, we shall want to count $(A\&_b B)\rightarrow_r A$ as a logical truth, but not $(A\&_r B)\rightarrow_r A$. We note that $\&_b$ is what is written as just "&" elsewhere in this book, and that $\&_r$ is what is elsewhere written as "$\circ$"; given our purposes, such terminological variation seems inescapable. Furthermore, note that here we are using "$\circ$" as an ambiguous structural connective.

Intuitionism is a one-family logic; intuitionist negation, however, is represented in Display logic not by $\sim_h$, but instead by another negation connective, $\neg_h$, introduced below in section §62.3.3. See §62.5.7 for details.

In order to avoid indices as much as possible, it is convenient to use the name of each generic connective also as a variable ranging over the specific connectives obtained by applying that generic connective to an index. Thus, "$\circ$" ranges over $\circ_b$, $\circ_{s4}$, etc. One more convention with the same purpose of reducing explicit mention of indices: whenever several names of generic connectives are used together in this way as variables, unless there is special indication to the contrary, they are to be taken as "ranging in tandem"; that is, they are all to be taken as denoting connectives (formula-connectives or structure-connectives) of the same family.

§62.2.3. Formulas, structures, and consecutions. There is a set of *variables*, among which some are distinguished as h-*variables* and some as e-*variables*. The former play a special role in the intuitionist family, the latter in the entailment family.

Formulas are defined as usual: a variable is a formula, and so is the result of applying a formula-connective of any family to an appropriate number of arguments.

Structures are defined inductively: a formula is a structure, and so is the result of applying a structure-connective of any family to the appropriate number of arguments. A substructure (in the obvious sense) of a structure is *positive* or *negative* according as it is inside an even or odd number of *s. We call "⊢" the *turnstile*, as in §7.2. If X and Y are structures, then X ⊢ Y is a *consecution*. X is its *antecedent*, and Y is its *consequent*. (As in §28.5 and §61, the locution "⊢ Y" is not admitted; here its work is done by "***I*** ⊢ Y", using the structural element ***I*** instead of the propositional constant ***t***.) An *antecedent* {*consequent*} *part* of a consecution is defined as a positive part of its antecedent or negative part of its consequent {positive part of its consequent or negative part of its antecedent}.

A *constituent* of a structure or consecution is an occurrence of a structure therein.

Variables are reserved as follows.

A, *B*, *C*, *M*	formulas
W, X, Y, Z	structures
S	consecutions

§62.2.4. Interpretation. Display logic is essentially a proof-theoretical tool; but some relations to semantic concepts are spelled out in §62.5. Furthermore, there are or can be many families, with a distinct "interpretation" for each, so all that is possible here is an indication of how the various elements of any one family are related. Each structural constituent of a consecution has an interpretation as a formula, depending on whether it is an antecedent part or a consequent part: ***I*** is interpreted as ***t*** when an antecedent part and ***f*** when a consequent part; * is always interpreted as negation ~; and $\circ$ is interpreted as & when an antecedent part and ∨ when a consequent part (all with matching markings). In other words, we define two functions a(X) and c(X) from structures to formulas as follows.

$a(A) = c(A) = A$ (where A is any formula)

$$\begin{array}{ll} a(\boldsymbol{I}) = \boldsymbol{t} & c(\boldsymbol{I}) = \boldsymbol{f} \\ a(X^*) = {\sim}c(X) & c(X^*) = {\sim}a(X) \\ a(X \circ Y) = a(X) \& a(Y) & c(X \circ Y) = c(X) \vee c(Y) \end{array}$$

Then the consecution $X \vdash Y$ is interpreted as saying that the formula a(X) implies the formula c(Y), where "implies" is here a family-independent concept. Warning: keep in mind that the interpretation of $\boldsymbol{I}$, as well as of $\circ$, is context-dependent (like the empty symbol, as well as commas, in Gentzen's calculus); it does *not* behave like a formula.

Furthermore, in each family $\rightarrow$, $\Box$, and $\Diamond$ may be interpreted as follows: $A \rightarrow B$ as ${\sim}A \vee B$, $\Box A$ as $\boldsymbol{t} \rightarrow A$, and $\Diamond A$ as ${\sim}\Box{\sim}A$. This leaves the following "kernel" connectives of each family as requiring explanation:

Kernel connectives: $\boldsymbol{t}$, $\boldsymbol{f}$, $\sim$, &, $\vee$.

Equivalently, and in some respects more perspicuously, one must explain the structural connectives: $\boldsymbol{I}$ in antecedent and consequent, *, and $\circ$ in antecedent and consequent. We repeat: these do not always have their usual meanings.

§62.3. Postulates for DL. We are setting out to construct a logic to be called "**DL**" for "Display logic." We will divide its postulates into four categories: (1) identity axioms, (2) display-equivalences, (3) connective postulates, and (4) structural postulates.

§62.3.1. Identity axioms. There is one schema:

$A \vdash A$

where A is any variable.

§62.3.2. Display-equivalence. The essence of Display logic is that, for *every* family, there are postulated certain bidirectional rules which allow any consequent {antecedent} part of a consecution S to be displayed as the entire consequent {antecedent} itself of a consecution equivalent to S (Theorem 3-2). Consecutions listed below are defined as *display-equivalent*$_1$ to the others listed on the same line. (Here and always when we state rules, we are supposing that indices match on those connectives which are explicitly presented.)

$$\begin{array}{lll} X \circ Y \vdash Z & X \vdash Y^* \circ Z & \\ X \vdash Y \circ Z & X \circ Y^* \vdash Z & X \vdash Z \circ Y \\ X \vdash Y & Y^* \vdash X^* & X^{**} \vdash Y \end{array}$$

Display-equivalence is defined as the reflexive, transitive, and symmetric closure of display-equivalence$_1$; that is, display-equivalent consecutions are those which are mutually inferable by application of the above bidirectional rules. Note that commutativity is postulated as a display-equivalence for $\circ$ only in the consequent.

FACT 3-1. $Y \vdash (X \circ Z^*)^*$ may be added as a display-equivalence to the first line, $X \circ Z^* \vdash Y$ to the second, and $X \vdash Y^{**}$ to the third. Also, items on the same line below are display-equivalent:

$$\begin{array}{ll} X^* \vdash Y & Y^* \vdash X \\ X \vdash Y^* & Y \vdash X^* \end{array}$$

PROOF. By diddling.

THEOREM 3-2. (DISPLAY THEOREM.) Each antecedent part X of a consecution S can be displayed as the antecedent (itself) of a display-equivalent consecution $X \vdash W$; and the consequent W is determined only by the position of X in S, not by what X looks like. Similarly for consequent parts of S.

That is, let f be an $(n+1)$-ary operation on structures definable by (only) composition from the structure-connectives of the various families. Let the first argument of f always appear as a positive substructure of the result of applying f to $n+1$ arguments. For each such f there is a composition-definable f′ such that, for all $X, Y_1, \ldots, Y_n$, and Z, the following are display-equivalent:

$$f(X, Y_1, \ldots, Y_n) \vdash Z \qquad X \vdash f'(Z, Y_1, \ldots, Y_n).$$

Analogously for the first argument of f negative, and for $Z \vdash f(X, Y_1, \ldots, Y_n)$, with X positive, or negative.

PROOF. Let X be an antecedent {consequent} part of S. Unless X is already the antecedent {consequent} of S, the "route" to X in S can only be through exactly one of $\circ$ in antecedent or consequent, or $*$ in antecedent or consequent. And, in the former cases, X must be either in the left or in the right. The display-equivalences postulated, or noted in Fact 3-1, suffice for all cases.

To *display* X is to find $X \vdash f'(Z, Y_1, \ldots, Y_n)$, or similarly with X as consequent. The possibility of displaying X is the defining feature of Display logic.

EXAMPLE. The constituent X (second occurrence) in a consecution $(X \circ Y)^* \circ (Z^* \circ X^*) \vdash W \circ Y$ can be displayed as consequent as follows by using the display-equivalences postulated or noted in Fact 3-1:

$$Z^* \circ X^* \vdash ((X \circ Y)^* \circ (W \circ Y)^*)^*;$$
$$Z^* \vdash X^{**} \circ ((X \circ Y)^* \circ (W \circ Y)^*)^*;$$
$$Z^* \circ ((X \circ Y)^* \circ (W \circ Y)^*)^{**} \vdash X^{**};$$
$$Z^* \circ ((X \circ Y)^* \circ (W \circ Y)^*)^{**} \vdash X.$$

§62.3.3. Connective postulates. There is as usual a pair of rules for each formula-connective. The Display theorem 3-2, however, allows us to insist as a special feature that the formula with the newly introduced formula-connective stand alone. It is a further striking feature of Display logic that the *same* set of formula-connective postulates is used for *every* family. This is Display logic's own way of making sense out of everyone's sense of family resemblance.

Subscripts on all formula-connectives and structure-connectives in the statement of a given rule are supposed to match.

$$(\boldsymbol{t}) \qquad \boldsymbol{I} \vdash \boldsymbol{t} \qquad\qquad \frac{\boldsymbol{I} \vdash X}{\boldsymbol{t} \vdash X}$$

$$(\boldsymbol{f}) \qquad \frac{X \vdash \boldsymbol{I}}{X \vdash \boldsymbol{f}} \qquad\qquad \boldsymbol{f} \vdash \boldsymbol{I}$$

$$(\&) \qquad \frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \& B} \qquad\qquad \frac{A \circ B \vdash X}{A \& B \vdash X}$$

$$(\vee) \qquad \frac{X \vdash A \circ B}{X \vdash A \vee B} \qquad\qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \circ Y}$$

$$(\rightarrow) \qquad \frac{X \circ A \vdash B}{X \vdash A \rightarrow B} \qquad\qquad \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X^{*} \circ Y}$$

$$(\sim) \qquad \frac{X \vdash A^{*}}{X \vdash \sim A} \qquad\qquad \frac{A^{*} \vdash X}{\sim A \vdash X}$$

$$(\neg) \qquad \frac{X \circ \boldsymbol{I} \vdash B^{*}}{X \vdash \neg B} \qquad\qquad \frac{B^{*} \vdash X}{\neg B \vdash \boldsymbol{I}^{*} \circ X}$$

"$\neg$" is in practice used only for intuitionist negation.

$$(\Box) \qquad \frac{X \circ \boldsymbol{I} \vdash A}{X \vdash \Box \mathrm{A}} \qquad\qquad \frac{A \vdash X}{\Box A \vdash \boldsymbol{I}^{*} \circ X}$$

$$(\Diamond) \qquad \frac{X \vdash A}{(X^{*} \circ \boldsymbol{I}^{*})^{*} \vdash \Diamond A} \qquad\qquad \frac{X^{*} \circ A \vdash \boldsymbol{I}^{*}}{\Diamond A \vdash X}$$

As an example of how the Display theorem 3-2 interacts with these postulates, we note the derivability of the form of the rule for → on the left of §28.5.1 and §62.1: from (1) $X \vdash A$ and (2) $\ldots B \ldots \vdash Z$ to infer $\ldots((A \rightarrow B) \circ X) \ldots \vdash Z$. First write (2) as $f(B) \vdash Z$, and, for the f' promised by the Display theorem, get $B \vdash f'(Z)$. Then $(A \rightarrow B) \vdash X^{*} \circ f'(Z)$ by the rule (→); $((A \rightarrow B) \circ X) \vdash f'(Z)$, by display-equivalence; and $f((A \rightarrow B) \circ X) \vdash Z = \ldots (A \rightarrow B) \circ X \ldots \vdash Z$, by the Display theorem.

THEOREM 3-3. (IDENTITY THEOREM.) Though in §62.3.1 $A \vdash A$ is postulated only for variables A, in fact $A \vdash A$ is provable for all formulas A.

We take the Identity theorem to constitute half of what is required to show that the "meaning" of formulas in Display logic is not context-sensitive, but that instead formulas "mean the same" in both antecedent and consequent position. (The Elimination theorem 4-4 below is the other half of what is required for this purpose.)

§62.3.4. Reduction. As we have seen, the postulates governing formula-connectives are the same for all families; in contrast, families differ in regard to their structural postulates: from among all the possible structural rules one might think of, some are postulated for this family, some for that. At this point to say which structural rules are postulated for which families would only get in our way, since the central theorems of §62.4.3 go through in total independence of this decision, as long as the choice of structural postulates satisfies quite general conditions elaborated below in §62.4.2. Consequently, we shall first list a set of potentially adoptable rules, all of which satisfy these conditions; then we will give a few examples of which rules are postulated for which families, leaving it pretty much up in the air what the full story is. (As usual, all indices of explicitly presented connectives must match.) Some of the names of the rules come via Meyer and Routley 1972 from those for the combinators in Curry and Feys 1958; some do not correspond to any combinator.

$$(\mathrm{I}+)\quad \frac{\mathrm{X} \vdash \mathrm{Y}}{\boldsymbol{I} \circ \mathrm{X} \vdash \mathrm{Y}}$$

(I+/e) Like (I+) but X must be an e-variable (§62.2.3)

$$(\mathrm{I}-)\quad \frac{\boldsymbol{I} \circ \mathrm{X} \vdash \mathrm{Y}}{\mathrm{X} \vdash \mathrm{Y}}$$

$$(\mathrm{I}-\mathrm{K})\quad \frac{\boldsymbol{I} \vdash \mathrm{Y}}{\mathrm{X} \vdash \mathrm{Y}}$$

$$(\mathrm{I}^*+)\quad \frac{\boldsymbol{I} \vdash \mathrm{X}}{\boldsymbol{I}^* \vdash \mathrm{X}} \qquad\qquad (\mathrm{I}^*-)\quad \frac{\boldsymbol{I}^* \vdash \mathrm{X}}{\boldsymbol{I} \vdash \mathrm{X}}$$

$$(\mathrm{B})\quad \frac{\mathrm{W} \circ (\mathrm{X} \circ \mathrm{Y}) \vdash \mathrm{Z}}{(\mathrm{W} \circ \mathrm{X}) \circ \mathrm{Y} \vdash \mathrm{Z}} \qquad\qquad (\mathrm{B}')\quad \frac{\mathrm{W} \circ (\mathrm{X} \circ \mathrm{Y}) \vdash \mathrm{Z}}{(\mathrm{X} \circ \mathrm{W}) \circ \mathrm{Y} \vdash \mathrm{Z}}$$

$$(\mathrm{C})\quad \frac{(\mathrm{W} \circ \mathrm{X}) \circ \mathrm{Y} \vdash \mathrm{Z}}{(\mathrm{W} \circ \mathrm{Y}) \circ \mathrm{X} \vdash \mathrm{Z}} \qquad\qquad (\mathrm{CI})\quad \frac{\mathrm{X} \circ \mathrm{Y} \vdash \mathrm{Z}}{\mathrm{Y} \circ \mathrm{X} \vdash \mathrm{Z}}$$

$$(\mathrm{C/h})\quad \frac{(\mathrm{W} \circ \mathrm{X}) \circ \mathrm{Y} \vdash \mathrm{Z}}{(\mathrm{W} \circ \mathrm{Y}) \circ \mathrm{X} \vdash \mathrm{Z}} \qquad\qquad (\mathrm{CI/I})\quad \frac{\mathrm{X} \circ \boldsymbol{I} \vdash \mathrm{Y}}{\boldsymbol{I} \circ \mathrm{X} \vdash \mathrm{Y}}$$

Restriction on (C/h): Y must be an h-variable (see §62.2.3).

(C∘) $\dfrac{W\circ(X\circ Y)\vdash Z}{X\circ(W\circ Y)\vdash Z}$

(W) $\dfrac{(X\circ Y)\circ Y\vdash Z}{X\circ Y\vdash Z}$ (WI) $\dfrac{X\circ X\vdash Y}{X\vdash Y}$ (⊢WI) $\dfrac{X\vdash Y\circ Y}{X\vdash Y}$

(W∘) $\dfrac{X\circ(X\circ Y)\vdash Z}{X\circ Y\vdash Z}$

(KI) $\dfrac{Y\vdash Z}{X\circ Y\vdash Z}$ (K) $\dfrac{X\vdash Z}{X\circ Y\vdash Z}$ (⊢K) $\dfrac{X\vdash Y}{X\vdash Y\circ Z}$

(KI ⊢ WI) $\dfrac{Y^*\vdash Y}{X\vdash Y}$ (K⊢WI) $\dfrac{X\circ Z^*\vdash Z}{X\circ Y\vdash Z}$

(Brw) $\dfrac{X\circ Y\vdash \boldsymbol{I}^*}{Y\circ X\vdash \boldsymbol{I}^*}$

We have not taken the rules I*+ and I*− as display-equivalences, but we have as a matter of fact postulated them for every family we have considered. On the other hand, we have avoided relying on these rules in any interesting way by using ***I**** instead of ***I*** in rules like (Brw) above; so as far as the families we have defined are concerned, the rules I*+ and I*− are useful only in showing that ***f*** and ***t*** are interchangeable. Only in Belnap 1990 does it become important that the underlying structure of Display logic permits the two to be distinguished.

For some purposes, e.g., annotations of multiple-family proofs, it is convenient to index rules as well as connectives; thus, the rule K_b sanctions the inference from $X\vdash Z$ to $X\circ_b Y\vdash Z$, whereas (say) K_r would sanction the inference from $X\vdash Z$ to $X\circ_r Y\vdash Z$. (The former is a rule of **DL**; as we see below, the latter is not postulated for **DL**.) But we avoid this indexing as much as possible.

In §62.5 below there is a full (but hardly complete) discussion of which of these rules are postulated for which families; in the meantime, here is a brief indication for the four families of Example 2-1.

EXAMPLE 3-4. (REDUCTION EXAMPLES.) For the Boolean family, the following rules are postulated, understood as applying when the displayed connectives are indexed with "b": I+, I−, I*−, K, W. See §62.5.1 below. For the **S4** family, indexed with "s4", one has KI, KI ⊢ WI, C∘, W∘, I*+, I*−, I−, CI/I, W, B (or B′). See §62.5.6 below. For the relevance family, indexed with "r", one has I+, I−, I*+, I*−, B, C, W. See §62.5.2 below. For the intuitionist family, indexed with "h", one has all the same postulates as for the **S4** family (but of course governing connectives indexed with "h" instead of "s4") together with C/h. See §62.5.7 below.

Suppose that it is settled which structural rules have been postulated for which families. Then we can say that the premisses of postulated rules *reduce*$_1$ to their respective conclusions. *Reduction* is then the reflexive and transitive closure of display-equivalence together with reduction$_1$. For example, X ⊢ Z reduces$_1$ to $X \circ_b Y \vdash Z$, but not to $X \circ_r Y \vdash Z$, since K is postulated for the b-family, but not for the r-family (in the alternative language introduced above, we postulate K_b but not K_r).

A relation of *reduction for structures* can be piggy-backed on that for consecutions. Definitions: X a⇒ X′ just in case X ⊢ Y reduces to X′ ⊢ Y, for all Y; that is, just in case X reduces to X′ in antecedent position. X c⇒ X′ just in case Y ⊢ X reduces to Y ⊢ X′, for all Y; that is, just in case X reduces to X′ in consequent position. X a⇔ X′ if both X a⇒ X′ and X′ a⇒ X; and similarly for X c⇒ X′; X⇔X′ just in case both X a⇔ X′ and X c⇔ X′.

FACT 3-5. (REDUCTION ANYWHERE.) Let X be an antecedent {consequent} part of S, and let S′ result from putting X′ for X in S. Then X a⇒ X′ {X c⇒ X′} implies that S reduces to S′. Hence, if X a⇔ X′ {X c⇔ X′, X⇔X′}, then X and X′ are interreplaceable in any antecedent context {any consequent context, any context}.

PROOF. The Display theorem 3-2 suffices. Pictorially: use the display-equivalences to move X to explicit antecedent {consequent} position; change it to X′; then use the reverse sequence of display-equivalences to put X′ where X was.

Except for specifying precisely which reduction rules go with which families, this completes the definition of the calculus: the identity axioms, the display-equivalences, and the formula-connective postulates, all of which are common to all families; and the separate reduction rules for each family. Call the system **DL** for "Display logic;" and let **DL**{*i*, *j*, . . .} be **DL** with its grammar restricted to the families associated with indices *i*, *j*, ...

§62.4. Subformula and elimination theorems. The Subformula theorem states that derivations are confined to subformulas of their conclusions (see §7.2); it is easy to verify, but the job is nevertheless done explicitly below.

The Elimination theorem (§7.2) states that the Elimination rule is admissible, where the Elimination rule appropriate for **DL** is the following:

$$\text{(ER)}\quad \frac{(1)\ \ \mathrm{X} \vdash M \qquad (2)\ \ M \vdash \mathrm{Y}}{(3)\ \ \mathrm{X} \vdash \mathrm{Y}}$$

(Henceforth a reference to *M* in (1) or (2) refers to the displayed occurrence.)

Evidently, for any family you like, (ER) suffices for the representation in **DL** of the rule of *modus ponens* in the form, from $\boldsymbol{I} \vdash A \rightarrow B$ and $\boldsymbol{I} \vdash A$ to infer

$I \vdash B$, provided the structural rules postulated for the family imply that $I \circ I \ \mathrm{a}{\Rightarrow}\ I$; for $(A \to B) \circ A \vdash B$ is available in every family.

It is worthwhile to note that (ER) can be strengthened in various ways in the direction of Gentzen's "mix"—but not in every way. First, there could be many M's in (2), and they could be located anywhere, as long as they were antecedent parts:

$$\frac{\mathrm{X} \vdash M \qquad (\ldots M \ldots M \ldots) \vdash \mathrm{Y}}{(\ldots \mathrm{X} \ldots \mathrm{X} \ldots) \vdash \mathrm{Y}}$$

This is the form of the rule given in §28.5.2. Second, keeping the right premiss as in the above example, the M in the left premiss can be buried, as long as it is a *single* consequent part:

$$\frac{\mathrm{X} \vdash (\text{---}M\text{---}) \qquad (\ldots M \ldots M \ldots) \vdash \mathrm{Y}}{(\ldots \mathrm{X}' \ldots \mathrm{X}' \ldots) \vdash \mathrm{Y}}$$

where $\mathrm{X}' \vdash M$ is display-equivalent to $\mathrm{X} \vdash (\text{---}M\text{---})$, as promised by the Display theorem 3-2. Also, duals of these and certain other small generalizations are easily available. In contrast, however, the following premisses

$$\frac{\mathrm{X} \vdash (\text{---}M\text{---}M\text{---}) \qquad (\ldots M \ldots M \ldots) \vdash \mathrm{Y}}{?}$$

have no conclusion in the general case—as far as we know. Observe that it is the Display theorem 3-2 which allows us to define a calculus for which the admissibility of our simple-minded (ER) is directly provable, without recourse to "mix" or the like.

The Subformula and Elimination theorems are proved in three parts. First we give an analysis of **DL** that defines the notions of parameter and congruence. Then we state eight conditions on such an analysis, and verify that **DL** in fact satisfies these conditions. Finally, we show that any calculus admitting an analysis satisfying these conditions must support the Subformula and Elimination theorems.

§62.4.1. Analysis, parameter, congruence. Think of a consecution calculus, for example, **DL**, as being determined by a family of *rules*, under each of which falls a family of *inferences*, each inference with finitely many, possibly zero, premisses (see §28.5.3). It appears that in any consecution-calculus inference there are some constituents that are "held constant" when passing from the premisses to the conclusion—these are called the "parameters"—and, among these so-called "parameters," some occurring in the premisses are "identified" with some in the conclusion—a relationship called "congruence." For some rules, e.g., the typical connective rules, there seems only one possible way of defining "parameter" and "congruence," whereas, for others, e.g., the structural axiom or the structural rules K, C, and W, there seem to be alternatives. In the terminology of this section, it is the job of an "analysis" to provide a definite decision on the matter. Abstractly put, for each inference *Inf* of a calculus, an *analysis* defines constituents of the various consecutions of *Inf* as *parametric* (for *Inf*) or not, and it defines an equivalence relation of *congruence* (for *Inf*) on the parameters of *Inf*. The *congruence class* of a constituent of *Inf* is the set of all constituents congruent to it.

Provision of such an analysis defining "parameter" and "congruence" for each inference sanctioned by its rules is the first step in showing that the consecution calculus **DL** defined in §62.3 admits rule (ER), regardless of which structural rules (drawn from §62.3.4) are postulated for which families. The following account is a bit sketchy, and convoluted by talk of "occurrences," but the idea should be clear.

Let *Inf* be an inference falling under a rule *Ru* of **DL**. *Inf* must then be obtained by assigning structures to the structure-variables and formulas to the formula-variables used in our statement of the rule *Ru*. (This is a remark about the procedure we employed above in describing the identity axioms, the display equivalences, the connective postulates, and the potentially adoptable reduction rules; it is *not* a remark about the nature of rules. See §28.5.3.)

DEFINITION 4-1. (ANALYSIS.) According to the *present analysis*, constituents occurring as part of occurrences of structures assigned to structure-variables (as they appear in our statement of the rules) are defined to be *parameters* of *Inf*; all other constituents are defined as nonparametric, including those assigned to formula-variables. Constituents occupying similar positions in occurrences of structures assigned to the same structure-variable are defined as *congruent* in *Inf*.

This definition, strictly speaking, presupposes that each inference falls under only one rule; but it doesn't matter, and here, in contrast to §61.4, we do not propose to worry about it. It is clear that congruence is an equivalence relation.

§62.4.2. Conditions on an analysis. The analysis 4-1 provided for **DL** satisfies eight crucial conditions. Our strategy will be to state and verify these eight conditions and then to prove that any consecution calculus possessing an analysis satisfying these conditions must admit the Elimination rule. The eight conditions are supposed to be reminiscent of those of Curry 1963 and should be compared with those of §28.5.3 and §61.4. In order to state the conditions, let *Ru* be one of **DL**'s rules, and let *Inf* be an inference falling under *Ru*. Recall that congruence is an equivalence.

C1. PRESERVATION OF FORMULAS. Each formula which is a constituent of some premiss of *Inf* is a subformula of some formula in the conclusion of *Inf*. That is, structure may disappear, but not formulas. In fact the rules so far given satisfy also the following:

(a) Preservation of parameters: each parameter is congruent to a constituent in the conclusion of *Inf*, with which it agrees in shape.

(b) Formula-connectives are introduced one at a time: that is, all nonparametric formulas in the premisses of *Inf* are immediate proper subformulas of a nonparametric formula in the conclusion of *Inf*. Definition: nonparametric formulas in the premisses are *components* of *Inf*.

This condition can be verified by eye. It comes to this: in our statement of the rules, no structure- or formula-variable is lost in passing from premiss to conclusion.

C2. SHAPE-ALIKENESS OF PARAMETERS. Congruent parameters are occurrences of the same structure.

This condition is inescapable, from the definition of "congruence" given by our analysis 4-1, without even peeking at the statement of the rules.

C3. NONPROLIFERATION OF PARAMETERS. Each parameter is congruent to at most one constituent in the conclusion of *Inf*; in other words (since congruence is an equivalence relation), no two constituents of the conclusion are congruent to each other. (There can be parameters that occur only in the conclusion; such a parameter is congruent only to itself.)

For verification of this condition by eye, observe that each structure-variable occurs exactly once in the conclusion of the statement of each rule. (Our argument would not apply to a calculus with the converse of the rule (WI), but it would apply to a calculus with the stronger "mingle rule": from $X \vdash Z$ and $Y \vdash Z$ to infer $X \circ Y \vdash Z$, as in §8.15.)

C4. POSITION-ALIKENESS OF PARAMETERS. Congruent parameters are either all antecedent parts or all consequent parts of their respective consecutions.

Verification by eye: this is true of each structure-variable used in our statements of the rules.

C5. DISPLAY OF PRINCIPAL CONSTITUENTS. If a formula is nonparametric in the conclusion of *Inf*, it is either the entire antecedent or the entire consequent of that conclusion. (This unusual condition is peculiar to Display logic; the Display theorem 3-2 permits us the luxury of insisting on it without our calculus's being too weak.) Definition: nonparametric formulas in the conclusion are *principal constituents* of *Inf*. Evidently there can be at most two such, but this happens only in the identity axiom $A \vdash A$; however, a formulation alternative to that of **S2** and **S3** in §62.5.6 below could involve replacing the rule ($\Box'$/I) there with an axiom $\Box' A \vdash \boldsymbol{t}$ which would also contain two principal constituents.

Verification: it is easy to see by eye that the only nonparametric formulas in conclusions are (a) the two values of A in the identity axiom, and (b) the formulas introduced by the connective postulates; these obviously satisfy the condition.

C6. CLOSURE UNDER SUBSTITUTION FOR CONSEQUENT PARAMETERS. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas that are *consequent* parts. That is, let *Inf* fall under *Ru*, and let M be a parametric *consequent* part of a consecution of *Inf*. Let *Inf'* result from putting some one structure X for all constituents of *Inf* in the congruence class of the consequent part M. Then *Inf'* also falls under *Ru*. Furthermore, constituents of the substituted X are all parametric in *Inf'*, and constituents not in substituted X are parametric or not in *Inf'* according as they are parametric or not in *Inf*.

Verification. As long as any rule is stated with the help of *unrestricted* structure-variables, this condition is bound to be satisfied. The only rules not so stated are (C/h) and (I+/e) of §62.3.4, and those rules are each all right, too, since no parametric *consequent* parts of the inferences falling under either (C/h) or (I+/e) are restricted by its "restriction." It is also clear that the "furthermore" clause is in order.

C7. CLOSURE UNDER SUBSTITUTION FOR ANTECEDENT PARAMETERS. Rules need not be wholly closed under substitution of structures for congruent formulas that are *antecedent* parts, but they must be closed enough. Let *Inf* fall under a rule *Ru*, and let M be a parametric *antecedent* part of a consecution of *Inf*. Also, let $\mathrm{X} \vdash M$ be the conclusion of an inference with M principal (not parametric). Let *Inf'* result by putting the structure X for all constituents of *Inf* in the congruence class of the antecedent part M. Then *Inf'* also falls under *Ru*. Furthermore (as in C6). (It suffices that C6 and C7 hold good for the union of all the rules of the calculus; but it is easier to verify the stronger rule-by-rule conditions stated.)

Verification. Again, rules stated without restrictions cannot fail to satisfy this condition; so we must consider only (C/h) and (I+/e). But those rules are all right too: Y in (C/h) must be an h-variable M; but structural axiom $M \vdash M$ is the only inference with principal (nonparametric) M as consequent when M is an h- (or indeed any) variable, and so C7 is trivially verified. And it is exactly the same for (I+/e).

The rules (C/h) and (I+/e) are the only examples in §62.3 of rules that are restricted and therefore interact much with C6 and C7; even they are clearly unusually simple cases. See §62.6.6 for additional and perhaps more typical examples of the use of restricted rules whose satisfaction of C6 and C7 requires careful consideration. Note in addition that C6 and C7 block structural rules that can operate only on formulas (e.g., Gentzen's own).

C8. ELIMINABILITY OF MATCHING PRINCIPAL CONSTITUENTS. If there are inferences Inf_1 and Inf_2 with respective conclusions (1) $\mathrm{X} \vdash M$ and (2) $M \vdash \mathrm{Y}$ (the premisses of (ER)) with M principal in both inferences (in the sense of C5), then either (3) $\mathrm{X} \vdash \mathrm{Y}$ (the conclusion of (ER)) is identical with one of (1) and (2), or else it is possible to pass from the premisses of Inf_1 and Inf_2 to (3) by means of inferences falling under the rules, together with the rule (ER):

$$\frac{\mathrm{X}' \vdash M' \qquad M' \vdash \mathrm{Y}'}{\mathrm{X}' \vdash \mathrm{Y}'}$$

for arbitrary X′, Y′, but with M' restricted to proper subformulas of M.

C8 needs checking in detail. The only possibilities are when (1) and (2) are both structural axioms (see §62.3.1)—but then (3) = (1) = (2) as required, and when (1) and (2) come by matching connective postulates (see §62.3.3). Verification of the latter case deserves highlighting.

FACT 4-2. When M is principal in both (1) and (2) for a connective postulate, (3) can be derived from their premisses using only (ER) on components, together with display-equivalence. In particular, none of the reduction rules need be involved.

PROOF. We provide an example:

$$\text{(ER)}\ \dfrac{(\to)\ \dfrac{\mathrm{X}\circ A \vdash B}{\mathrm{X} \vdash A \to B} \qquad \dfrac{\mathrm{Y} \vdash A \qquad B \vdash \mathrm{Z}}{A \to B \vdash \mathrm{Y}^* \circ \mathrm{Z}}}{\mathrm{X} \vdash \mathrm{Y}^* \circ \mathrm{Z}}$$

Use (ER) and display-equivalence as follows (with one premiss indicating display-equivalence, and with two indicating (ER)):

$$\dfrac{\dfrac{\mathrm{Y} \vdash A \qquad \dfrac{\dfrac{\mathrm{X}\circ A \vdash B \qquad B \vdash \mathrm{Z}}{\mathrm{X}\circ A \vdash \mathrm{Z}}}{A \vdash (\mathrm{X}\circ \mathrm{Z}^*)^*}}{\mathrm{Y} \vdash (\mathrm{X}\circ \mathrm{Z}^*)^*}}{\mathrm{X} \vdash \mathrm{Y}^* \circ \mathrm{Z}}$$

§62.4.3. Proofs of subformula and elimination theorems. There are two principal consequences of these eight conditions.

THEOREM 4-3. (SUBFORMULA THEOREM.) With respect to **DL** or any other calculus that, like **DL**, admits an analysis satisfying C1, a derivation of $X \vdash Y$ contains no formulas that are not subformulas of constituents of $X \vdash Y$.

PROOF needs no more than a citation of C1.

The interest of a Subformula theorem such as 4-3 evidently depends inversely on what might disappear, namely, the structural elements. In **DL** there are three (or five) in each family, which is a lot. Some may not be expressible in standard vocabularies, e.g., Boolean negation in intuitionism. For another example, consider the modal family for **S4**, §62.5.6, and note that one must have $X \circ Y$ holding at y just in case: Y holds at y, Rxy, and X holds at x, for some x (so $\circ$, in effect, moves us backwards). This $X \circ Y$ is not expressible in standard **S4** vocabularies; but of course is expressible in **S5**, where R is symmetric.

The presence of the extra structure, whether expressible in standard vocabularies or not, counts as a technical demerit, inasmuch as it weakens the force of the Subformula theorem 4-3; the Subformula theorem does not by itself take us very far toward decision procedures. Furthermore, there is not even a *general* guarantee that derivations of $X \vdash Y$ will contain only structure-connectives (as opposed to formula-connectives) from families already represented, through either formula- or structure-connectives, in $X \vdash Y$. Separate, piecemeal argument appears to be needed in each case (and is often available).

The presence of structural elements not expressible in standard vocabularies seems to us not, in contrast, any sort of demerit; for the standard vocabularies are defined historically, not by "logic itself." It may well be that the connectives suggested by Display logic will turn out to have their uses; see Saarinen 1978 for a suggestion about "backwards-looking operators" such as the modal connective & noted in §62.5.6.

THEOREM 4-4. (ELIMINATION THEOREM.) With respect to **DL** or any other calculus that, like **DL**, admits an analysis satisfying C2–C8, the Elimination Rule (ER) is admissible.

We follow Curry 1963 in dividing the proof into three stages, each of which may be taken as a separate lemma. For all stages, suppose as common hypothesis that C2–C8 of §62.4.2 hold.

STAGE 1. Assume as hypotheses of this stage that (H1a) $X \vdash M$ is derivable, and that (H1b), for all X′, if there is a derivation of $X' \vdash M$ ending in an inference in which displayed M is *not* parametric then $X' \vdash Y$ is derivable. Then $X \vdash Y$ is derivable.

STAGE 2. Assume as hypotheses of this stage that (H2a) $M \vdash \mathrm{Y}$ is derivable, that (H2b) for all Y', if there is a derivation of $M \vdash \mathrm{Y}'$ ending in an inference in which displayed M is *not* parametric, then $\mathrm{X} \vdash \mathrm{Y}'$ is derivable, and that (H2c) $\mathrm{X} \vdash M$ is the conclusion of some inference in which M is not parametric. Then $\mathrm{X} \vdash \mathrm{Y}$ is derivable.

STAGE 3. Assume as hypotheses of this stage that, for each of (H3a) $\mathrm{X} \vdash M$ and (H3b) $M \vdash \mathrm{Y}$, there are derivations ending in inferences in which the respective displayed M's are *not* parametric, and that (H3c), for all X', Y', and proper subformulas M' of M, $\mathrm{X}' \vdash \mathrm{Y}'$ is derivable if $\mathrm{X}' \vdash M'$ and $M' \vdash \mathrm{Y}'$ are. Then $\mathrm{X} \vdash \mathrm{Y}$ is derivable.

PROOF OF THE THEOREM. It is evident that the theorem follows from the three stages. The grand simplicity of Curry's proof is, however, concealed rather than revealed by any prose we have so far conceived or seen; so we choose to make it manifest by display in the natural deduction format of Fitch 1952:

1	(ER) is admissible for proper subformula of M.	
2	$\mathrm{X} \vdash M$ is derivable	hyp (=H1a)
3	$M \vdash \mathrm{Y}$ is derivable	hyp (=H2a)
4	$\mathrm{X}' \vdash M$ is derivable (M nonparametric)	hyp
5	$M \vdash \mathrm{Y}'$ is derivable (M nonparametric)	hyp
6	$\mathrm{X}' \vdash \mathrm{Y}'$ is derivable	Stage 3: 4 5 1
7	H2b (but with X′ for X)	5–6
8	H2c (but with X′ for X)	4
9	$\mathrm{X}' \vdash \mathrm{Y}$ is derivable	Stage 2: 3 7 8
10	H1b	4–9
11	$\mathrm{X} \vdash \mathrm{Y}$ is derivable	Stage 1: 2 10
12	(ER) is admissible for M	2–11
13	(ER) is admissible Induction:	1–12

PROOF of Stage 1 relies on C2, C3, C4, C5, and C6. Let D_1 be a derivation of $\mathrm{X} \vdash M$ as promised by hypothesis H1a of Stage 1. If M in $\mathrm{X} \vdash M$ is not parametric in the inference ending D_1, the conclusion of this stage is immediate, by H1b. Suppose then that M is parametric in that inference. Define a set Q inductively by first putting that occurrence of M into Q and then working up D_1, adding, for each inference *Inf* in D_1, each constituent of a premiss of *Inf* that is congruent (with respect to *Inf*) to a constituent of the conclusion of *Inf* which is already in Q. (The members of Q are sometimes called "parametric ancestors" of M.) For each consecution S in D_1, let S′ result from putting Y for each constituent of S which lies in Q. $(\mathrm{X} \vdash M)' = \mathrm{X} \vdash \mathrm{Y}'$, so it suffices to show that S′ is derivable for all S in D_1, assuming as inductive hypothesis that the primes of all premisses of S in D_1 are derivable.

Let S be $W \vdash Z$, and let the inference of D_1 leading to S be

$$\mathit{Inf} \qquad \frac{S_1, \ldots, S_n}{W \vdash Z}$$

Assume that *Inf* falls under rule *Ru*. Let $W' \vdash Z'$ be (not $(W \vdash Z)'$ but) the result of putting Y for all members of Q in S which *in addition* are parametric in *Inf*, and consider

$$\mathit{Inf}' \qquad \frac{S_1', \ldots, S_n'}{W' \vdash Z'}$$

By the bottom-upward definition of Q, together with C3, Q must contain all of a congruence class of *Inf* if it contains any member at all. And, by C4, all members of Q are consequent parts; so, by C6, *Inf′* falls under the same rule *Ru* as does *Inf*. The inductive hypothesis gives us the derivability of all the premisses (if any) S_i', and so $W' \vdash Z'$ is derivable by *Ru*. If each member of Q in S is parametric for *Inf*, $W' \vdash Z'$ is S′, and we are done. Otherwise, by C2, C4, and C5, $Z = Z' = M$, and, by the "furthermore" part of C6, that M is not parametric in *Inf′*. Then we have the derivability of $W' \vdash Y$, which is S′, by the hypothesis H1b of the Stage.

PROOF of Stage 2 relies on C2, C3, C4, C5, and C7. The proof of this Stage closely follows that of Stage 1. There are two differences: we deal with antecedent parts in Q, and we have the additional hypothesis that M is not parametric in an inference concluding with $X \vdash M$. We may use this information to rely on C7 instead of C6.

PROOF of Stage 3. C8 suffices.

This completes the proof that the Elimination rule (ER) is admissible in **DL**.

§62.5. Some families and logics. Display logic is in a way a schema; you can include such families as you like, differentiating among them by postulating whatever structural rules you like—so long as they satisfy C1–C8 of §62.4.2. Then you can make up hybrid logics by mixing families as you wish. For definiteness, however, we show how to represent in **DL** a few well-known logics by introducing appropriate families with well-chosen structural postulates. In some cases, what follows are definitions, not claims. Given the Elimination theorem 4-4, it is easy to see that **DL** is strong enough to represent the desired logic with the help of the family or families indicated, but verification of the companion claim of not-too-strong is sometimes on the list of future projects. We will in each case say what claims are plausible and what is known about those claims.

§62.5.1. Boolean family and two-valued logic. Two-valued logic is based in **DL** on a single family, the Boolean family, indexed with "b". For this family, we postulate structural rules

I+, I−, I*−, K, W.

All others follow. Recall that the index for the Boolean family is "b", and so the structure-connectives are *I*, *, and ∘ indexed with "b"; but to avoid having to index asterisks in print, we write just "−" for the Boolean negative structure-connective. Thus, the structure-connectives for the Boolean family are I_b, −, and $\circ_b$. We further remind the reader that we sometimes subscript rule names in accord with the convention of §62.3.4; e.g., W_b.

The connectives of this family are to be given their standard Boolean readings. The following claims are obviously true.

First, consider **DL**{b}, with its grammar consisting of only the Boolean family. $X \vdash Y$ is provable in **DL**{b} just in case X tautologically implies Y, and Y is a tautology just in case $I_b \vdash Y$ holds in **DL**{b}. The techniques of any standard proof will show this.

Second, **DL**, with all its families, is a conservative extension of **DL**{b}.

Third, **DL** does not give much information about separation in **DL**{b}, since the interpretation (in the sense of §62.2.4) of the structural elements, which are not subject to the subformula theorem, comprise all of ***t***, ***f***, *, &, and ∨ in the Boolean family; but of course there are additional separation results known by other techniques.

§62.5.2. Relevant implication. The original "standard" vocabulary of §27.1.1 for the calculus **R** of relevant implication contains ~, →, &, and ∨—the first two being thought of as "intensional" and the last two as "extensional." In **DL**, **R** will therefore be hybrid, involving, first, a family indexed by "r" for the "intensional" or "relevant" connectives and, second, the Boolean family for the "extensional" connectives. Turning first to the kernel connectives (in the sense of §62.2.4) of the relevance family, we note that $\&_r$ and $\vee_r$ are definable by (omitting subscripts) $\sim(A \to \sim B)$ and $\sim A \to B$ respectively, whereas t_r has to be added—conservatively, by §45.1. This addition has always seemed in the spirit of the relevance enterprise—see §27.1.2 and §62.1.

The situation with regard to the Boolean family is technically similar but philosophically different. The mathematical fact is that one can add the remaining Boolean kernel connectives ***t***, ***f***, and ~ to **R** conservatively; this yields the "classical relevant logic" **CR***, as in Meyer and Routley 1974a. This calculus can also be axiomatized by adding to **R** the axioms $A \&_b \sim_b A \to_r B$ and $A \to_r B \vee_b \sim_b B$ as in Meyer 1979b where it is called $\mathbf{R}^{\urcorner}$ (here we should write “$\mathbf{R}^{\sim b}$”). Even philosophically there is no complaint about the addition of Boolean $\boldsymbol{t}$ and $\boldsymbol{f}$, since these are anyhow definable, with the help of propositional quantifiers, as $\exists pp$ and $\forall pp$; but the addition of Boolean negation is controversial—see §80.2 for discussion. The problem of finding a consecution calculus for **R** *without* Boolean negation remains open. The difficulty is that relevance logic has things to say about “Boolean” conjunction and disjunction, so we need the Boolean family to help us out; but the techniques of this section do not permit us to enjoy “Boolean” conjunction and disjunction without carrying along Boolean negation as well.

In any event, to repeat, **R** in **DL** is hybrid; its Boolean connectives are governed by the postulates of §62.5.1, while for its relevant family we postulate the following rules drawn from §62.3.4:

I+, I−, I*+, B, C, W.

DL as a whole is obviously a conservative extension of **DL**{r, b}, since all other connectives can be interpreted as if Boolean; hence, we now confine attention to formulas and structures in the vocabulary of **DL**{r, b}.

The relation of **DL**{r, b} to the “classical relevant logic” **CR*** is given by: $\boldsymbol{I}_r \vdash A$ holds in **DL**{r, b} just in case A is a theorem of **CR***. One can show that **DL**{r, b} is strong enough by establishing $\boldsymbol{I}_r \vdash A$ for each axiom A of **CR***, and similarly for its rules; and one can show that **DL**{r, b} is not too strong by verifying its axioms and rules in the semantics of Meyer and Routley 1974a.

In **CR*** the relevance family is taken as somehow primary (witness the role of $\boldsymbol{I}_r$ in the scheme relating **CR*** to **DL**), while the special Boolean theorems are marked by “$\boldsymbol{t}_b \rightarrow_r A$,” with $\boldsymbol{t}_b$ Boolean but $\rightarrow_r$ relevant. Since $\boldsymbol{t}_b$ is itself a theorem of **CR***, it is clear that we are marking off a subset of the theorems of **CR*** as Boolean. As will often be the case for hybrid logics with non-equivalent $\boldsymbol{I}$s, one obtains an equally satisfying calculus answering to **DL**{r, b} by taking the Boolean family as primary: calling the Hilbert calculus “**RC**”, define A to be a theorem of **RC** just in case $\boldsymbol{I}_b \vdash A$ holds in **DL**{r, b}. Then the special relevance theorems A are marked by “$\boldsymbol{t}_r \rightarrow_b A$”, with (this time) $\boldsymbol{t}_r$ relevant but $\rightarrow_b$ Boolean. Here A itself would not always be a theorem of the Hilbert calculus **RC** when $\boldsymbol{t}_r \rightarrow_b A$ was (since $\boldsymbol{t}_r$ is not a theorem of **RC**), but $\boldsymbol{t}_r \rightarrow_b A$ would be a theorem for each theorem A. So the marking, in effect, enlarges the set of (call them) quasi-theorems. From the present point of view, the two procedures for finding a Hilbert calculus corresponding to **DL**{r, b} are distinct but interchangeable on mathematical grounds.

In this special case, the $\boldsymbol{I}_r$ of the relevance family and the $\boldsymbol{I}_b$ of the Boolean family are comparable: $\boldsymbol{I}_b \ \mathrm{a}\!\Rightarrow \boldsymbol{I}_r$. But in the general case, when there are various families, each with its own $\boldsymbol{I}$, and all incomparable, we can only say that each choice of $\boldsymbol{I}$ defines a Hilbert calculus via the schema “A is a theorem just in case $\boldsymbol{I} \vdash A$ holds in **DL**,” and that each of the others is marked therein by appropriate "$t\to A$"—the t corresponding to the other I, the arrow corresponding to our chosen I. This discussion assumes that the rules I+ and I− of §62.3.4 are postulated for both Is; for, otherwise, it would seem that I does not sufficiently resemble Gentzen's empty symbol to warrant a role in defining a notion of theoremhood.

One more illustrative fact. In §28.3.2 we showed that all the pure arrow theorems of **R** are derivable from its pure arrow axioms (as given in §R2) by means of *modus ponens*. **DL**{r} is strong enough to prove those axioms and *modus ponens*; so it is strong enough to prove all pure arrow theorems of **R** without detours. Here is an example (all connectives are in the relevance family):

$$\begin{array}{cl}
A \vdash A \qquad B \vdash B & \\ \hline
A\to B \vdash A^{*}\circ B & (\to) \\ \hline
I\circ(A\to B) \vdash A^{*}\circ B & (\mathrm{I}+) \\ \hline
(I\circ(A\to B))\circ A \vdash B & \text{Display} \\ \hline
(I\circ A)\circ(A\to B) \vdash B & (\mathrm{C}) \\ \hline
(I\circ A) \vdash (A\to B)\to B & (\to) \\ \hline
(I \vdash A\to((A\to B)\to B) & (\to)
\end{array}$$

Here is a verification of a postulate of **R** involving connectives from both the Boolean and the relevance families; in this example, Boolean connectives are marked with "b", but relevance connectives have been left unmarked.

$(A\to B)\circ A \vdash B$ and $(A\to C)\circ A \vdash C$, by 3-3 and $\to$;
$((A\to B)\circ A)\circ_b((A\to C)\circ A) \vdash B\&_b C$, by $\&_b$;
$(((A\to B)\circ_b(A\to C))\circ A)\circ_b(((A\to B)\circ_b(A\to C))\circ A) \vdash B\&_b C$, by K_b to
introduce $(A\to C)$ and by KI_b to introduce $(A\to B)$;
$((A\to B)\circ_b(A\to C))\circ A \vdash B\&_b C$, by WI_b;
$((A\to B)\&_b(A\to C))\circ A \vdash B\&_b C$, by $\&_b$;
$I_r \vdash ((A\to B)\&_b(A\to C))\to(A\to(B\&_b C))$, by $\mathrm{I}+_r$ and $\to_r$.

Observe that I+ is the only structural postulate required for the relevance family. (We have subscripted rule names in accordance with the convention of §62.3.4.)

§62.5.3. Entailment. The calculus **E** of entailment (Chapter IV and §R2) is hybrid in exactly the same way as **R**; in this case, however, it is not known whether the addition of Boolean negation to **E**, in the way most directly suggested by our Display-logic treatment of **R**, is or is not conservative. (Both Meyer and Giambrone confirmed our ignorance on this matter in 1982, but no further light seems to have appeared in the intervening years.) For this reason we will present two different ways of "displaying" **E**, the first and simpler of which will run up against the just-mentioned problem, whereas the second will avoid it.

Because **E** is hybrid, we need two families, the Boolean for its extensional connectives, and an appropriately marked intensional family for the distinctive intensional connectives. Since we are going to present two versions of **E** here, we will use "e_1" as the index for the first, simple version which is not known to be adequate, reserving plain "e" for the index of the second version which we show to be just what is wanted.

For the first, simpler version, the following rules from §62.3.4 are postulated:

$$\mathrm{I}+,\ \mathrm{I}-,\ \mathrm{I}^*+,\ \mathrm{I}^*-,\ \mathrm{CI/I},\ \mathrm{B}',\ \mathrm{W},\ \vdash \mathrm{WI}.$$

From these B, improbably, follows: $(\mathrm{X}\circ(\mathrm{Y}\circ\mathrm{Z}))$ a$\Rightarrow$ $(\boldsymbol{I}\circ(\mathrm{X}\circ(\mathrm{Y}\circ\mathrm{Z})))$ a$\Rightarrow$ $((\mathrm{X}\circ\boldsymbol{I})\circ(\mathrm{Y}\circ\mathrm{Z}))$ a$\Rightarrow$ $((\mathrm{Y}\circ(\mathrm{X}\circ\boldsymbol{I}))\circ\mathrm{Z})$ a$\Rightarrow$ $(((\mathrm{X}\circ\mathrm{Y})\circ\boldsymbol{I})\circ\mathrm{Z})$ a$\Rightarrow$ $((\boldsymbol{I}\circ(\mathrm{X}\circ\mathrm{Y}))\circ\mathrm{Z})$ a$\Rightarrow$ $((\mathrm{X}\circ\mathrm{Y})\circ\mathrm{Z})$.

Certainly these postulates for **E**'s intensional connectives, together with the Boolean postulates for its extensional connectives, are strong enough to prove the set of axioms and rules for **E** of §21.1 (or §R2) in the form $\boldsymbol{I}_e \vdash A$. For example, to prove E7, start with $\boldsymbol{I}_e \vdash A\rightarrow A$ and $A \vdash A$, and then obtain $((A\rightarrow A\rightarrow A)\&_b(B\rightarrow B\rightarrow B))\circ\boldsymbol{I}_e \vdash A$ by $\rightarrow$, K_b, and $\&_b$. Obtain the same antecedent turnstiling B, and accordingly $((A\rightarrow A\rightarrow A)\&_b(B\rightarrow B\rightarrow B))\circ\boldsymbol{I}_e \vdash A\&_bB$. Put this together with $A\&_bB \vdash A$, using $\rightarrow$, and similarly put it together with $A\&_bB \vdash B$, and combine the results by $\&_b$ to obtain

$$(A\&_bB \rightarrow .A\&_bB)\circ((A\rightarrow A\rightarrow A)\&_b(B\rightarrow B\rightarrow B))\circ\mathrm{I}_e) \vdash A\&_bB.$$

Now, CI/I applied to the right portion of the antecedent, followed by B′, yields a consecution that gives E7 by $\rightarrow$.

Because, however, we do not know that $\mathbf{DL}\{e_1, b\}$ is a conservative extension of **E**, we also do not know that it is not too strong, permitting the proof of formulas in the vocabulary of **E** that are not provable in **E** itself. We therefore offer a slightly different calculus, which can be seen to be a Display-logic formulation of **E** without needing to solve the above problem of conservative extension.

The idea is easy: we simply replace the rule I+ with the rule I+/e as given in §62.3.4: from $\mathrm{X} \vdash \mathrm{Y}$ to infer $\boldsymbol{I}\circ\mathrm{X} \vdash \mathrm{Y}$, *provided* X is an e-variable (§62.2.3). Here is how this change helps. In the first place, let us be clear on grammar. We are considering $\mathbf{DL}\{e, b\}$, with "e" the index for the entailment family and "b" for the Boolean family. This usage implies a large stock of structures and formulas, with every mixture permitted as indicated in §62.2. Second, let us be clear what we are postulating: the Identity axioms, Display equivalences, and Connective postulates of §§62.3.1–3 for all formulas; the structural rules for the Boolean family as listed in §62.5.1; and finally, the structural rules for the entailment family as listed at the beginning of this section, except that we postulate I+/e in place of I+.

To formulate the claim that $\mathbf{DL}\{e, b\}$ does a workmanlike job of displaying **E**, we begin by noticing that the *original* connectives of **E** are $\{\rightarrow_e, {\sim}_e,$

$\&_b$, $\vee_b\}$, a stock inherited from Ackermann 1956, and we call any formula an *e-formula* if it is made from e-variables by means of these connectives. (We could have added more connectives if we liked; but we could *not* have added Boolean negation $\sim_b$ as a builder of e-formulas.) The calculus **DL**{e, b} contains many formulas that are not e-formulas—a matter of some interest. What we wish to show, however, is that, if A is any e-formula, A is provable in **E** just in case $\mathbf{I}_e \vdash A$ is provable in **DL**{e, b}.

FACT 1. The Elimination theorem holds. In particular, the restricted rule I+/e satisfies all the conditions C2-8. Therefore, the rule *modus ponens* is verified in the usual form: $\mathbf{I}_e \vdash A \to_e \mathrm{B}$ and $\mathbf{I}_e \vdash A$ yield $\mathbf{I}_e \vdash B$. Obviously, the rule of conjunction introduction is verified in a strictly analogous form.

FACT 2. $\mathbf{I}_e \circ_e A \vdash A$ holds whenever A is an e-formula (but not necessarily when it is not). Proof by easy induction on the structure of e-formulas. From the Elimination theorem it follows that, whenever A is an e-formula, $A \vdash \mathrm{X}$ yields $\mathbf{I}_e \circ_e A \vdash \mathrm{X}$; $\mathbf{I}_e \vdash A \to_e A$ is therefore provable for e-formulas.

FACT 3. If an e-formula A is an axiom of **E**, then $\mathbf{I}_e \vdash A$ is provable in **DL**{e, b}. Fact 2 is needed for the choice of A as E1 or E7 of §R2, in which identities play a special role.

FACT 4. If an e-formula A is a theorem of **E**, then $\mathbf{I}_e \vdash A$ is provable in **DL**{e, b}; that is, the Display logic **DL**{e, b} is strong enough. From Facts 1 and 3.

For the converse, we need some semantics. Let us take (K, 0, R, *, $\vDash$) to be an **E**-model in the sense of §48.6, being careful to observe that the definition requires only the *Atomic* Hereditary condition of §48.3. (We obtain the required valuation clauses for $\to_e$, $\&_b$, $\vee_b$, and indeed for $\&_e$ from that same section, and a clause for $\sim_e$ from §48.5. But we should not use the clause for $\mathbf{t}$ from §48.5, since it is inappropriate for $\mathbf{t}_e$.) In accordance with the discussion in §62.2.4, we need to supply a semantic interpretation for the remainder of the kernel connectives of the two families, which we do as follows, calling the result the *display extension* of the original **E**-model.

$\mathbf{t}_e$: true at just those set-ups a such that Za (§48.6).
$\mathbf{f}_e$: $\sim_e \mathbf{t}_e$.
$\vee_e$: $A \vee_e B$ is defined as $\sim_e A \to_e B$.
$\mathbf{t}_b$: true everywhere in K.
$\mathbf{f}_b$: false everywhere in K.
$\sim_b$: $\sim_b A$ is true just where A is not (Boolean negation).

By §64.2.4, this is enough to impose an interpretation on all other formula- and structure-connectives of **DL**{e, b}. A consecution X ⊢ Y holds in a display extension of an **E**-model just in case, for each of its set-ups, if X holds therein, so does Y.

FACT 5. All postulates of **DL**{e, b} are verified in the display extension of each **E**-model; so, by contraposition, if X ⊢ Y is not verified in the display extension of some **E**-model, it is not provable in **DL**{e, b}. We take up two specially sensitive cases.

First, I+/e. It suffices to show for each e-variable p that, if $\boldsymbol{t}\&_e p$ holds at a then so does p. Assume the antecedent, which existentially gives us b and z such that (1) p is true at b and (2) $\boldsymbol{t}$ is true at z and (3) Rzba. Item (2) implies that Zz, which, with (3), implies (4) R0ba. But (1) and (4) now give the desired result by the Atomic Hereditary condition.

Second, CI/I. It suffices to show that if $\boldsymbol{t}\&_e$X holds at a then so does X$\&_e\boldsymbol{t}$. Assume the former; then, existentially, we have (1) Rzxa, (2) $\boldsymbol{t}$ true at z, and (3) X true at x. By (2) we have that Zz, and so (4) R0xa. By postulate 3.1 of §48.6, there is a z′ such that (5) Zz′ and (6) Raz′a. From (4) and (6) and the monotony condition 2 of §48.6, we have (7) Rxz′a, and (5) tells us that (8) $\boldsymbol{t}$ is true at z′. Now (3), (8), and (7) are enough to warrant the truth at a of X$\&_e\boldsymbol{t}$, as desired.

FACT 6. If A is an e-formula then, if A is unprovable in **E**, $\boldsymbol{I}_e \vdash A$ is unprovable in **DL**{e, b}. For suppose A is unprovable in **E**. Then A is false at 0 in some **E**-model according to the Routley-Meyer result as reported in §48.6. Form the display extension of that **E**-model. Then, since $\boldsymbol{I}_e$ as antecedent is true at 0, $\boldsymbol{I}_e \vdash A$ is false. But now Fact 5 guarantees that $\boldsymbol{I}_e \vdash A$ is not provable in **DL**{e, b}.

BIG FACT. **DL**{e, b} is a conservative extension of **E**. By Facts 4 and 6.

PROBLEM. Is **DL**{e, b} complete in the stated semantics?

PROBLEM. Is **DL**{e_1, b} a conservative extension of **E**?

See §62.6.6 below for another possible formulation of **E** in **DL**.

§62.5.4. Ticket entailment. The calculus **T** of ticket entailment of §27.1.1 or §R2 is hybrid in precisely the same sense as **R**; and since it is known that Boolean negation can be added to **T** as it can be added to **R**, our state of information is precisely analogous. (Letter from S. Giambrone, April 27, 1981. The result appears in Giambrone 1983.) The following are the postulates for the intensional family of connectives of **T**:

I+, I−, I*+, I*−, B, B′, W, ⊢ WI.

§62.5.5. Semantics of relevance logics. One paradigmatic form of semantic investigation of relevance logics such as **R**, **E**, and **T** has been based on a three-termed relation (see §48); this form appears to fit well with Display logic; but the matter has been little investigated; so we here present only some suggestive definitions and no facts.

A *model set* is a quadruple (K, D, R, *) satisfying the "display postulates": (R*) Rxyz only if Rxz*y*, and (**) x** = x.

Warning: this star is used for historical reasons, and has nothing whatsoever to do with * in **DL**; indeed, it is more of an identity operator than a negation operator.

The kernel connectives are evaluated as follows, where "A_x" means that A is true at x: $\boldsymbol{t}_x$ iff x is in D; $\boldsymbol{f}_x$ iff x* is not in D; $(\sim A)_x$ iff not A_{x*}; $(A\&B)_z$ iff, for some x, y in K, Rxyz and A_x and B_y; $(A \vee B)_x$ iff, for all y and z in K, if Rxy*z then either A_y or B_z.

It is easy to see that the display equivalences are sound on this semantics, when one interprets all connectives via the kernel connectives as in §62.2.4 and interprets X ⊢ Y as: for all x in K, X_x only if Y_x. Presumably completeness is also at hand, but time is finite; still, the matter ought to be pursued because, although *so far* this semantics seems to have little intuitive appeal, we certainly know that it has great technical power.

§62.5.6 Modal logics. We discuss only modal logics based on a binary relational structure. In **DL** these logics are hybrid: their extensional connectives are part of the Boolean family of §62.5.1, while for their modal connectives the idea is to add a family interpreted in a relational structure (K, D, R), with K the set of all points, D a set of "normal" points, and R a binary relation on K (as in Kripke 1965). The "kernel" connectives of §62.2.4 are explained as follows. $\boldsymbol{t}_x$ for just the points x in D, and $\boldsymbol{f}_x$ just for x in K − D. $(\sim A)_x$ just in case not A_x; $(A\&B)_y$ just in case, for some x in K, Rxy, A_x, and B_y; $(A \vee B)_x$ just in case, for every y in K such that Rxy, either A_y or B_y.

This induces the following explanation of the structure-connectives. ***I*** in antecedent {consequent} position holds {doesn't hold} at all points in D. In antecedent position, (X∘Y) holds at a point y just in case, for some x in K, Rxy, X holds at x, and Y holds at y. In consequent position, (X∘Y) holds at x just in case, for every y in K such that Rxy, either X holds at y or Y holds at y. X* holds at x just in case X doesn't hold at x.

The induced account of →, □, and ◇ agrees with that of Kripke 1965 only for "normal" logics where D = K. Nonnormal logics are discussed below. The *modal* connective & is not always definable in the "standard" vocabulary; there is a discussion of this point following Theorem 4-3 in §62.4.3 above.

For every modal family discussed in this section, we postulate

$$\mathrm{KI},\ \mathrm{KI} \vdash \mathrm{WI},\ \mathrm{C}\circ,\ \mathrm{W}\circ,\ \mathrm{I}^*+,\ \mathrm{I}^*-.$$

So much for what is common to the modal families of all the modal logics of the sort we are treating. In addition, we are supposing that each such logic is fitted out with the Boolean family and that the connectives of this family are given their usual extensional interpretation in a relational structure (K, D, R). These postulates (both modal and Boolean) are valid, and the display equivalences preserve validity, where to say that $X \vdash Y$ is valid is to say that X_x implies Y_x, for all x in K, for each relational structure (K, D, R). Presumably completeness is available, probably easily, but this claim is on a long list of future projects.

Before dealing with individual modal systems, we offer a few facts applying to any family that satisfies the modal postulates listed above. Let $(\boldsymbol{I}, {}^*, \circ)$ be the modal structure-connectives, and recall that $(\boldsymbol{I}_b, -, \circ_b)$ are Boolean.

FACT. I+ is a special case of KI.

FACT. From $\mathrm{KI} \vdash \mathrm{WI}$ and KI we obtain $\mathrm{K} \vdash \mathrm{WI}$ as follows: $X \vdash Z\circ Z$; $Z^* \vdash (X\circ Z^*)$; $X\circ Z^* \vdash (X\circ Z^*)^*$, by KI; $(X\circ Z^*)^{**} \vdash (X\circ Z^*)^*$; $Y \vdash (X\circ Z^*)^*$ by $\mathrm{KI} \vdash \mathrm{WI}$; $X\circ Y \vdash Z$.

FACT. Also, given only $\mathrm{KI} \vdash \mathrm{WI}$, we can calculate that modal * is just Boolean negation: $X^* \Leftrightarrow X-$. Start with $X^* \vdash Y$; $(X\circ_b Y)^* \vdash (X\circ_b Y)$, by Boolean moves; $X- \vdash (X\circ_b Y)$, by $\mathrm{KI} \vdash \mathrm{WI}$; $X- \vdash Y$, by Boolean moves. Now start with $X \vdash Y^*$; $(X\circ_b Y) \vdash (X\circ_b Y)^*$, by Boolean moves; $Y-{}^* \vdash (X\circ_b Y)^*$, by $\mathrm{KI} \vdash \mathrm{WI}$; $(X\circ_b Y) \vdash Y-$; $X \vdash Y-$, by Boolean moves. So $X^*\ \mathrm{a}\!\Rightarrow X-$ and $X^*\ \mathrm{c}\!\Rightarrow X-$. The first of these implies that $X-\ \mathrm{c}\!\Rightarrow X^*$, and the second that $X-\ \mathrm{a}\!\Rightarrow X^*$; so $X^* \Leftrightarrow X-$ as required.

FACT. Given $\mathrm{KI} \vdash \mathrm{WI}$ and KI, $(X\circ Y)\,\mathrm{a}\!\Leftrightarrow ((X\circ \boldsymbol{I}_b)\circ_b Y)$. Start with $X\circ Y \vdash Z$; $X\circ Y^{**} \vdash Z$; $X\circ(K^*\circ_b Z)^* \vdash (Y^*\circ_b Z)$, by Boolean moves; $X\circ \boldsymbol{I}_b \vdash (Y^*\circ_b Z)$, by $\mathrm{K} \vdash \mathrm{WI}$; $((X\circ \boldsymbol{I}_b)\circ_b Y) \vdash Z$. Now start with $((X\circ \boldsymbol{I}_b)\circ_b Y) \vdash Z$; $((X\circ \boldsymbol{I}_b)\circ_b (X\circ Y)) \vdash Z$, by KI; $X\circ Y \vdash Z$, by Boolean moves.

FACT. Consequently, given $\mathrm{KI} \vdash \mathrm{WI}$ and KI, in the presence of the Boolean family, C∘ and W∘ are redundant.

For the normal logics, where all points are normal (D = K), add $(\mathrm{I}-\mathrm{K})$ as a postulate. This clearly suffices to identify $\boldsymbol{I}$ and $\boldsymbol{I}_b$.

For von Wright's **M** (Kripke 1965), add the "reflexivity" postulates I$-$, CI/I, and W. (I$-$K) follows, using KI, CI/I, I$-$, and so do WI and $\vdash$WI. Query: can the "reflexivity" postulates be *usefully* simplified?

For **S4**, add the "reflexivity" postulates I$-$, CI/I, and W, and a transitivity postulate, either B or B$'$. (I$-$K), WI, $\vdash$WI, and the other one of B and B$'$ follow.

For the Brouwerische logic (D$=$K, R reflexive and symmetric) add I$-$, CI/I, W, and Brw. (I$-$K), WI, and $\vdash$WI follow. Here is a proof of the Brouwerische postulate ($A \vdash \Box\Diamond A$): $A \vdash A$; $(A^{*}\circ \boldsymbol{I}^{*})^{*} \vdash \Diamond A$, by ($\Diamond$); $(\Diamond A)^{*} \vdash (\mathrm{A}^{*}\circ \boldsymbol{I}^{*})$; $(\Diamond A)^{*}\circ A \vdash \boldsymbol{I}^{*}$; $A\circ\Diamond A^{*} \vdash \boldsymbol{I}^{*}$, by (Brw); $A\circ \boldsymbol{I} \vdash \Diamond A$; $A \vdash \Box\Diamond A$, by ($\Box$).

For **S5** (D $=$ K, R an equivalence relation) add I$-$, CI/I, W, Brw, B or B$'$. WI, $\vdash$WI, and the other of B and B$'$ follow.

These welcome spinoffs from Display logic appear to shed new light on modal logics: we do not know of another consecution formulation of **M** or of **S4** (for which an Elimination theorem is provable) (a) with unrestricted rules, or (b) with rules for possibility as a primitive; we do not know of another consecution formulation of the Brouwerische logic; and we do not know of another consecution formulation of **S5** that uses only techniques already introduced for other purposes—and for which an Elimination theorem is provable (see Sato 1980 for a survey regarding **S5**).

For **S2** and **S3**, where the relation is reflexive on the points in D (the normal points), but not all points are normal, complications are required. In particular, the truth conditions of some of the modal connectives are altered (see Kripke 1965). To avoid confusion, we introduce $\Box' A$, $\Diamond' A$, and $A \rightarrow' B$ as variants of $\Box$, $\Diamond$, and $\rightarrow$.

$(\Box' A)_{\mathrm{x}}$ if x is in D and if Rxy implies that A_{y}, for all y in K;
$(\Diamond' A)_{\mathrm{x}}$ if x is not in D or if Rxy and A_{y} for some y in K;
$(A \rightarrow' B)_{\mathrm{x}}$ if x is in D and if Rxy and A_{y} only if B_{y}, for all y in K.

These variant connectives are the usual connectives of **S2** and **S3**. Their cousins, with truth conditions as given at the beginning of this section, are definable; see §11.

We are speaking of "nonnormal" systems, and postulates for these connectives must refer to normality. We give two ways of doing so. One, which adds a premiss X$\vdash \boldsymbol{t}$ or $\boldsymbol{f} \vdash$X to certain rules, requires us to think of $\boldsymbol{t}$ and $\boldsymbol{f}$ as "automatic subformulas" in order to satisfy condition C1 for the Subformula theorem 4-3 (or one could just weaken the theorem explicitly instead of indirectly). The other adds a conjoined or disjoined (by the Boolean $\circ_{\mathrm{b}}$) modal $\boldsymbol{I}$ in the conclusion and, hence, involves two families. We present both ways at once, with the understanding that only one of the curly bracketed pieces is to be included in each rule. (These rules are to be added to the basic modal postulates given at the beginning of this section.)

$$(\Box'/\mathrm{I})\quad \frac{\boldsymbol{I} \vdash \mathrm{X}}{\Box' A \vdash \mathrm{X}} \qquad\qquad (\rightarrow'/\mathrm{I})\quad \frac{\boldsymbol{I} \vdash \mathrm{Y}}{A \rightarrow' B \vdash \mathrm{X}}$$

$$(\Diamond'/\mathrm{I})\quad \frac{\mathrm{X} \vdash \boldsymbol{I}}{\mathrm{X} \vdash \Diamond' A}$$

$$(\Box')\quad \frac{\mathrm{X} \vdash A \circ A \qquad \{\mathrm{X} \vdash \boldsymbol{t}\}}{\mathrm{X}\{\circ_{\mathrm{b}} \boldsymbol{I}\} \vdash \Box' A} \qquad\qquad \frac{A \vdash \mathrm{Y}}{\Box' A \vdash \mathrm{X} \circ \mathrm{Y}}$$

$$(\Diamond')\quad \frac{\mathrm{X} \vdash A}{(\mathrm{X}^* \circ \mathrm{Y})^* \vdash \Diamond' A} \qquad\qquad \frac{\mathrm{X}^* \circ A \vdash A^* \qquad \{\boldsymbol{f} \vdash \mathrm{X}\}}{\Diamond' A \vdash \mathrm{X}\{\circ_{\mathrm{b}} \boldsymbol{I}\}}$$

$$(\rightarrow')\quad \frac{X \circ A \vdash B \qquad \{\mathrm{X} \vdash \boldsymbol{t}\}}{\mathrm{X}\{\circ_{\mathrm{b}} \boldsymbol{I}\} \vdash A \rightarrow' B} \qquad\qquad \frac{\mathrm{X} \vdash A \qquad B \vdash \mathrm{Y}}{A \rightarrow' B \vdash \mathrm{X}^* \circ \mathrm{Y}}$$

The **S2** structural postulates are modified versions of the "reflexivity" postulates I−, CI/I, and W; the modification corresponds to requiring that reflexivity hold only for points in D (normal points) instead of for all points in K. As above, the curly brackets indicate a choice of ways to deal with normality.

$$(\mathrm{I}-/\boldsymbol{t})\quad \frac{\boldsymbol{I} \circ \mathrm{X} \vdash \mathrm{Y} \qquad \{\mathrm{X} \vdash \boldsymbol{t}\}}{\mathrm{X}\{\circ_{\mathrm{b}} \boldsymbol{I}\} \vdash \mathrm{Y}}$$

$$(\mathrm{CI}/\mathrm{I}/\boldsymbol{t})\quad \frac{\mathrm{X} \circ \boldsymbol{I} \vdash \mathrm{Y} \qquad \{\boldsymbol{I} \circ \mathrm{X} \vdash \boldsymbol{t}\}}{(\mathrm{I} \circ \mathrm{X})\{\circ_{\mathrm{b}} \mathrm{I}\} \vdash \mathrm{Y}}$$

$$(\mathrm{W}/\boldsymbol{t})\quad \frac{(\mathrm{X} \circ \mathrm{Y}) \circ \mathrm{Y} \vdash \mathrm{Z} \qquad \{\mathrm{X} \circ \mathrm{Y} \vdash \boldsymbol{t}\}}{(\mathrm{X} \circ \mathrm{Y})\{\circ_{\mathrm{b}} \boldsymbol{I}\} \vdash \mathrm{Z}}$$

It needs to be verified that these postulates satisfy the conditions C1–C8. C1 either is not quite satisfied or requires the "automatic subformulahood" of $\boldsymbol{t}$ and $\boldsymbol{f}$, as noted, for the exta-premiss version. The rest, except for C8, are straightforward. But verifying C8 for ($\vdash \Box'$) and ($\Diamond' \vdash$) requires appeal to (K ⊢ WI), and so Fact 4-2 no longer holds. Furthermore, verification of C8 for the extra-premiss version requires adding the following pair of rules, which use the "automatic subformulahood" of $\boldsymbol{t}$ and $\boldsymbol{f}$ (in order to satisfy C1—they satisfy C2–C8 as well) in a particularly repugnant (but still technically harmless) fashion:

$$(\mathrm{ER}/\boldsymbol{t})\ \frac{\mathrm{X} \vdash \boldsymbol{t} \qquad \boldsymbol{t} \vdash \mathrm{Y}}{\mathrm{X} \vdash \mathrm{Y}} \qquad\qquad (\mathrm{ER}/\boldsymbol{f})\ \frac{\mathrm{X} \vdash \boldsymbol{f} \qquad \boldsymbol{f} \vdash \mathrm{Y}}{\mathrm{X} \vdash \mathrm{Y}}$$

For **S3**, add a transitivity postulate B or B′ (§62.3.4) to the **S2** postulates.

Theoremhood of A in **S2** or **S3** is defined by $\boldsymbol{I} \vdash A$, with the $\boldsymbol{I}$ from the modal family (truth at all *normal* points). **E2** and **E3** of Lemmon 1957, cited by Kripke 1965, are obtained by taking the Boolean $\boldsymbol{I}_{\mathrm{b}}$ (instead of the modal $\boldsymbol{I}$) as the sign of theoremhood: $\boldsymbol{I}_{\mathrm{b}} \vdash A$ (truth at *all* points).

One can hope that there is a set of postulates for these "nonnormal systems" which neither involves two-family postulates nor cheats on the Subformula theorem even a little; nevertheless, those who have seen other consecution formulations of **S2** or **S3** will agree that the present formulation represents an improvement with respect to simplicity. For example, Zeman 1973, p. 114, has the following proviso for a key rule: "The left premiss is optional; however, if the rule is once applied with the left premiss not holding, it may not be used again at all in the same proof string."

Deontic logics. "Obligatory implies permitted" is obtained by postulating the rule: from $(\mathrm{X}\circ(\mathrm{Y}\circ\boldsymbol{I}))\vdash\boldsymbol{I}^*$ to infer $\mathrm{X}\vdash\mathrm{Y}^*$.

§62.5.7. Intuitionist logic. An *h-formula* is constructed from h-variables (§62.2.3) by the standard intuitionist connectives $\&_{\mathrm{h}}$, $\vee_{\mathrm{h}}$, $\rightarrow_{\mathrm{h}}$, and $\neg_{\mathrm{h}}$. Plain negation $\sim_{\mathrm{h}}$ is explicitly excluded (see §62.3.3 for the two negations; as indicated above (§62.5.6, third Fact), given $\mathrm{KI}\vdash\mathrm{WI}$, $\sim_{\mathrm{h}}$ agrees with Boolean $\sim_{\mathrm{b}}$). Intuitionist logic is given semantically (Kripke) by the conditions $\mathrm{D}=\mathrm{K}$, R reflexive and transitive, all intuitionistic formulas "persistent": Rxy and A_{x} imply A_{y} for A an h-formula. This is a single-family logic; consequently, for the rest of this section we save subscripts by assuming that all connectives are indexed with "h" and that all formulas are h-formulas. To obtain the structural postulates for the h-family, add (restricted) C/h to the postulates for **S4** of §62.5.6. We observe that Display logic permits the happy coexistence of intuitionist and Boolean logic.

As an example of how things go, we will verify all the Łukasiewicz postulates for intuitionist logic as presented in Prior 1955. But first an essential

LEMMA 5-1. (PERSISTENCE.) $A\circ\boldsymbol{I}\vdash A$. Hence, by the Elimination theorem 4-4, $A\ \mathrm{a}\!\Rightarrow A\circ\boldsymbol{I}$.

PROOF. We argue inductively on the definition of "h-formula."

Suppose A is an h-variable. $(\boldsymbol{I}\circ\boldsymbol{I})\circ A\vdash A$, by the Identity theorem, 3-3 and KI; $(\boldsymbol{I}\circ A)\circ\boldsymbol{I}\vdash A$, by C/h; $A\circ\boldsymbol{I}\vdash A$, by $\mathrm{I}-$.

$A\vdash A$, by Theorem 3-3, and $B\circ\boldsymbol{I}\vdash B$ by inductive hypothesis (IH); $A\circ(B\circ\boldsymbol{I})\vdash A\&B$, by &; $(A\circ B)\circ\boldsymbol{I}\vdash A\&B$, by B; $(A\&B)\circ\boldsymbol{I}\vdash A\&B$, by &.

$A\circ\boldsymbol{I}\vdash A$, by IH; $A\circ(\boldsymbol{I}\circ B^*)\vdash A$, by $\mathrm{I}-\mathrm{K}$; $(A\circ\boldsymbol{I})\circ B^*\vdash A$, by B; $A\vdash\boldsymbol{I}^*\circ(A\circ B)$; $B\vdash\boldsymbol{I}^*\circ(A\circ B)$, similarly; $A\vee B\vdash(\boldsymbol{I}^*\circ(A\circ B))\circ(\boldsymbol{I}^*\circ(A\circ B))$, by $\vee$; $(A\vee B)\circ\boldsymbol{I}\vdash A\circ B$, by $\vdash$WI and Display theorem 3-2; $(A\vee B)\circ\boldsymbol{I}\vdash A\vee B$, by $\vee$.

$(A\rightarrow B)\circ A\vdash B$, by Theorem 3-3 and $\rightarrow$; $(A\rightarrow B)\circ(\boldsymbol{I}\circ A)\vdash B$, by KI; $((A\rightarrow B)\circ\boldsymbol{I})\circ A\vdash B$, by B; $(A\rightarrow B)\circ\boldsymbol{I}\vdash A\rightarrow B$, by $\rightarrow$.

$A^*\vdash A^*$, Theorems 3-3 and 3-2; $\neg A\vdash\mathrm{I}^*\circ A^*$, by $\neg$; $\neg A\circ\boldsymbol{I}\vdash A^*$; $\neg A\circ(\boldsymbol{I}\circ\boldsymbol{I})\vdash A^*$, by $\mathrm{I}+$; $(\neg A\circ\boldsymbol{I})\circ\boldsymbol{I}\vdash A^*$, by B; $\neg A\circ\boldsymbol{I}\vdash\neg A$, by $\neg$.

The postulates as numbered in Prior 1955, p. 308, may now be verified by straightforward calculation.

1. $A\circ\mathbf{I}\vdash A$ by the Persistence lemma 5-1; $A\circ B\vdash A$, by I−K;
   $\mathbf{I}\vdash A\to(B\to A)$, by I+ and →.

2. $((A\to(B\to C))\circ A)\circ B\vdash C$, by Theorem 3-3 and →;
   $((A\to(B\to C))\circ A)\circ((A\to B)\circ A)\vdash C$, by Theorem 3-3 and →;
   $((A\to B)\circ((A\to(B\to C))\circ A))\circ A\vdash C$, by B;
   $(((A\to B)\circ(A\to(B\to C)))\circ A)\circ A\vdash C$, by B;
   $((A\to B)\circ(A\to(B\to C)))\circ A\vdash C$, by W;
   $\mathbf{I}\vdash(A\to B)\to((A\to(B\to C))\to(A\to C))$, by I+ and →.

3. $A\circ\mathbf{I}\vdash A$, by Persistence lemma 5-1; $A\circ B\vdash A$, by I−K; $\mathbf{I}\vdash(A\&B)\to A$, by &, I+, and →.

4. $A\circ B\vdash B$, by Theorem 3-3 and KI; $\mathbf{I}\vdash(A\&B)\to B$ by &, I+, and →.

5. $A\circ B\vdash A\&B$, by Theorem 3-3 and &; $\mathbf{I}\vdash A\to(B\to(A\&B))$, by I+ and →.

6. $A\circ\mathbf{I}\vdash A$ by Lemma 5-1; $A\circ B^*\vdash A$, by I−K; $A\vdash A\circ B$ by Display theorem 3-2; $\mathbf{I}\vdash A\to(A\vee B)$ by ∨, I+, and →.

7. $A\vdash A\circ B$, as for 6: $A\vdash B\circ A$, by Theorem 3-2; $\mathbf{I}\vdash A\to(B\vee A)$, by ∨, I+, and →.

8. $(A\to C)\circ A\vdash C$, by Theorem 3-3 and →;
   $A\vdash((A\to C)\circ C^*)^*$, by Theorem 3-2;
   $B\vdash((B\to C)\circ C^*)^*$, similarly;
   $(A\vee B)\vdash((A\to C)\circ C^*)^*\circ((B\to C)\circ C^*)^*$, by ∨;
   $(A\vee B)\vdash(((A\to C)\circ(B\to C))\circ C^*)^*\circ(((A\to C)\circ(B\to C))\circ C^*)^*$, by Lemma 5-1, with I−K to introduce $(B\to C)$ and KI to introduce $(A\to C)$;
   $(A\vee B)\vdash(((A\to C)\circ(B\to C))\circ C^*)^*$, by ⊢WI;
   $((A\to C)\circ(B\to C))\circ(A\vee B)\vdash C$, by Theorem 3-2;
   $\mathbf{I}\vdash(A\to C)\to((B\to C)\to((A\vee B)\to C))$, by I+ and →.

9. $B^*\vdash B^*$, by Theorems 3-3 and 3-2;
   $\neg B\circ\mathbf{I}\vdash B^*$, by ¬ and Theorem 3-2;
   $\mathbf{I}\circ\neg B\vdash B^*$, by CI/I;
   $B\vdash(\neg B)^*$, by I− and Theorem 3-2;
   $(A\to B)\circ A\vdash(\neg B)^*$, by 3-3 and →;
   $(A\to B)\circ\neg B\vdash A^*$, by Theorem 3-2;
   $(A\to B)\circ((A\to\neg B)\circ A)\vdash A^*$, by Theorem 3-3 and →;
   $((A\to B)\circ(A\to\neg B))\circ A\vdash A^*$, by B;
   $((A\to B)\circ(A\to\neg B))\circ A^{**}\vdash A^*$, by Theorem 3-2;
   $((A\to B)\circ(A\to\neg B))\circ\mathbf{I}\vdash A^*$, by K⊢WI;
   $(A\to B)\circ(A\to\neg B)\vdash\neg A$ by ¬;
   $\mathbf{I}\vdash(A\to B)\to((A\to\neg B)\to\neg A)$, by I+ and →.

10. $A\circ\boldsymbol{I}\vdash A$ by Persistence lemma 5-1; $A^*\vdash(\mathrm{A}\circ\boldsymbol{I})^*$; $\neg\mathrm{A}\vdash\boldsymbol{I}^*\circ(A\circ\boldsymbol{I})^*$, by $\neg$; $\neg A\circ(A\circ\boldsymbol{I})\vdash\boldsymbol{I}^*$; $(A\circ\neg A)\circ\boldsymbol{I}\vdash\boldsymbol{I}^*$, by B; $(A\circ\neg A)\circ B^*\vdash\boldsymbol{I}^*$, by I−K; $(A\circ\neg A)\circ\boldsymbol{I}\vdash B$; $A\circ\neg A\vdash B$, by CI/I and I−; $\boldsymbol{I}\vdash A\rightarrow(\neg A\rightarrow B)$, by I+ and →.

Modus ponens. Suppose $\boldsymbol{I}\vdash A$ and $\boldsymbol{I}\vdash A\rightarrow B$; but $(A\rightarrow B)\circ A\vdash B$, by 3-3 and →; so $\boldsymbol{I}\circ\boldsymbol{I}\vdash B$, by the Elimination theorem 4-4 and the Display theorem 3-2; so $\boldsymbol{I}\vdash B$, by I−.

See §62.6.6 for an alternative formulation of intuitionism within **DL**.

§62.5.8. Interfamilial relations. Here we record a staccato of interfamilial facts. We will let $(\boldsymbol{I}_\mathrm{b}, {}^\dagger, \circ_\mathrm{b})$ be of the Boolean family of §62.5.1, for which I+, I−, I*−, K, and W, and hence all structural rules hold, and we will let $(\boldsymbol{I}, {}^*, \circ)$ be of some other family. See §62.3.4 for the notation "$\Leftrightarrow$" and its cousins.

FACT 5-2. (UNIQUENESS OF BOOLEAN FAMILY.) There is only one Boolean family: if I+, I−, I*−, K, and W are postulated for $(\boldsymbol{I}, {}^*, \circ)$, then $\boldsymbol{I}\Leftrightarrow\boldsymbol{I}_\mathrm{b}$, $(\mathrm{X}\circ\mathrm{Y})\Leftrightarrow(\mathrm{X}\circ_\mathrm{b}\mathrm{Y})$, and $\mathrm{X}^*\Leftrightarrow\mathrm{X}^\dagger$.

It seems likely (but we do not have a proof) that this is the only uniqueness result of its kind: if a set of structural rules on the connectives of some one family is enough to confer uniqueness for all three elements of the family, then that set of rules implies I+, I−, K, and W. Otherwise put, we conjecture that, for every family other than the Boolean, it is possible to have multiple families satisfying the same one-family structural postulates without interreplaceability.

FACT 5-3. (BINARY REDUCTION.) Suppose WI {or ⊢WI} for ∘. Then $(\mathrm{X}\circ\mathrm{Y})\ \mathrm{a}{\Rightarrow}\ (\mathrm{X}\circ_\mathrm{b}\mathrm{Y})$ {or $(\mathrm{X}\circ\mathrm{Y})\ \mathrm{c}{\Rightarrow}\ (\mathrm{X}\circ_\mathrm{b}\mathrm{Y})$}. (The only features required of the Boolean family are K (hence ⊢K) and KI.)

FACT 5-4. (STAR DISTRIBUTION OVER BOOLEAN FAMILY.) The * of any family distributes over the Boolean connectives: $\boldsymbol{I}_\mathrm{b}^*\Leftrightarrow\boldsymbol{I}_\mathrm{b}$, $(\mathrm{X}\circ_\mathrm{b}\mathrm{Y})^*\Leftrightarrow(\mathrm{X}^*\circ_\mathrm{b}\mathrm{Y}^*)$, and $\mathrm{X}^{\dagger *}\Leftrightarrow\mathrm{X}^{*\dagger}$.

The following give us $\boldsymbol{I}_\mathrm{b}^*\Leftrightarrow\boldsymbol{I}_\mathrm{b}$:

$$\boldsymbol{I}_\mathrm{b}^*\ \mathrm{a}{\Rightarrow}\ (\boldsymbol{I}_\mathrm{b}\circ_\mathrm{b}\boldsymbol{I}_\mathrm{b}^*)^*\ \mathrm{a}{\Rightarrow}\ \boldsymbol{I}_\mathrm{b}^{**}\ \mathrm{a}{\Rightarrow}\ \boldsymbol{I}_\mathrm{b};\ \boldsymbol{I}_\mathrm{b}\ \mathrm{a}{\Rightarrow}\ (\boldsymbol{I}_\mathrm{b}\circ_\mathrm{b}\boldsymbol{I}_\mathrm{b}^*)\ \mathrm{a}{\Rightarrow}\ \boldsymbol{I}_\mathrm{b}^*;$$

and exactly similar moves yield $\boldsymbol{I}_\mathrm{b}^*\ \mathrm{c}{\Leftrightarrow}\ \boldsymbol{I}_\mathrm{b}$ as well.

Next consider $(\mathrm{X}\circ_\mathrm{b}\mathrm{Y})^*$ and $(\mathrm{X}^*\circ_\mathrm{b}\mathrm{Y}^*)$. For right to left, each of X and Y reduces to $(\mathrm{X}\circ_\mathrm{b}\mathrm{Y})$ in both antecedent and consequent positions; so WI and

$\vdash$ WI now suffice. For left to right, each of X and Y reduces to $(X^*\circ_b Y^*)^*$ in both antecedent and consequent positions; so again WI and $\vdash$ WI now suffice.

Lastly, consider $X^{*\dagger}$ and $X^{\dagger *}$. Start with $X^{*\dagger} \vdash Y$. Then: $Y^\dagger \vdash X^*$; $X \vdash Y^{\dagger *}$; $X \vdash (Y^{*\dagger}\circ_b Y^{\dagger *})$; $Y^* \vdash (X^\dagger \circ_b Y^{\dagger *})$; $(X^\dagger\circ_b Y^{\dagger *})^* \vdash Y$; $(X^{\dagger *}\circ_b Y^{\dagger **}) \vdash Y$ (by the distribution of star over $\circ_b$, just proved); $(X^{\dagger *}\circ_b Y^\dagger) \vdash Y$; $X^{\dagger *} \vdash (Y\circ_b Y)$; $X^{\dagger *} \vdash Y$; so $X^{*\dagger}$ a$\Rightarrow$ $X^{\dagger *}$. And $X^{*\dagger}$ c$\Rightarrow$ $X^{\dagger *}$, by an analogous argument. The first of these implies that $X^{\dagger *}$ c$\Rightarrow$ $X^{*\dagger}$, and the second that $X^{\dagger *}$ a$\Rightarrow$ $X^{*\dagger}$; so we are done with proving that $X^{*\dagger} \Leftrightarrow X^{\dagger *}$.

These arguments were uncovered by reflecting on the proof of Meyer 1976c that the relevance and Boolean negations permute.

Fact 5-5. (Star distribution with CI.) If CI holds, then

$$(X\circ Y)^* \Leftrightarrow (X^*\circ Y^*).$$

Hence, under the same assumption, if the Boolean family is the only other family present, and assuming the rules I*+ and I*−, all *s may be pushed inside to formulas. (But note: it does not follow that the Boolean negative structuring, †, can be pushed inside structure-connectives from other families.)

Fact 5-6. (Equivalence of *I*s.) Let I+, I−, I*+, I*−, CI/I, and KI hold for each of two (e.g., modal) families. Then their *I*s are equivalent.

§62.6. Further developments. This section raises some possibilities and questions.

§62.6.1. Demarcation. It would be a matter of great interest to characterize those logics which can and those which cannot be codified by means of the techniques of Display logic. On the other hand, we do not think that Display logic should be viewed as itself setting the boundary of the province of logic (Kneale 1956) in the style of Hacking 1979. Logic is that discipline which tries to shed light on the problem of separating the good inferences from the bad; we do not therefore propose to use some technical property not closely connected with that aim to mark off Logic from Nonlogic, or to use such a property to defend a historically given logic as somehow privileged. For example, of those logics offered as philosophically interesting, quantum logic is one that we see no way of catching by the techniques of Display logic (it also eludes Hacking 1979). This is equally true of the logic answering to the theory of modular lattices, which presents a somewhat simpler version of the same problem. But we should *not* conclude that quantum logic is not a logic. Whether it is or is not of significance in sorting good from bad arguments must be argued on quite other grounds.

§62.6.2. Quantifiers. Quantifiers may be added with the obvious rules:

$$(\mathrm{UQ})\ \frac{Aa \vdash \mathrm{X}}{\forall xAx \vdash \mathrm{X}} \qquad \frac{\mathrm{X} \vdash Aa}{\mathrm{X} \vdash \forall xAx}$$

provided, for the right-hand rule, that *a* does not occur free in the conclusion. (The rule for the intuitionist universal quantifier, however, would involve ***I***.) The rule for the existential quantifier would be dual. The abstract details of C6, C7, and C8 would need complicating, but not the ideas. One might talk about *variants*' of inferences being isomorphic with respect to the analysis into parameters and congruence classes.

On the other hand, as yet this addition provides no extra illumination, doubtless because these rules for quantifiers are "structure free" (no structure-connectives are involved; see also §62.6.5). One upshot is that adding these quantifier rules to modal logic brings along the Barcan formula and its converse (see Hughes and Cresswell 1968) willy-nilly, which is an indication of an unrefined account; alternatives therefore need investigating. Introducing a family for each constant helps.

§62.6.3. Interpolation. Since both interpolation (see §15.2) and the Elimination theorem 4-4 require "enough connectives," we had hoped that Display logic could have been used for an interpolation theorem. But in July 1989 (as reported in an address to the Third Logic Biennial at Chaika, Bulgaria, in June 1990), Urquhart proved by the geometric methods of §65 that interpolation fails between **T** and **KR**.

§62.6.4. Algebra. Evidently algebra is in the air, especially residuation; see §28.2. The most immediate inspiration for the algebraic flavor is Meyer and Routley 1972.

If one did not have *, one would have some residuals in each family, using the Display theorem 3-2 as a guide. For example, suppose that we replace * in each family by a pair of binary structural connectives $\mathrm{X}-\mathrm{Y}$ and $\mathrm{X}--\mathrm{Y}$, thinking of X as positive and Y as negative substructures. Then the following equivalences would (for example) suffice: $\mathrm{X} \vdash \mathrm{Y}\circ\mathrm{Z}$ and $\mathrm{X} \vdash \mathrm{Z}\circ\mathrm{Y}$, as before; $\mathrm{X} \vdash \mathrm{Y}\circ\mathrm{Z}$ and $\mathrm{X}-\mathrm{Y} \vdash \mathrm{Z}$ and $\mathrm{X}--\mathrm{Y} \vdash \mathrm{Z}$ (the two new connectives are not different on the left); $\mathrm{X}\circ\mathrm{Y} \vdash \mathrm{Z}$ and $\mathrm{X} \vdash \mathrm{Z}-\mathrm{Y}$ and $\mathrm{Y} \vdash \mathrm{Z}--\mathrm{X}$.

In the same spirit, one might look at the case when one refuses to postulate commutativity for $\circ$ on the right of the turnstile.

§62.6.5. Other connectives. One sees that the basic three-place relation is $\mathrm{X}\circ\mathrm{Y} \vdash \mathrm{Z}$, or, with equal fundamentality, $\mathrm{X} \vdash \mathrm{Y}\circ\mathrm{Z}$. So, for the premiss for the rule for a binary connective in which the components are together, there are two possibilities: in the place of X and Y (or Z), and in the place of Y and Z. When one adds * to get the effect of positive and negative, one gets many possibilities. Only some are directly realized in our formula-connective rules; for example, we miss an arrow $A \rightarrow B$ with rule $A \circ \mathrm{X} \vdash B$ yielding $\mathrm{X} \vdash A \rightarrow B$. Of course in the presence of CI such an arrow would not be much of an addition. There are also other possibilities involving I.

There is also the possibility of "structure-free" formula-connectives, the rules for which involve no structure-connectives; for example, the rules of Gentzen 1934 for conjunction were such:

$$\frac{A \vdash \mathrm{X}}{A \& B \vdash \mathrm{X}} \qquad \frac{B \vdash \mathrm{X}}{A \& B \vdash \mathrm{X}} \qquad \frac{\mathrm{X} \vdash A \qquad \mathrm{X} \vdash B}{\mathrm{X} \vdash A \& B}$$

Such formula-connectives should doubtless be specially marked (or unmarked) to indicate their independence of any family. These connectives seem to be thought central to the "linear logic" enterprise of Girard 1987; see §83.2 for some additional references. We think that conjunction and disjunction (with dual rules) are the only two possibilities; in particular, that there is no structure-free negation connective, nor any structure-free implication.

Note that distribution cannot be obtained for these formula-connectives without appeal to structural elements and that, in the presence of the Boolean family, not only is distribution forthcoming, but these structure-free formula-connectives agree with the corresponding Boolean connectives. (This is a paradigm case of failure of conservative extension in **DL**.)

In any event, the spirit of **DL** suggests that only those formula-connective rules be postulated which allow Fact 4-2 to go through, thus strengthening C8 by forbidding use of any but display-equivalences in reducing the complexity of the formula eliminated in (ER). But see the treatment of **S2** and **S3** in §62.5.6 for some rules that do not follow this suggestion.

§62.6.6. Restricted rules. Curry 1963 and others (including ourselves in §61.2) obtain modal logic by restricting the rule for $\Box$ on the right—requiring every formula on the left (thinking only of commas) to have the form $\Box B$. The Elimination theorem 4-4 survives in the presence of such rules. That is, instead of adding a structural family for modality, one can keep the nonmodal family only, say the Boolean family or the relevance family, and instead place restrictions on the rules. Exactly how these restrictions have to go is controlled by conditions C6 and C7 (§62.4.2).

For example, obtaining a **DL** version of **R** with an **S4** necessity (see §27.1.3 and §R2) based on a modal family of its own seems to require interfamilial postulates. But one can obtain it instead by using just the relevance and Boolean families and restricting the rule for $\Box A$ on the right as follows: the parameter X on the left may contain no formula as negative part, and each formula it does contain must have the form $\Box B$. It can then be seen that this rule satisfies C6 (trivially—there are no parametric formulas that are consequent parts) and C7 (not quite so trivially, but still easily) of §62.4.2. It does not seem possible to add an **S5** necessity (Bacon 1966) in the same way; positing a separate family appears to be the only way.

For intuitionism, instead of omitting structural rules from the full Boolean set, one can restrict the rules for introducing the intuitionistic connectives on the right. The restriction would be this: the antecedent of the conclusion may contain no formulas as negative parts; and each formula it does contain must be an h-formula (§62.5.7). Again verification of C6 and C7 is straightforward.

We can still show that **DL** is a conservative extension of **DL**{h} as follows: by the Subformula theorem 4-3, we need pay attention only to consecutions involving h-variables and intuitionistic formula-connectives (but with the possibility of structure-connectives from other families). Re-interpret all such consecutions in this way: S means that the conjunction of all its formula antecedent parts implies the disjunction of all its formula consequent parts. Then the restrictions guarantee that all rules are verified intuitionistically. (That is, we do not need to give a separate interpretation to * at all.)

For either formulation of **E** of §62.5.3 above, one would not have "the Ackermann property" discussed in §§5.2.1, 12, 22.1.1, and 45, according to which one does not have a theorem $A\to_e(B\to_e C)$ unless (in the "standard" vocabulary) A contains some implicative formula; for of course there is $\boldsymbol{I}_e \vdash A\to_e(B\to_e(A\&_e B))$. To restore this (we would say) happy property, one might restrict the rule for implication on the right in the manner suggested by the above discussion.

Let us be more definite. Let the family indexed with "e′" be just like the family indexed with "e" that we described in §62.5.3, except that the rule for introducing → on the right of the turnstile (§62.3.3) is restricted as follows: X may contain no formulas as negative parts, and each formula it does contain must have the form $C\to D$. The following points are all obvious.

(a) We can still prove all the axioms of **E** of §21.1 (or §R2) in the form: $\boldsymbol{I}_{e'} \vdash A$, provided A is in the standard vocabulary $\{\to_{e'}, \&_b, \vee_b, \sim_{e'}\}$.

(b) The Elimination theorem 4-4 still goes through, since the rules satisfy the conditions C2–C8 of §62.4.2 as before; in particular, the amended version of → does not violate C7. Accordingly we can prove the rules of **E**, and hence all its theorems—in the standard vocabulary.

(c) $\boldsymbol{I}_{e'} \vdash A\to_{e'}.B\to_{e'}.A\&_{e'}B$ is unprovable when A is a propositional variable.

Since the calculus we have defined is properly weaker than that of §62.5.3, it is possible—we do not say likely—that the question of conservative extension raised there is more easily decidable here than there.

§62.6.7. Incompatibility. There is some value in working through the "incompatibility version" of the above proceedings. This corresponds to (but does not imitate) the "left-handed systems" explained in §60.2.

The idea is straightforward: define an incompatibility relation $X|Y$ as $X \vdash Y^*$. Evidently the relation is family-relative, unlike the turnstile—which makes the whole thing less interesting. In the single-family case, however, or in the case in which the Boolean logic is taken as "primary," it is worth while working through what things look like in this new guise. For one thing, * tends to disappear except on formulas, and a new positive binary structure-connective $(X{:}Y) = (X^* \circ Y^*)^*$ turns out to do some work.

Since there is such a close relation between "analytic tableaux" and one-sided consecution calculuses, perhaps this suggests that the proper way to arrive at an analytic-tableau formulation for **DL** on the model of §60 would be to use an essentially relational idea, as in §50.

§62.6.8. Binary structuring and infinite premiss sets. Why didn't Gentzen 1934 use a binary structure-connective instead of polyvalent commas? (The idea is due to Meyer 1976c.) Of course, for the fellow who leaps Platonistically to thinking of the stuff on the left of the turnstile as intending a *set*, there would be no point to binary structuring. And, even if one thought of what is on the left of the turnstile as a *sequence*, in the abstract sense, binary structuring would not be likely to emerge. Perhaps this was Gentzen's picture; for he was careful in his formalistic way to *postulate* the rules WI (contraction) and CI (permutation), while evading the necessity of worrying about an associativity rule such as B only by the gimmick of using commas as polyvalent. (Not to be misleading, let us note that B in fact follows from WI, K, and KI, in contrast to the definability situation in combinatory logic.)

Perhaps Gentzen did not much worry about the theory of the grammar of his L calculuses. For example, although Gentzen 1934 once speaks of his comma as an auxiliary symbol (2.3, p. 71), he does not list it with the two parentheses and the arrow when he is officially listing the "auxiliary symbols" of his language (1.1, p. 70). (References are to the Szabo translation.)

There is tension here, and several ways to resolve it. One is by construing the left as a set name from the beginning, as some have done. That misses possibilities, but is coherent. Another way is to use the notion of a "fireset," as in McRobbie 1979. ("Firesets," or "finitely repeatable sets" are more commonly known as (finite) "multisets.") That is also coherent, but again misses possibilities. The only device that misses nothing is to take structuring as binary instead of polyvalent. And we think on reflection, that this course is more in the spirit of Gentzen's cautious postulation of WI and CI than are the later leaps to sets or firesets.

We are arguing not that binary structuring is more intuitive, but instead that it is more satisfactory from a mathematical point of view. We are recommending binary structuring on quite the same grounds that have led nearly everyone to prefer binary conjunction in formal systems to a polyvalent ("run on") conjunction.

It might be objected that the limitation to binary structuring prevents generalization to infinite sets of premisses; but this is not so. To guide imagination, picture a structure X as a tree; now (while keeping at most binary forking at each node) let its branches be infinite. Why not?

§62.6.9. Priority of the right? Others (Curry 1963, p. 173, cites Lorenzen) have been able to find a special priority for the rules introducing connectives on the right. It might appear that we share this vision, given the asymmetry in the conditions C6 and C7 (§62.4.2) and the related asymmetry between Stages 1 and 2 (§62.4.3). But the appearance is illusory: although one of the logics we treat, namely intuitionism, is asymmetrical in this way (it is the only one of the logics treated in §62.5 that requires the asymmetry; but see §62.6.6 for others), the method is not in itself asymmetrical. That is, there could well be another logic that required giving priority to Stage 2 over Stage 1, a kind of dual of intuitionism. These methods could treat that logic equally well, but could not treat both that logic and intuitionism at the same time. (See Belnap 1990 for a reworking of the Stages that obliterates even the appearance of priority by relying on conditions that are entirely symmetrical as between left and right. In this way Display logic is given the ability to handle simultaneously a richer variety of logics than is possible with the present conceptualization.)

Perhaps it is worth noting here that our primary treatment of modal logics **S4** and **S5**, in §62.5.6, does *not* involve an asymmetry—none of the rules are restricted in any way.

A related view is that the left rule for a connective can somehow be "deduced" from the right rule. Some weak version of this is likely correct, but the rule ($\Box'$) for **S2**–**S3** in §62.5.6 comes close to providing counterevidence. Nor does the possibility of this "deduction" suggest an asymmetry, unless one were prepared to argue that the reverse "deduction" was not equally possible.

§63. Decidability: Survey. For almost thirty years the decision problems for the various propositional calculus fragments of the principal relevance logics remained unacceptably open (though Meyer early on showed the decidability of the "semi-relevant" system **RM**—see §29.3.2). Only with the work of Urquhart reported in §65 do we know that they uniformly have a negative answer: there is no mechanical procedure by which to decide whether a candidate is or is not a theorem of the calculus **E** of entailment (and similarly for the other calculuses in the neighborhood). This negative answer is all the more interesting because of the truth, when written, of the remark of Harrop 1965 that "all philosophically interesting propositional calculi for which the decision problem has been solved have been found to be decidable." It is certainly not too much to attribute undecidability to the relevance intuitions themselves (in contrast, say, to modal or constructive intuitions), since the absence of a decision procedure is invariant over various tinkerings with the postulational structures in the field of relevance logic. Undecidability was, furthermore, from at least one point of view to be "expected," since relevance insights have always been taken to be essentially *relational*, and one knows that it is in the presence of relations (in contrast to mere properties) that undecidability seems to be found.

In any event, undecidability of logical truth of formulas involving relevance connectives is a matter of fact; see §65 for details. There are, however, a number of positive decidability results for calculuses that are in some sense or another *partial*; without claiming any sort of completeness where none is possible, we undertake to survey enough of these results to create an overall picture.

§63.1. Decidability of fragments limited by degrees. As in §15, "degree" refers to the degree of nesting of arrows; one may secure decidability by limiting degree.

Zero degree formulas in the sense of §15 are just formulas without arrows, hence with only the standard truth-functional connectives. The zero degree fragment of **E** (or of any of the relevance logics) can be decided by the usual two-valued truth tables or by any other equivalent procedure. Ho hum. One simple proof-theoretical procedure, closely tied to relevance considerations, is described in §24.1.

First degree entailments are entailments between truth-functional (zero degree) formulas. The provable ones (in any of the relevance logics) are the "tautological entailments" of §15 and, more generally, of Chapter III. There you will find both proof-theoretical procedures, including a normal-form argument (§15.2), and semantic procedures, including a simple application of a four-valued matrix due to Smiley (§15.3). In that section the matrix is given almost purely as an abstract structure, but it nevertheless appears to be closely bound to relevance. It recurs with considerable frequency in studies guided by that consideration—we think most recently of an application of it to the Barwise-Perry "situation logic" which was described in a 1984 talk by Fenstad, almost twenty years after its earliest concrete formulation in Dunn 1966.

First degree formulas are truth functions of first degree entailments and zero degree formulas; that is, the first degree formulas are those with no nesting of arrows inside other arrows. One decision procedure is presented in §19 and another closely related procedure in §40; both rely on products of the eight-valued matrix M_0 presented in §18.4 and are considerably more combinatorial than the cases described above.

Second degree formulas permit arrows to occur within the scope of arrows, but do not permit additional nesting: no arrows within arrows within arrows. §64 outlines the argument of Meyer that the decision problem for each of the relevance logics reduces to the problem for its second degree fragment; and that section also provides a positive solution for the special case

$$(A_1 \to B_1) \& \ldots \& (A_n \to B_n) \to . \; C \to D$$

of a conjunction of first degree entailments entailing an entailment.

Though a little more is doubtless possible, the last-mentioned result completes our analysis by degrees.

§63.2. Decidability of fragments limited by connectives. We now start another tack. Rather than look at fragments delimited by complexity of formulas, we instead consider fragments delimited by the connectives that they contain.

Implication fragments. The earliest result of this kind is due to Kripke 1959b, who gave a decision procedure for the implicational fragments of **E** and **R** which was based on a Gentzen consecution calculus. This result is in effect presented in §13.

The "merge" consecution calculuses of §7 were invented in order to try to contribute to the solution of the various decision problems, but, as reported in §7.5, they did not succeed in doing so. The following questions from that section remain open:

PROBLEM. Can decision procedures be based upon the merge formulations?

PROBLEM. Is the implicational fragment $\mathbf{T}_\to$ of **T** decidable? (The question is equally open for richer fragments of **T**.)

Implication and negation fragments. The technique of Kripke 1959b carries over at once to the implication-negation fragment of **R**, but some combinatorial work needs to be supplied in order to adapt it to the implication-negation fragment of **E**; see §13.

Implication-conjunction fragments. Meyer 1966 showed that Kripke's technique extends easily to the implication-conjunction fragment of **R** (incidentally, contraction can be dropped without affecting the arguments). The idea is to add to the Kripke consecution formulation LR the rules:

$$(\&\vdash) \quad \frac{\alpha, A \vdash C}{\alpha, A\&B \vdash C} \quad \frac{\alpha, B \vdash C}{\alpha, A\&B \vdash C} \qquad (\vdash\&) \quad \frac{\alpha \vdash A \qquad \alpha \vdash B}{\alpha \vdash A\&B}$$

Note that it is important that the rule (&⊢) is stated in two parts, and not as one "Ketonen form" rule:

$$(\mathrm{K}\&\vdash) \quad \frac{\alpha, A, B \vdash C}{\alpha, A\&B \vdash C}.$$

The reason is that, without weakening (§7.2), it is impossible to derive the rule(s) (&⊢) from (K&⊢). These techniques apply straightforwardly when fusion is added with the rules:

$$(\circ\vdash)\quad \frac{\alpha, A, B \vdash C}{\alpha, A\circ B \vdash C} \qquad\qquad (\vdash\circ)\quad \frac{\alpha \vdash A \qquad \beta \vdash B}{\alpha, \beta \vdash A\circ B}$$

Also, the techniques are unaffected by the addition of the sentential constant $\boldsymbol{t}$ with the axiom $\vdash \boldsymbol{t}$ and the rule:

$$(\boldsymbol{t})\quad \frac{\alpha \vdash A}{\alpha, \boldsymbol{t} \vdash A}$$

Implication-disjunction fragments. Nothing is known about these (see §28.3.2 for the briefest of mentions), nor do we think the question likely to be interesting.

Positive fragments. An upshot of §65 is that the positive fragments of all the principal relevance logics are undecidable: leaving out negation and contenting oneself with →, &, and ∨ doesn't make the least trifle simpler the problem of separating the good guys of relevance logic from the bad guys. None of the Gentzen control of §§28.5 and 61 helps at all.

§63.3. Decidability of neighbors. We do not presume to survey the decidability of the furious farrago of systems arising from the logicians' love of tinkering; instead we mention an *ad hoc* list of topics.

First, **R** *without distribution.* If the distributive axiom is subtracted from the usual list of postulates for **R** (§R2), the resulting calculus is decidable. The result, mentioned in Meyer 1966, is an easy extension of Kripke 1959b: just add both some conjunction and some disjunction rules to Kripke's weakening-free formulation of the implication fragment of **R**, as mentioned in §63.2 above (also fusion, its dual "fission," and $\boldsymbol{t}$ and $\boldsymbol{f}$ can be added with natural rules). (We think no one has done all the homework to verify that the same is true for **E**.) Incidentally, the presence or absence of contraction does not affect the arguments. (These old results might matter for computer-science considerations; see the end of §84, and especially the work of Girard 1987, Avron 1987, and Rezus 198+a.)

Second, *two undecidable neighbors of* **R**. Though perhaps not meriting a secure place among the Forms, the system of Meyer and Routley 1973a deserves special mention: though it was made up to be undecidable (in virtue of harboring the word problem), the system looks sensible, with only the smallest flavor of the *ad hoc*. It is a historical marker on the road to Urquhart's general undecidability result as reported in §65. That result pushes undecidability aloft to the system **KR**, also defined by Meyer, that results when the negation of **R** is given the irrelevant property, $A\&\bar{A}\to B$. Since the implication-conjunction fragment of **R** is, as we said, decidable, it is worth adding that the implication-conjunction fragment of **KR** is undecidable (§65.2.5).

Third, *contraction subtraction*. The contraction axiom

$$(A\to.A\to B)\to.A\to B$$

has always put difficulties in the way of decidability; for example, in the context of Gentzen consecution calculuses it forces premisses to be longer than their conclusions and thus puts the threat of infinite searches. What happens if we consider systems resulting from the more familiar systems by deletion of this bugbear?

Implicational fragments without contraction. As we mention in §63.2, the decision problem for the implicational fragment of **T** is open; but §66 decides that fragment without contraction. (An alternative proof is mentioned in §7.5.) Since §63.2 indicates that other implicational systems are decidable even with contraction, that observation can complete this part of our report.

Positive fragments without contraction. §67 shows that subtracting contraction from the positive fragment of **T** or of **R** suffices for decidability. §65 shows that adding a variant of a *modus ponens* axiom is enough to restore undecidability. Incidentally, there is a potentially confusing subtlety. For $\mathbf{T}^{+}-\mathbf{W}$, the problem of *theoremhood* is solvable (§67), whereas the problem of deducibility (from finite premiss sets) is *not* (see §65.2.3 and §65.2.5 for definitions and results). Similarly, the *deducibility* problem for $\mathbf{E}_{+}$ is unsolvable (see §65.2.5). But the question of the decidability of the contraction-free positive fragment of **E** remains open (see §67.6).

E, **T**, *and* **R** *without contraction.* The questions of decidability for these systems, whether interesting or not, remain open.

Fourth, *further weakenings* permit decision by the model-theoretic methods of §51, which should be consulted.

Fifth, *strengthening in ways we deem irrelevant*—as we say in §29.5—leads to **RM**, which is shown decidable in §29.3.2. All its normal extensions are also decidable, as shown in §29.3.3.

Sixth, the addition of *monadic quantifier formulas* is shown in §41 to lead at once to undecidability.

§64. Which entailments entail which entailments? We offer a procedure for deciding when a conjunction of entailments provably entails a single entailment. First, some context.

The context from below is chiefly supplied in §19 and §24.3 (see also §40.7), where we showed how to decide provability for *first degree formulas* (no nesting of arrows). From above, the context is provided by Meyer 1979, who shows by a surprisingly simple argument that the decision question for *second degree formulas* (arrows within arrows O.K., but no arrows within arrows within arrows) is equivalent to the decision question for the entire calculus—for just about any calculus you can think of. Since we know from §65 below that the principle relevance logics are one and all undecidable, we cannot hope to settle the *general* decision problem for second degree formulas. This is what makes the result reported here for a special kind of second degree formula have some interest.

We use §64.1 to sketch with great brevity the argument of Meyer 1980 for the reducibility of the decision problem to the second degree. Then in §64.2 we show how to decide the positive case of a conjunction of entailments entailing an entailment, and in §64.3 we add what is necessary to carry out the argument in the presence of negation.

§64.1. Reducibility of the decision question to the second degree. We use Meyer 1980. Let $\boldsymbol{t}$ be characterized as in §R2 so that it is provable and provably implies all instances $A \rightarrow A$ of identity. Let the horseshoe be material "implication":

$$A \supset B =_{\text{df}} {\sim} A \vee B.$$

Then it is perfectly clear that, for every calculus S we have looked at, the following hold interchangeably:

1 $\vdash_{\mathrm{S}}(\ldots A \ldots)$

2 $\vdash_{\mathrm{S}}[(\boldsymbol{t}\&(p \rightarrow A)\&(A \rightarrow p)] \supset(\ldots p \ldots).$

In fact, given 1, it is easy to see by the Light of Natural Reason that we can establish 2 not only as a material "implication" but even as a real implication,

2′ $\vdash_{\mathrm{S}}[(\boldsymbol{t}\&(p \rightarrow A)\&(A \rightarrow p)] \rightarrow(\ldots p \ldots),$

in any of the calculuses we have considered; the Light shows that it is a matter of having the right sort of replacement principles. ($\boldsymbol{t}$ is needed to supply instances of $A \rightarrow A$ which are perhaps needed to help in making replacements in conjunctive or disjunctive contexts and, in the weaker calculuses, to yield $(\ldots . A \ldots)$ itself; we skip the details.) And, given 2′, one can move to 2 by easy steps. The reverse direction, from 2 to 1, involves first a substitution of A for p, and then the rule (γ)—detachment for material "implication"—as established in §42 for *all* the systems we consider.

It is also perfectly clear that, if we choose A in 1 and 2 as a formula $A_1 \rightarrow A_2$ where A_1 and A_2 contain *no* arrows, we can gradually reduce the amount of nesting we need to consider to that represented by $p \rightarrow (A_1 \rightarrow A_2)$ and $(A_1 \rightarrow A_2) \rightarrow p$; that is, to the second degree. And $\boldsymbol{t}$ itself can be replaced as in §45.1 by a conjunction of identities $q \rightarrow q$ between propositional variables.

§64.2. The positive case. In order to highlight the main line of our argument, we first address the positive case of the question, When does a conjunction of entailments entail an entailment? We answer this form of the question by supplying a common decision procedure for all systems between $\mathbf{B}_+$ +Conjunctive transitivity, that is, $(A\to B)\&(B\to C)\to.A\to C$, and $\mathbf{R}_+$. ($\mathbf{B}_+$ was defined in §48.5, and $\mathbf{R}_+$ in §27.1.1 or §R2.) The decision procedure was found in 1966; the present version, which dates from a decade later, translates the semantic basis of the procedure from an algebraic form to a form based on the three-termed relational semantics described in §48.

First, some notational conventions.

A_i, B_i, C, and D range over zero degree (arrow-free) formulas.
$P =_{\mathrm{df}} (A_1\to B_1)\& \ldots \&(A_n\to B_n)$
$U =_{\mathrm{df}} P\to.C\to D$
$\mathrm{N} =_{\mathrm{df}} \{1, \ldots, n\}$
W, X, Y, Z range over nonempty proper subsets of N.
$\vee A_{\mathrm{X}} =_{\mathrm{df}} A_{i_1}\vee \ldots \vee A_{i_p}$, for $\mathrm{X} = \{\mathrm{i}_1, \ldots, \mathrm{i}_p\}$.
$\& A_{\mathrm{X}} =_{\mathrm{df}} A_{i_1}\& \ldots \& A_{i_p}$, for X = ditto.

Define a set J of formulas to be a U_+-*set* iff

(1) every formula in J has one of the forms $C\to\vee A_{\mathrm{X}}$, $\&B_{\mathrm{Y}}\to\vee A_{\mathrm{X}}$, and $\&B_{\mathrm{Y}}\to D$, and there is a binary relation C on J such that

(2) if *FCG* then *F* has one of the first two forms above (so its consequent is $\vee A_{\mathrm{X}}$), *G* has one of the second two forms (so its antecedent is $\&B_{\mathrm{Y}}$), and $\mathrm{X}\cup\mathrm{Y} = \mathrm{N}$;

(3) for some X, Y, $(C\to\vee A_{\mathrm{X}})\mathrm{C}(\&B_{\mathrm{Y}}\to D)$; and

(4) C is *strongly dense* in J: if *FCG* then, for some $H\in\mathrm{J}$, both *FCH* and *HCG*. Hereafter, by "dense" we mean strongly dense.

Define a set J of formulas to be *provable* in a given system if some disjunction of members of the set is provable.

THEOREM. Let $\mathbf{S}_+$ be any system between $\mathbf{B}_+$ + Conjunctive transitivity and $\mathbf{R}_+$. Then a negation-free *U* is provable in $\mathbf{S}_+$ just in case so also are $C\to\vee A_{\mathrm{N}}$, $\&B_{\mathrm{N}}\to D$, and every U_+-set.

This will provide a decision procedure for *U*, because (a) there are only finitely many formulas of the sort specified in (1) of the definition of U_+-set; (b) checking whether a subset J of these meets clauses (2)–(4) is effective; (c) J is provable just in case one of its members is (by §19.5 and §24.3); (d) a member is provable just in case it is a tautological entailment (by §24.2); and (e) this is decidable (by §15.1 or §15.3 or §17).

PROOF. For sufficiency of the provability of $C\to\vee A_{\mathrm{N}}$ and $\&B_{\mathrm{N}}\to D$, together with all the U_+-sets, for the provability of *U*, we observe the derivability, in the weakest $\mathbf{S}_+$ considered, of the following two rules ($-\mathrm{X} =_{\mathrm{df}} \mathrm{N}-\mathrm{X}$).

Rule 1. Conclusion: U. Premisses:

$$(C\to\bigvee A_{\mathrm{X}})\vee(P\to.\&A_{-\mathrm{X}}\to\bigvee B_{-\mathrm{Y}})\vee(\&B_{\mathrm{Y}}\to D)$$

is a premiss for each X, Y, neither being N, such that $\mathrm{X}\cup\mathrm{Y}=\mathrm{N}$. There are two more premisses: $C\to\bigvee A_{\mathrm{N}}$ and $\&B_{\mathrm{N}}\to D$.

Rule 2. Conclusion: $P\to.\&A_{-\mathrm{X}}\to\bigvee B_{-\mathrm{Y}}$, with neither X nor Y being N, and with $\mathrm{X}\cup\mathrm{Y}=\mathrm{N}$. Premisses:

$$(P\to.\&A_{-\mathrm{X}}\to\bigvee B_{-\mathrm{W}})\vee(\&B_{\mathrm{W}}\to\bigvee \mathrm{A}_{\mathrm{Z}})\vee(P\to.\&A_{-\mathrm{Z}}\to\bigvee B_{-\mathrm{Y}})$$

is a premiss for each W, Z, neither being N, such that $\mathrm{X}\cup\mathrm{W}=\mathrm{N}$ and $\mathrm{Y}\cup\mathrm{Z}=\mathrm{N}$.

We may justify Rule 1 as follows. Assume all its premisses, and choose one disjunct from each for a Big Distribution argument. Let $\{\mathrm{X}_i\}$ be the set of index-sets on the chosen $C\to\bigvee A_{\mathrm{X}_i}$ disjuncts and let $\{\mathrm{Y}_j\}$ be the set of index-sets on the chosen $\&B_{\mathrm{Y}_j}\to D$ disjuncts (these will be nonempty). Where $\{\mathrm{X}'_k\}$ is the set of all selection-sets over $\{\mathrm{X}_i\}$ (i.e., each X'_k has a nonempty intersection with each X_i) and where $\{\mathrm{Y}'_m\}$ is the set of selection-sets over $\{\mathrm{Y}_j\}$, we have (by modest distributions) both

$$C\to(\dots\vee(\&A_{\mathrm{X}'_k})\vee\dots)$$

and

$$(\dots\&(\bigvee B_{\mathrm{Y}'_m})\&\dots)\to D.$$

To obtain the conclusion of Rule 1 it suffices to show every

$$P\to.\ \&A_{\mathrm{X}'_k}\to\bigvee B_{\mathrm{Y}'_m}.$$

We have this whenever $\mathrm{X}'_k\cap\mathrm{Y}'_m\neq\varnothing$ by the definition of P. And when $\mathrm{X}'_k\cap\mathrm{Y}'_m=\varnothing$, consider that, since $-\mathrm{X}'_k\cup-\mathrm{Y}'_m=\mathrm{N}$, we must have

$$(C\to\bigvee A_{-\mathrm{X}'_k})\vee(\mathrm{P}\to.\&A_{\mathrm{X}'_k}\to\bigvee B_{\mathrm{Y}'_m})\vee(\&B_{-\mathrm{Y}'_m}\to D)$$

among the premisses of Rule 1. Because X'_k and Y'_m are selection-sets over $\{\mathrm{X}_i\}$ and $\{\mathrm{Y}_j\}$, respectively, our initial choice of disjunct for the Big Distribution must have been $P\to.\&A_{\mathrm{X}'_k}\to\bigvee B_{\mathrm{Y}'_m}$ (for no set can select from its own complement).

Our justification of Rule 2 is similar. With $\{-\mathrm{W}_i\}$ being all the index-sets (on the *B*s) of chosen first disjuncts and $\{-\mathrm{Z}_j\}$ being all the index-sets (on the *A*s) of chosen third disjuncts, we have, for the families of all selection-sets $\{\mathrm{W}'_k\}$ and $\{\mathrm{Z}'_m\}$ over $\{-\mathrm{W}_i\}$ and $\{-\mathrm{Z}_j\}$, respectively,

$$P\to.\ \&A_{-\mathrm{X}}\to(\dots\vee(\&B_{\mathrm{W}'_k})\vee\dots)$$

and

$$P\to.(\dots\&(\bigvee A_{\mathrm{Z}'_m})\&\dots)\to\bigvee B_{-\mathrm{Y}'}.$$

Now consider that we must have as one of the premisses of Rule 2 the instance with W'_k for W and Z'_m for Z; and, as for Rule 1, in each such case we must have chosen the middle disjunct

$$\&B_{W'_k} \to \vee A_{Z'_m}.$$

These suffice, with what we already have, to yield the conclusion of Rule 2.

Having established Rules 1 and 2 as derivable in even the weakest calculus S_+ considered in the Theorem, we return to the sufficiency of the provability of $C \to \vee A_N$, $\&B_N \to D$, and all the U_+-sets, for the provability of U; and we proceed by contraposition: suppose that U is unprovable. Then so is some premiss of Rule 1, and, if it is either $C \to \vee A_N$ or $\&B_N \to D$, we are home free. Otherwise we are going to find a U-set by constructing a *directed graph* G—i.e., a collection of *nodes* and *edges*, each edge having a node as *source* and a node (not necessarily distinct) as *target.* Furthermore, every edge will be *labeled.* Distinct edges might have the same label, but never both the same source and the same target.

Begin G by using "otherwise" to choose some unprovable premiss of Rule 1 having the form of the displayed three-termed disjunction. Put in the outside disjuncts as nodes. Connect them by an edge from the left to the right. Label the edge with the middle disjunct.

To proceed, let us say that an edge E is *densed* in a graph if E is not a counterexample to strong density: i.e., E is densed iff there is in the graph a node such that there is an edge from the source of E to that node and an edge from that node to the target of E. If at a stage of the construction every edge in the graph so far constructed is densed, stop. Otherwise, choose some undensed edge E. Its label will be unprovable, and will be a fit conclusion of Rule 2. Choose an unprovable premiss of Rule 2. The middle disjunct F of the chosen premiss provides a node (possibly new, possibly already in the graph). Enter an edge from the source of E to F (unless there already is one), labeling it with the left disjunct of the chosen premiss, and also an edge from F to the target of E (unless there already is one), labeling it with the right disjunct of the chosen premiss.

This construction is bound to stop, since there are only finitely many possible nodes, hence only finitely many possible edges. The desired graph G has then been constructed. The set J of its nodes is clearly unprovable, and also clearly a U_+-set, defining C by: FCG just in case F and G are nodes in G such that there is in G an edge from F to G. Which finishes on the side of sufficiency.

Now for the converse. Suppose first that $C \to \vee A_N$ is unprovable. Take any $\mathbf{R}_+$ model structure (§48.3) with R0ab, a $\neq$ b. Use The Way Up of §42.1–2 to find a prime $\mathbf{R}_+$-containing $\mathbf{R}_+$-theory containing C but not $\vee A_N$, and call it $S(C \to \vee A_N)$. Make variables true at a iff in $S(C \to \vee A_N)$, and false every place else. Since all A_i are false everywhere, P is true at 0. But C is true at a, and D is false at b, which makes U false at 0, hence unprovable in $\mathbf{R}_+$. The argument when $\&B_N \rightarrow D$ is unprovable is similar.

For the rest, let J be an unprovable U_+-set. We need to show U unprovable in $\mathbf{R}_+$. First, some definitions, and then a lemma relating the strong density feature of U_+-sets to $\mathbf{R}_+$ model structures.

DEFINITIONS. Given a binary relation C on a set J,

$C^i = C/C \ldots C/C$ (i Cs; "/" for relative product).

$C^\exists$ is the transitive closure of C, so $aC^\exists b$ iff aC^ib, for some i (hence the notation; we need to reserve the more usual *).

C is (*strongly*) *dense in* J ("mediated" in Belnap 1967) iff aCb implies aC^2b.

CONVENTION. a, b, c, d, e, f range over J.

DEFINITIONAL FACTS (used only silently):

If aC^ib then $aC^\exists b$.

If $aC^\exists b$ then aC^ib, for some i.

If $aC^{i+j}b$ then aC^ic and cC^jb, for some c.

If $aC^\exists b$ and $bC^\exists c$ then $aC^\exists c$.

DENSITY FACTS. If C is strongly dense in J,

1. If aC^ib then aC^jb whenever $i \leq j$.
2. If $aC^\exists b$ then for each j there is a c such that: $aC^\exists c$ and cC^jb.
3. If $aC^\exists b$ then for each j there is a c such that: aC^jc and $cC^\exists b$.
4. If $aC^\exists b$ then there is a c such that: $aC^\exists c$ and $cC^\exists b$.

DENSITY LEMMA. Let C be strongly dense in J. Then there is an $\mathbf{R}_+$ model structure $\langle K, R, 0\rangle$ such that $J \subseteq K$, and there is a point $p_1 \in K$ such that, for a, b $\in$ J,

Rp_1ab iff aCb.

PROOF. First define K by adding to J a denumerable family of points, all distinct from those in J and from each other: $0, p_1, \ldots, p_i, \ldots$. 0 is defined as 0. Now define R on K as follows. (Recall that, by convention, a, b, c, d, e, f range over J.)

1. $R0xy$ iff $x = y$
2. $Rx0y$ iff $x = y$
3. $Rxy0$ iff $x = y = 0$
4. Rp_ip_jx iff $x =$ some p_k, and $k \leq i + j$
5. Rp_iap_j is TRUE
6. Rap_ip_j is TRUE

7. Rp_iab iff aC^ib
8. Rap_ib iff aC^ib
9. $Rabp_i$ is TRUE
10. Rabc iff one of
10.1. a = b = c
10.2. aC^3c
10.3. bC^3c.

(If it is desired to keep the $\mathbf{R}_+$ model structure finite, this can be done when there is a longest C^3-chain in J; i.e., when there is an n such that $C^{n+1} = C^n$. Then it is necessary only to add $n + 1$ points $0, p_1, \ldots, p_n$, which keeps the $\mathbf{R}_+$ model structure finite if J is. Nothing would need changing in the definition of the R relation above. And the verification below of Pasch's postulate would go through, too, except that, when asked to choose p_{i+j} with $i+j$ over the maximum n, we choose p_n instead.)

It is obvious that Rp_1ab iff aCb. But to see that this is an $\mathbf{R}_+$ model structure, we need to verify five items from §48.3. Identity and Monotony are trivial in virtue of 1–2. Commutativity (permutation of the first two arguments of R) is trivially built in. Idempotence, Rxxx, is trivial; so Pasch is the only problem. (We verify Pasch instead of Associativity by historical accident; the two principles are in context equivalent.)

For Pasch, we are given R^2wxyz, i.e., Rwxg and Rgyz—we sometimes call g "the *g*iven link." We want R^2wyxz, i.e., for some m (we call it "the *m*issing link"), Rwym and Rmxz.

Trivial arguments suffice when any of w, x, g, y, z are 0.

Let $z = p_m$. If any of w, x, y are in J, the wanted R^2wyxz holds quite generally—by 4, 5, 6, 9. The remaining subcase begins with $R^2p_ip_jp_kp_m$. A little calculation using 4 shows that we can choose the missing link as p_{i+k}.

In the remaining cases, we are assuming z∈J.

Case 1. $y = p_k$. The given link cannot by 4 be p_i; so, with g∈J, we must have gC^kz.

Case 1.1. $x = p_j$. Observe that, since the given link is not p_i, w cannot be p_m. So we must have w∈J, and wC^jg, hence $wC^{j+k}z$. Choose the missing link so that wC^jm and mC^kz.

Case 1.2. x∈J.

Case 1.2.1. $w = p_i$; so xC^ig; so $xC^{i+k}z$. Choose the missing link as p_{i+k}.

Case 1.2.2. w∈J. Cases on Rwxg are given by 10. If w = x = g, xC^kz; so missing link can be p_k. If wC^3g, then wC^3z. Use missing link promised for *k* by Density fact 3. If xC^3g then xC^3z. Use p_m where xC^mz.

Case 2. y∈J.

Case 2.1. The given link is p_k; so $\mathrm{yC}^k\mathrm{z}$. Four subcases. If $\mathrm{w} = \mathrm{p}_i$ and $\mathrm{x} = \mathrm{p}_j$, $k \leq i + j$. By Density fact 1, $\mathrm{yC}^{i+j}\mathrm{z}$. Choose the missing link m so that $\mathrm{yC}^i\mathrm{m}$ and $\mathrm{mC}^j\mathrm{z}$. If $\mathrm{w} \in \mathrm{J}$ and $\mathrm{x} = \mathrm{p}_j$, use the missing link promised for j by Density fact 2. If $\mathrm{w} = \mathrm{p}_i$ and $\mathrm{x} \in \mathrm{J}$, use the missing link promised for i by Density fact 3. If w, $\mathrm{x} \in \mathrm{J}$, use the missing link promised by Density fact 4.

Case 2.2. The given link is in J; so not both $\mathrm{w} = \mathrm{p}_i$ and $\mathrm{x} = \mathrm{p}_j$, and cases for Rgyz are determined by 10. If $\mathrm{w} \in \mathrm{J}$ and $\mathrm{x} = \mathrm{p}_j$, we shall have either $\mathrm{wC}^{\exists}\mathrm{z}$ or $\mathrm{yC}^{\exists}\mathrm{z}$, and can use the missing link promised for j by Density fact 2. If Rgyz holds because $\mathrm{yC}^{\exists}\mathrm{z}$, we can use either Density fact 3 for i (if $\mathrm{w} = \mathrm{p}_i$) or Density fact 4 (if w, $\mathrm{x} \in \mathrm{J}$). We examine the remaining cases.

Case 2.2.1. $\mathrm{w} = \mathrm{p}_i$, $\mathrm{x} \in \mathrm{J}$; so $\mathrm{xC}^i\mathrm{g}$. If $\mathrm{g} = \mathrm{y} = \mathrm{z}$, $\mathrm{xC}^i\mathrm{z}$, and missing link can be chosen as p_i. If $\mathrm{gC}^{\exists}\mathrm{z}$, then $\mathrm{wC}^{\exists}\mathrm{z}$, and missing link can be chosen as p_m where $\mathrm{wC}^m\mathrm{z}$.

Case 2.2.2. w, $\mathrm{x} \in \mathrm{J}$; so cases for Rwxg are given by 10. Suppose $\mathrm{w} = \mathrm{x} = \mathrm{g}$. If also $\mathrm{g} = \mathrm{y} = \mathrm{z}$, the matter is trivial. If $\mathrm{gC}^{\exists}\mathrm{z}$, then $\mathrm{wC}^{\exists}\mathrm{z}$, and Density fact 4 can be used. Suppose next $\mathrm{wC}^{\exists}\mathrm{g}$. Then, in either of the remaining cases on Rgyz, $\mathrm{wC}^{\exists}\mathrm{z}$ holds; so Density fact 4 can be used. Suppose lastly, $\mathrm{xC}^{\exists}\mathrm{g}$. Then in either remaining case on Rgyz, $\mathrm{xC}^{\exists}\mathrm{z}$ holds; so the missing link can be chosen as a p_i such that $\mathrm{xC}^i\mathrm{z}$.

Cases closed.

Returning now to the proof of the theorem, we have let J be an unprovable U_+-set, and we need to show that U is unprovable in $\mathbf{R}_+$.

First, invoke the Density lemma to get the $\mathbf{R}_+$ modal structure there promised. We need to find a valuation that will turn this into an $\mathbf{R}_+$ model falsifying U. For each formula $J = J_1 \rightarrow J_2 \in \mathrm{J}$, use the Pair Extension lemma of §42.2 on $(\{J_1\}, \{J_2\})$ to find a prime $\mathbf{R}_+$-theory, call it $\mathrm{S}(J)$, including its antecedent J_1 and excluding its consequent J_2. This can be done because J is unprovable, and accordingly all members J of J are also unprovable. Let $\mathrm{S}(J)$ determine the value of each variable p at J. Further, set the value of p at points in $\mathrm{K} - \mathrm{J}$ as always true. (By prime $\mathbf{R}_+$-theoryhood, the antecedent of J is true at J, and its consequent is not true at J.)

We show that $U = P \rightarrow . C \rightarrow D$ is false at 0 by showing P true at p_1 and $C \rightarrow D$ false at p_1.

$C \rightarrow D$ is false at p_1 in virtue of $\mathrm{Rp}_1\mathrm{ab}$, where $\mathrm{a} = (C \rightarrow \vee A_{\mathrm{X}})$ and $\mathrm{b} = (\& B_{\mathrm{Y}} \rightarrow D)$, as promised by (3) of the definition of a U_+-set. For C must be true at a, and D false at b. Of course $\mathrm{Rp}_1\mathrm{ab}$ holds because (3) promises aCb, and $\mathrm{Rp}_1\mathrm{ab}$ was defined as holding just when aCb.

P is true at p_1 because, for each i, $A_i \rightarrow B_i$ is true at p_1. For suppose $A_i \rightarrow B_i$ were false at p_1; then $\mathrm{Rp}_1\mathrm{xy}$, where A_i true at x and B_i false at y. y cannot be a point 0 or p_k, since all (positive zero degree) formulas are true at all of these points in K − J. Since y ∈ J, by the definition of R, x must also ∈ J; and indeed we must have xCy. But then, by clause 2 of the definition of U_+-set, (the very end of that clause), either A_i is a disjunct of the consequent of x, hence false at x, or B_i is a conjunct of the antecedent of y, hence true at y. Which is all most absurd.

This completes the proof.

§64.3. The case with negation. Having given a straightforward answer to the positive case of the question of which entailments entail which entailments, we now indicate the complications induced by negation. The only interesting one is the graph-theoretical fact referred to below as the Dense Graph lemma.

We want to decide

$$U' = (A_1 \to B_1) \& \ldots \& (A_m \to B_m) \to . C \to D,$$

where, although the *A*s and *B*s remain zero degree, they can now involve negation. To relate this problem as much as possible to the notation outlined for the positive case in §64.2, define $n = 2m$, and $A_{m+i} = \overline{B_i}$, and $B_{m+i} = \overline{A_i}$ $(1 \le i \le m)$. Then we use without change the definitions given for the positive case of *P*, *U*, N, W, X, Y, Z, $\vee A_X$, and $\& A_X$. In particular,

$$P = (A_1 \to B_1) \& \ldots \& (A_m \to B_m) \& (\overline{B_1} \to \overline{A_1}) \& \ldots \& (\overline{B_m} \to \overline{A_m}).$$

Obviously, the original question for *U′* is by contraposition, equivalent to the question for *U*. Adding now to the definitions for the positive case, define J to be a *U*-set if it is a U_+-set satisfying one further condition: (5) the transitive closure $C^\exists$ of C in J is "weakly connected" in J: for $F \ne G \in J$, either $FC^\exists G$ or $GC^\exists F$ (i.e., either $FCH_1CH_2C \ldots CH_pCG$, or vice versa).

THEOREM. Let **S** be a calculus between ($\mathbf{B}_+$ + Conjunctive transitivity + R12 (contraposition) + R13 (double negation) of §R2) and **R**. Then *U* is provable in **S** just in case so also is $C \to \vee A_N$, $\& B_N \to D$, and every *U*-set.

PROOF. Suppose *U* unprovable. Dismiss $C \to \vee A_N$ and $\& B_N \to D$ as before. Otherwise using the graph construction of the positive case, obtain a graph G, which is a graph of a U_+-set but not necessarily a *U*-set. To proceed, we need some graph terminology and a lemma.

Since we are conceiving of a graph as a set of nodes and edges (assuming that membership of an edge guarantees membership of its source and target as nodes), by a *subgraph* we can mean just a graph which is a subset.

A G-*path from* a *to* b is a sequence of edges $\langle a, x_1\rangle, \langle x_1, x_2\rangle, \ldots \langle x_{n-1}, x_n\rangle, \langle x_n, b\rangle$, all of which are in G. By G(a)—*the* G-*leaf of* a (Ore 1962)—we mean a together with all edges and nodes of edges that lie on some G-path from a to a. Every node a in G is a member of exactly one

G-leaf in G, though perhaps only of an edgeless leaf containing just a itself. Note that G(a) = G(b) just in case b ∈ G(a), and also just in case either a = b or there is both a G-path from a to b and a G-path from b to a. A leaf G(a) is said to be *in* a subgraph H of G if it is itself a subgraph of H. For G(a) and G(b) both in a subgraph H of G, G(a) H-*precedes* G(b) if there is an H-path from a to b, but none from b to a; and G(a) *immediately* H-*precedes* G(b) if G(a) H-precedes G(b) but there is no G(c) in H between them (in the sense of H-precedence). All these relations are independent of the choice of representatives of G(a), G(b), G(c).

We note that, if G is dense (-strongly dense), so is every G-leaf G(a), since if an edge ⟨b, c⟩ lies on a G-path from a to a, so do the edges ⟨b, d⟩ and ⟨d, c⟩ known by density to be in G.

Dense Graph lemma. Let G be a graph that (1) contains an edge ⟨c, d⟩ and (2) is dense. Then G has a subgraph G′ that (1) contains ⟨c, d⟩, (2) is dense, and (3) is weakly connected, where by saying that any H is *weakly* connected we mean that for each pair of distinct nodes a, b in H, there is either an H-path from a to b or an H-path from b to a.

Proof. Preparing for Zorn's lemma, let Γ be the family of all subgraphs H of G such that:

Γ1. H includes ⟨c, d⟩, hence c and d.

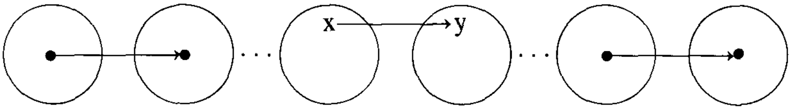
Γ2. If a is in H, so is G(a).

Γ3. The set of G-leaves in H is simply ordered by H-precedence.

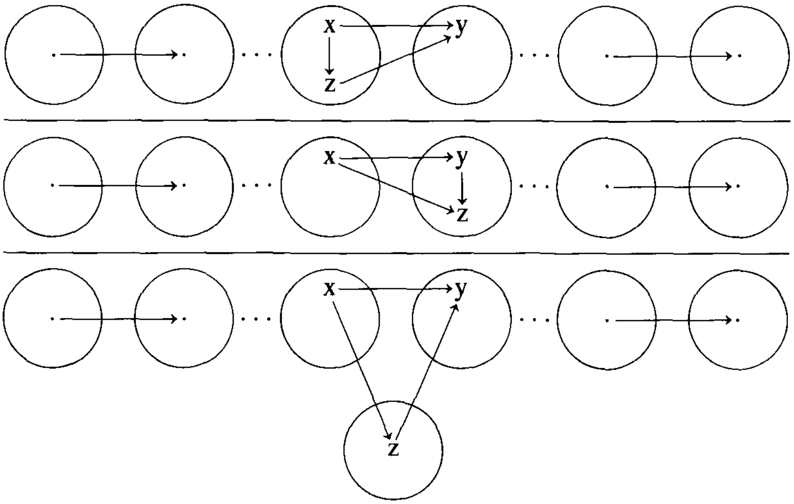
Γ4. If an edge ⟨a, b⟩ in H is undensed in H then (1) G(a) immediately H-precedes G(b), and (2) no other edge in H from a node in G(a) to a node in G(b) is undensed in H.

Γ is nonempty by virtue of containing the subgraph consisting of exactly G(c), G(d), and the edge ⟨c, d⟩. And it can be verified that the union of every nonempty chain in Γ is itself a member of Γ. So, by Zorn's lemma, Γ has a maximal member G′. By Γ1–Γ3, G′ is evidently a weakly connected subgraph of G containing ⟨c, d⟩. We show that maximality leads to density.

Because G′ belongs to Γ, it must have a picture like this, where we are supposing for *reductio* that the edge from x to y is undensed in G′ (the other displayed edges are supposed to represent arbitrary other undensed edges, taking account of Γ4).



G′-precedence of leaves is from left to right; note Γ3. By the density of G, there are edges ⟨x, z⟩ and ⟨z, y⟩ in G. Define G″ as the result of adding these two edges and also G(z) to G′. Because of the new edges, G(z) cannot G″-precede G(x) or be G″-preceded by G(y). Consequently, it must either be G(x), be G(y), or lie between them (in G″). So there are three cases for G″, as faithfully represented by the following pictures.



Because ⟨x, y⟩ was not densed in G′, the displayed new edges outside of leaves must really be new; so G″ is a proper supergraph of G′. And, recalling that edges within a leaf are one and all densed, one can see almost by inspection that G″ is in Γ. This contradicts the maximality of G′, and finishes the Dense Graph lemma.

Choosing c = $C\to\bigvee A_X$ and d = $\& B_Y\to D$, apply the lemma to the graph G of the U_+-set provided by the construction of the positive case (§64.2), getting G′. Define J as the set of nodes in G′, and let FCG hold just in case $\langle F, G\rangle$ is an edge in G′. Evidently J is a U-set; and an unprovable one. So, if U is unprovable, so is some U-set (or $C\to\bigvee A_N$ or $\& B_N\to D$).

The part of the converse involving $C\to\bigvee A_N$ and $\& B_N\to D$ is left to the reader. For the rest, suppose J is an unprovable U-set. We first invoke the

DENSITY-CONNEXITY/**R** LEMMA. Let C be dense in J, and let its transitive closure $C^{\exists}$ be weakly connected in J. Then there is an **R** model structure (**K**, R, 0, *) in the sense of §48.5 such that J ⊆ K, and there is a point $p_1 \in K$ such that, for a, b ∈ J, Rp_1ab iff aCb.

PROOF. Let * be a function mapping J one-one onto some disjoint set J*, and define J! as J∪J*. Let $0, p_1, \ldots, p_i, \ldots; p_1^*, \ldots, p_i^*, \ldots$ be all distinct and not in J!, and let K be the result of adding these to J!. Let * on K be an extension of * on J such that 0* = 0, $\mathrm{p}_i^* = \mathrm{p}_i^*$ (so to speak), $\mathrm{p}_i^{**} = \mathrm{p}_i$, and, for a* ∈ J*, a** = a ∈ J. Note that x** = x. Extend C to all of J! by declaring, for a, b ∈ J, never aCb*, never a*Cb, and a*Cb* iff bCa. Note that, in general for a, b ∈ J!, aCb iff b*Ca*, and similarly when C is replaced by C^i or $\mathrm{C}^\exists$. Evidently C is dense in J!. Define R on K as follows.

R1. R0xx, Rx0x, and Rxx*0 all hold.

R2. $\mathrm{Rp}_i^*\mathrm{xy}^*$, $\mathrm{Rxp}_i^*\mathrm{y}^*$, and Rxyp_i all hold if neither x nor y is 0 and if at least one of x or y is not some p_j.

R3. $\mathrm{Rp}_i\mathrm{p}_j\mathrm{p}_k$, $\mathrm{Rp}_i\mathrm{p}_k^*\mathrm{p}_j^*$, and $\mathrm{Rp}_k^*\mathrm{p}_i\mathrm{p}_j^*$ all hold if $k \leq i+j$.

R4. $\mathrm{Rp}_i\mathrm{xy}$, $\mathrm{Rxp}_i\mathrm{y}$, and $\mathrm{Rxy}^*\mathrm{p}_i^*$ all hold if x, y ∈ J! and if $\mathrm{xC}^i\mathrm{y}$.

R5. Rxyz holds if x, y, z ∈ J!, and if either (x = y = z or x = y* = z or x = y* = z*) or ($\mathrm{xC}^\exists\mathrm{z}$ or $\mathrm{yC}^\exists\mathrm{z}$ or $\mathrm{xC}^\exists\mathrm{y}^*$).

R6. Rxyz does not hold if not by 1–5.

One may verify that (K, R, 0, *) is indeed an **R** model structure. The only really new case—i.e., the only case not taken care of either automatically or via density—is the following instance of Pasch: Rxx*0 and R0yy imply $\mathrm{R}^2\mathrm{xyx}^*\mathrm{y}$, when x, y ∈ J! and when neither x = y nor x = y*. In this case we can argue, by the weak connexity of $\mathrm{C}^\exists$ in J and, hence, in J*, that one of the following holds: $\mathrm{xC}^\exists\mathrm{y}$ or $\mathrm{yC}^\exists\mathrm{x}$ (if x and y are either both in J or both in J*) or $\mathrm{xC}^\exists\mathrm{y}^*$ or $\mathrm{y}^*\mathrm{C}^\exists\mathrm{x}$ (if x∈J and y∈J*, or vice versa). And in each of these cases an appropriate missing link is available—a member of J! in the former cases, and a p_i^* or p_i in the latter. (We note for its interest that the weak connexity of J is a bit stronger than required; it would be all right, too, if either $\mathrm{xC}^\exists\mathrm{x}$ or $\mathrm{yC}^\exists\mathrm{y}$. That is, we really need only the connectedness of distinct points neither of which is self-connected. Further, if in different circumstances J and J* were being developed together so that there could be interesting C-relations between them, and assuming xCy iff y*Cx*, then the form of connectedness required of J! is precisely that, when x ≠ y and x ≠ y* and neither x nor y is self-connected by $\mathrm{C}^\exists$, then either x or x* bears $\mathrm{C}^\exists$ to either y or y*.)

Resuming the proof of the theorem where we left off, we have an unprovable *U*-set J and need to show *U* unprovable in **R**. The Density-Connexity/**R** lemma gives us an appropriate **R** model structure. For its valuation we find for each formula *J*∈J a prime **R**-theory S(*J*) including its antecedent and excluding its consequent. Let membership in S(*J*) determine the value of each variable at *J*∈J. And let *p* be true at a*, for a∈J, just in case $\bar{p} \notin$ S(a). Lastly, let *p* be true at all members of K − J!. Calculate that *E* is true at *J*∈J iff *E* ∈ S(*J*).

The argument that *p*→.*C*→*D* is false at 0 proceeds as before, except for showing that one cannot have $\mathrm{Rp}_1\mathrm{xy}$ with A_i true at x and B_i not true at y. Consider only $1 \leq i \leq m$, noting that other conjuncts of *P* ($m+1 \leq i \leq 2m$) are contrapositives of these. As before, we may restrict attention to x, y both in J!, and the argument does not change if x, y $\in$ J. Since one cannot, under the hypothesis Rp_1xy (hence xCy), have one of x, y in J and the other in J*, the only remaining case is when both are in J*—and furthermore y*Cx* (with both y*, x* $\in$ J). In this case we note that $A_{i+m} = \overline{B_i}$ and $B_{i+m} = \overline{A_i}$; so, by clause (2) of the definition of *U*-set, either $\overline{B_i}$ is a disjunct of the consequent of y*, hence false at y*, hence making B_i true at y; or, in a parallel way, A_i is made false at x. Which is even more absurd.

This completes the proof.

Observations. This general type of decision method can be extended in various ways to treat of somewhat more complex formulas, but nothing of much interest appears to emerge. For a different type of method in a closely related setting, see Meyer 1979b.

It might turn out to be interesting to look on the rule: from $C \rightarrow \vee A_N$ and $\& B_N \rightarrow D$ and all *U*-sets (each construed as a disjunction) to infer U' (i.e., $(A_1 \rightarrow B_1) \& \ldots \& (A_m \rightarrow B_m) \rightarrow . C \rightarrow D$—noting *m*, not *n*), as a kind of Gentzen rule. In contrast with Rules 1 and 2, it is "cut-free" in the sense that no constituent occurs as both antecedent and consequent part. It is to be noted that the rule is derivable, hence usable inside of disjunctive contexts, but not itself an entailment, so not usable inside intensional contexts.

§65. The undecidability of all principal relevance logics (by Alasdair Urquhart). The principal purpose of this section is to show that the logics **E** of entailment, **R** of relevant implication, and **T** of ticket entailment are undecidable. The decision problem is also shown to be unsolvable in an extensive class of related logics. The main tool used in establishing these results is an adaptation of the von Neumann coordinatization theorem for modular lattices. A secondary but almost equally prominent purpose is to explain some of the interesting connections between relevant implication and projective geometry which have emerged from the present attack on the decision problem.

This section divides rather sharply into two. In §65.1, in an effort to highlight the geometrical ideas involved, we deal only with the logic **KR** (defined therein), and we suppress numerous details. In §65.2 we adapt the geometrical insights to prove the undecidability of the principal relevance logics.

§65.1. Relevant implication and projective geometry. In this section we give an exposition of the recently discovered connections between relevant implication and projective geometry. One of the consequences of this connection is a simple proof that the propositional logic **KR** (an extension of the logic **R**) is undecidable. This proof can be generalized to a proof that any logic between the positive system $\mathbf{T}_+$ of ticket entailment and **KR** is undecidable, as we shall see in §65.2. The proof of this result, however, is of necessity long, complex, and formal. We present here a separate proof of the undecidability of **KR**, which is intuitively easy to grasp. An understanding of the section is excellent preparation for reading §65.2. A more important motivation for the section is to point out the great wealth of ideas, problems, and constructions that flow from the connection between geometry and relevance logics, which turns out to be surprisingly intimate.

§65.1.1. Models for relevance logics.

The present advances in the understanding of **R** came about (like many advances in logic) by the discovery of a new method for constructing models. Although the basic semantical analysis of **R** has been around for over a decade, until quite recently disappointingly few *examples* of **R** model structures were known. If you omit negation, then you can use semilattices to model $\mathbf{R}_+$ (§47). However, semilattice models fail in the worst possible way to extend to the full system **R**; only the one-element semilattice can be used to validate all of **R**. In the early 1970s only the following models for **R** were known: the Sugihara matrix and its finite versions (§26.9) and various small matrices derived by fiddling with many-valued truth tables, one of which is generalized to an infinite family of models in §40. The list of small models was enormously extended by a computer search using some remarkable programs written by Slaney, Meyer, Pritchard, Abraham, and Thistlewaite (for an early progress report on this research effort the reader is referred to Slaney 1980). These programs churned out huge quantities of **R** matrices and model structures of all shapes and sizes. Clearly, there are lots and lots of **R** model structures out there! But what are they like? Can we classify them in some intelligent fashion? Are there general constructions that produce interesting examples? The answer to the first two questions is still obscure, though clearer than it was. The answer to the last question is an emphatic "yes!".

We confess here to an old antipathy (now abandoned) to the Routley/Meyer semantics. Our dislike of the model theory was based on the unexamined prejudice that it was impossible to "get a picture" of **R** model structures, in seeming contrast to semilattice models and Kripke-style modal semantics. The main purpose of this section is to convince you that it is extremely easy to "get a picture" of **R** model structures. In a literal sense, these models have been staring us in the face for a long time.

§65.1.2. The logic KR.

To those who have taken the trouble to read the literature on relevance logic rather than fulminate against it, it has been a familiar fact since the early 70s that there are two conceptually distinct classes of "paradoxes of material implication." The archetype of the first class (paradox of consistency) is $(A \& {\sim} A) \to B$. The archetype of the second (paradox of relevance) is $A \to (B \to A)$. It is easy to devise systems of entailment that omit one but not the other. Thinking about the system **R**, we can see immediately that if we add $A \to (B \to A)$ then the result is classical logic with paradoxes of both types. However, the consequences of adding $(A \& {\sim} A) \to B$

to **R** are not so clear. Here we have a system of relevance logic with regular classical Boolean negation, satisfying all the natural postulates of **R**-style negation, including contraposition. The credit for investigating the resulting system **KR** belongs to R. K. Meyer and A. Abraham (see Routley with Plumwood, Meyer, and Brady 1982 for details of their investigations).

Parenthetically, it should be noted that **KR** is *not* the same as the classical relevance logic **CR** investigated by Meyer and Routley 1973 (see §62.5.2). That system adds to **R** a classical negation operator which is distinct from the negation proper to **R**. In **KR**, classical negation and relevant negation are *identified*. One's initial reaction to **KR** is that it is probably a trivial system, if it doesn't simply collapse into classical logic. As we shall see, this reaction could hardly be wider of the mark. The first indication that **KR** is indeed nontrivial came from the computer, which churned out reams of interesting **KR** matrices. In retrospect, this is hardly surprising, because we now know that **KR** models can be manufactured *ad lib* from projective geometries.

First, though, some definitions. A **KR** model structure (krms) is a three-place relation Rabc on a set containing a distinguished element 0, satisfying the postulates:

1. R0ab iff a = b
2. Raaa
3. If Rabc then (Rbac and Racb) (total symmetry).
4. If (Rabc and Rcde) then ∃f (Radf and Rfbe) (Pasch's postulate).

Note that a krms is just an rms in the sense of Routley and Meyer, or an **R**-frame in the sense of §48.5, except that we have imposed total symmetry by setting a* = a for all a. We define truth and falsity with respect to a krms exactly as for an **R**-frame, except for negation (actually, even that is the same if we take seriously the suggestion that a* = a). Writing "A is true at a" as "a $\models A$," the crucial clauses are:

$\mathrm{a} \models \sim A$ iff $\mathrm{a} \not\models A$
$\mathrm{a} \models A \to B$ iff $(\forall \mathrm{bc})$(if ($\mathrm{b} \models A$ and Rabc) then $\mathrm{c} \models B$).

A slight modification of the usual completeness proof for **R** shows that **KR** is complete with respect to the class of all **KR** model structures.

The total symmetry condition seems especially odd on first acquaintance. To explain how we can construct such strange models in profusion, we turn to the theory of projective geometry.

§65.1.3. Projective spaces. In this section we give a summary of standard material on projective spaces. There are numerous good textbooks on projective geometry. We found the books of Garner 1981, Hartshorne 1967, and Mihalek 1972 helpful; also, the classic by Veblen and Young 1910 is inspiring reading. For the lattice-theoretic approach to projective geometry Birkhoff 1948 (3rd edition) and Grätzer 1978 should be consulted.

DEFINITION 3.1. A projective space consists of a set of *points* P and a collection of subsets of P called *lines*, satisfying the two conditions:

P1. Two distinct points a, b lie on (i.e., belong to) exactly one line a + b.
P2. If a, b, d, e are distinct points such that some point c lies on both a + b and d + e, then there is a point f lying on both a + d and b + e (see Fig. 1).

A projective space is said to be irreducible if it satisfies:

P3. No line contains exactly two points.

We shall also make use of the additional postulate:

P4. No line contains exactly three points.

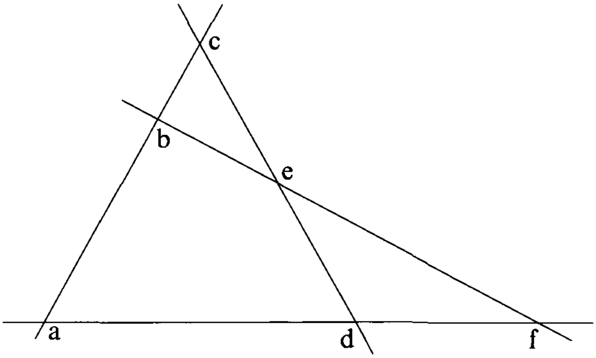


Figure 1

The most familiar example of a projective space is ordinary Euclidean 3-space, enriched by the addition of a point at infinity for each parallelism class in ordinary 3-space, together with the plane and lines at infinity. This is real projective 3-space.

We now define the notion of collinearity. Various ways of doing this are possible. The method adopted by most texts is to define points a, b, c to be collinear if they all lie on a single line. This definition, however, is not suitable for our purposes, because it is "too fat"; it counts as collinear any triple containing repeated points. Instead we use:

DEFINITION 3.2. If P is a projective space, the collinearity relation Cabc in P is defined by:

Cabc iff (a) $a = b = c$, or (b) a, b, c are distinct and lie on a common line.

Note that, if we define $a + a = \{a\}$, Cabc can be given the symmetric definition: Cabc iff $(a + b = b + c = a + c)$.

LEMMA 3.3. Let P be a projective space satisfying P4. Then the collinearity relation on P satisfies:

1. if Cabc then (Cbac and Cacb);
2. Cabb iff a = b;
3. if (Cabc and Ccde and $a \neq d$) then $\exists$f (Cadf and Cfbe);
4. if (Cabc and Cbcd and $a \neq d$) then Cabd.

PROOF. A straightforward calculation. The postulate P4 is needed to validate the instances of 3, where a = b = c.

A set of points X in a projective space P is a (linear) subspace of P if:

if a, b $\in$ X and Cabc then c $\in$ X.

The family of all linear subspaces of P forms a complete lattice, ordered by containment, in which the lattice join of two subspaces X and Y is

$X+Y = \bigcup(a+b\colon a\in X \text{ and } b\in Y)$.

Projective spaces can be characterized by means of their linear subspaces.

DEFINITION 3.4. A modular geometric lattice is a complete lattice $\langle L, \wedge, +\rangle$ satisfying:

1. if $a \geq c$ then $a\wedge (b+c) = (a\wedge b)+c$;
2. every element of L is a join of atoms in L;
3. every atom in L is compact; that is, $a \leq \Sigma X$ implies $a \leq \Sigma Y$ for some finite $Y \subseteq X$.

(Here, a + b denotes the least upper bound of a, b; ΣX denotes the least upper bound of $X \subseteq L$.)

FACT 3.5. The subspace lattices of projective spaces are exactly the modular geometric lattices (up to isomorphism).

For these and other classical results on projective spaces, see Grätzer chapter IV and Birkhoff chapter IV.

Given a modular geometric lattice, a projective space can be defined by taking the points as the atoms and the lines all sets of atoms of the form $\{x\colon x \leq a+b\}$, where $a \neq b$. The concepts of projective space and modular geometric lattice are thus completely equivalent.

An enormous variety of projective spaces can be constructed from vector spaces. Let V be a vector space over a division ring (= skew field; see

Birkhoff and Mac Lane 1941). Define a point to be a one-dimensional subspace and a line to be a two-dimensional subspace of V. The result is a projective space. This construction applied to the space of four vectors over the real numbers produces real projective 3-space.

The language of pure lattice theory is the equational language containing variables $x_1, x_2, \ldots$, constants 0 and 1, and symbols for lattice meet $x \wedge y$ and lattice join $x+y$. If K is a class of lattices, the word problem for K is the problem of determining whether or not a given equation can be deduced from a given finite set of equations in the equational theory determined by K.

The undecidability result for **KR** is based on the following important result proved independently by Hutchinson 1973 and Lipshitz 1974.

Fact 3.6. Let K be a class of modular lattices which contains the subspace lattice of an infinite-dimensional projective space. Then the word problem for K is unsolvable.

§65.1.4. Model structures constructed from projective spaces. With the notion of collinearity in a projective space to hand, it is not hard to see how we can construct examples of **KR** model structures. The following construction is general enough for our needs.

Lemma 4.1. Let C be the collinearity relation in an irreducible projective space satisfying P4. Add a new element 0, and define R to be the smallest totally symmetric relation on $P \cup \{0\}$ containing $C \cup \{\langle 0, x, x\rangle : x \in P \cup \{0\}\}$. Then $\langle P \cup \{0\}, R\rangle$ is a krms.

Proof. An easy verification, using Lemma 3.3.

We now have a copious supply of highly nontrivial **KR** model structures. But how general is the construction? The following lemma answers this question to some extent by showing that the connection between **KR** and projective geometry is very intimate. It is from this simple but powerful lemma that all the undecidability results flow.

The definition of linear subspace used in the previous section carries over directly to **KR** model structures, substituting Rabc for Cabc in the definition.

Lemma 4.2. Let M be a **KR** model structure. The nonempty linear subspaces of M form a modular lattice (Def. 3.4(1)). If M satisfies the condition:

(*) if Rabb then $a = 0$ or $a = b$,

then the subspace lattice is geometric (Def. 3.4(2, 3)).

Proof. Before proceeding to the proof proper, we pause to note that the lattice join and meet can be expressed in the language of **KR**. Lattice meet corresponds to conjunction $A\&B$; join corresponds to $A\circ B = {\sim}(A\rightarrow{\sim}B)$. More precisely, let $\|A\|$ stand for the set of points in the model structure at which A is true. Then, if A, B are formulas of **KR** and A, B are linear subspaces of a **KR** model structure such that $\|A\| = \mathrm{A}$ and $\|B\| = \mathrm{B}$, we have:

$$\mathrm{A}\wedge\mathrm{B} = \|A\&B\|$$
$$\mathrm{A}+\mathrm{B} = \|A\circ B\|.$$

$A\circ B = {\sim}(A\rightarrow{\sim}B)$ is the definition of the fusion connective (§27.1.4, R1), which, as Meyer observed some time ago, is the key connective in relevance logics, rather than →. Here it turns up as a lattice join operation—a somewhat surprising role for the connective to play, since the definition above is the classical definition of *conjunction*.

Now for modularity. Let A, B, and C be subsets of M, with $\mathrm{C}\subseteq\mathrm{A}$ and A a linear subspace of M. If $\mathrm{x}\in\mathrm{A}\wedge(\mathrm{B}+\mathrm{C})$, then $\mathrm{x}\in\mathrm{A}$ and $\mathrm{x}\in\mathrm{B}+\mathrm{C}$. Thus there exist y, z such that Rxyz, $\mathrm{y}\in\mathrm{B}$, $\mathrm{z}\in\mathrm{C}$; hence $\mathrm{z}\in\mathrm{A}$. Since $\mathrm{x}\in\mathrm{A}$, $\mathrm{z}\in\mathrm{A}$, and Rxyz, $\mathrm{y}\in\mathrm{A}+\mathrm{A}=\mathrm{A}$ (remember, A is a linear subspace). Thus $\mathrm{y}\in\mathrm{A}\wedge\mathrm{B}$; so $\mathrm{x}\in(\mathrm{A}\wedge\mathrm{B})+\mathrm{C}$.

The second part of the lemma follows easily from the fact that if M satisfies (*) then any set of the form $\{0, \mathrm{a}\}$ is a linear subspace.

Let's take stock! At this point we have shown that every **KR** model structure has associated with it in a natural way a modular lattice, which in an important special case is the lattice of subspaces of a projective space. This is enough to give us undecidability. To state the result precisely we need to specify the translation of lattice equations into the language of **KR**. The translation $(\varphi = \psi)^{\mathrm{t}}$ of a lattice equation $(\varphi = \psi)$ is defined as follows:

$$(\mathrm{x}_i)^{\mathrm{t}} = p_i, \qquad (\varphi\wedge\psi)^{\mathrm{t}} = \varphi^{\mathrm{t}}\&\psi^{\mathrm{t}}, \qquad (\varphi+\psi)^{\mathrm{t}} = \varphi^{\mathrm{t}}\circ\psi^{\mathrm{t}},$$
$$(\varphi = \psi)^{\mathrm{t}} = \varphi^{\mathrm{t}}\rightleftarrows\psi^{\mathrm{t}}, \qquad 0^{\mathrm{t}} = \boldsymbol{t}, \qquad 1^{\mathrm{t}} = \boldsymbol{T}.$$

Now let I = "if $((\varphi_1 = \psi_1)$ and ... and $(\varphi_n = \psi_n))$ then $\delta = \epsilon$" be an implicational statement of pure lattice theory. The translation of I is:

$$\mathrm{I}^{\mathrm{t}} = (\mathrm{L}(p_1)\&\ldots\&\mathrm{L}\,(\mathrm{p}_m)\&(\varphi_1 = \psi_1)^{\mathrm{t}}\&\ldots\&(\varphi_n = \psi_n)^{\mathrm{t}})\&\boldsymbol{t}\rightarrow(\delta = \epsilon)^{\mathrm{t}}$$

where L(A) abbreviates $(((A\circ A)\rightleftarrows A)\&\boldsymbol{t}\rightarrow A)$ and $p_1, \ldots, p_m$ contain all the variables in the translation of the lattice equations.

§65.1.5. Undecidability. Now all we have to do is put together Fact 3.6 and Lemma 4.1, and we have proved the undecidability of **KR**. Actually, we have proved a good deal more. Let P be an irreducible projective space satisfying P4, and let L(P) be the logic determined by the model structure constructed from P using the procedure of Lemma 4.1.

THEOREM 5.1. Let L be a logic intermediate between **KR** and L(P), where P is an infinite-dimensional irreducible projective space satisfying P4. Then L is undecidable.

PROOF. Let M be a krms constructed from a projective space P. It is easy to see that the lattice of linear subspaces of M is isomorphic to the lattice of linear subspaces of P. Furthermore, an implication I holds in this lattice if and only if its translation I^t is valid in the model structure. The undecidability of L now follows immediately from Fact 3.6.

This theorem is a powerful result. It suggests that we can easily get an undecidability proof for **R** by some little trick, like embedding **KR** in **R**. Originally we thought we had such an embedding, but Meyer soon disabused us of *that* idea. The basic trouble is that the conditions $A\&{\sim}A \rightleftarrows F \rightleftarrows B\&{\sim}B$ do not extend inductively to $(A \circ B)$, so there is apparently no simple way to embed **KR** in **R**.

To get an undecidability result for **R**, we have to pull aside the rug that conceals the trap door leading to the hidden treasures on a lower level. In Theorem 5.1 we have simply used the Hutchinson/Lipshitz result without examining its proof. To deal with **R**, we need to dig a bit deeper and look at their actual construction, which turns out to be very interesting. They used the von Neumann coordinatization theorem for modular lattices, a powerful technique whose history we now briefly sketch.

Let P be any projective space satisfying the Desargues theorem. Then the lattice of subspaces of P is isomorphic to the subspace lattice of a vector space over a division ring D. This classical result was proved by von Staudt by the "algebra of throws." To coordinatize the space P we single out a fixed line in the space and choose three distinct points on the line as the zero, unit, and point at infinity. We can then define multiplication and addition for points on the line using purely geometrical constructions (see Veblen and Young 1910, vol. 1, chapter 6, and Grätzer 1978, pp. 208–210, for details). The resulting algebra is the division ring D. Thus we have constructed an algebra from purely geometrical material; to those familiar with the history of geometry it should come as no surprise that the ancestry of the von Staudt constructions can be traced to the Eudoxian theory of proportion. (Compare the opening pages of *La géométrie* of Descartes with

Euclid VI, Prop. 11 and Prop. 13.) Figure 2 illustrates the definition of multiplication, expressed in this section by "x.y."

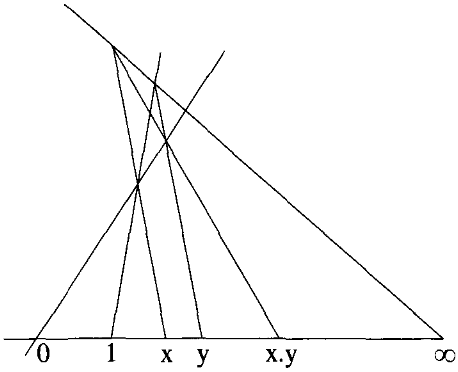


Figure 2

A far-reaching generalization of the von Staudt construction was introduced by von Neumann in the 1930s, building on the work of Birkhoff and Menger. He observed that the use of the Desargues theorem could be avoided by postulating the existence of an appropriate coordinate frame. With this modification we can construct a ring with which we can coordinatize any complemented modular lattice containing a three-dimensional coordinate frame satisfying certain added conditions; for a detailed account of this result the reader is referred to the classic von Neumann 1960.

Von Neumann's proof that the ring multiplication is well-defined and associative uses only the modularity of the given lattice. This observation leads directly to the proof of Fact 3.6, because the existence of a coordinate frame can be expressed in terms of pure lattice equations, and any countable semigroup can be embedded in the multiplicative semigroup of the von Neumann ring constructed from an infinite-dimensional projective space.

To prove undecidability for **R**, we have to generalize the construction still further. First, let's go back and examine the proof of modularity in Lemma 4.2. We can extend the proof to **R** in a simple way to derive the following result: in an **R** model structure if A is a linear subspace satisfying $(A \wedge {\sim} A) = \varnothing$ and if $C \subseteq A$ then, for any B, $A \wedge (B+C) = (A \wedge B)+C$; that is, A is *modular*. Now if we turn to the multiplication operation in the von Neumann construction, we find that the proof that the operation is well-defined and associative uses only a finite number of instances of modularity; to be precise, von Neumann needs modularity only for elements of the coordinate frame. The combination of these two observations gives us undecidability for **R**.

The extension to **E**, **T** and the positive systems is a messy business which employs the Glivenko double-negation construction. The unpleasant details are all to be found in §65.2. This particular mix of ingredients is enough to give us the general result:

THEOREM 5.2. Let **L** be a positive logic (expressed in terms of &, v, →) intermediate between $\mathbf{T}_+$ and the positive part of L(P), where P is an infinite-dimensional projective space satisfying P3 and P4. Then **L** is undecidable.

This theorem seems to be just about the best that can be done using these geometrical techniques. Any further advance on the decision problems for relevance logics (such as the still open decision problem for the semilattice system of §47) would seem to require some new ideas.

§65.1.6. More geometrical ruminations. As we said at the beginning, one of the main aims of this section is to make logicians aware of the rich possibilities offered by the techniques of classical synthetic geometry in the field of relevance logics. As far as decidability goes, the techniques are probably close to exhaustion; but elsewhere the surface has hardly been scratched. For example, how can we axiomatize the logic determined by the class of all models constructed from projective spaces?

In conclusion, we give one more small result which shows the kind of theorems that can be proved by simply adapting known techniques from geometry and lattice theory.

THEOREM 6.1 There are continuum-many logics intermediate between **KR** and classical logic.

PROOF. For p a prime number, let L(p) be the subspace lattice of the projective plane coordinatized by the p-element field. By a result of Baker (see Grätzer 1978, pp. 239–240), distinct subsets of $\{L(p)\colon p \text{ a prime}, p>2\}$ generate distinct equational classes. It follows that the logics determined by the classes of model structures constructed from such distinct subsets are distinct extensions of **KR**.

The connection between projective geometry and relevance logics is both simple and natural, and it makes sense to ask why it was not investigated earlier. In the early 1970s the connection with geometry was clear to JMD, who christened the crucial condition "Pasch's postulate." [Note by principal authors: the author of this section is overly generous with his use of "clear." In fact JMD spent many fruitless hours trying to find some natural geometric models for **R** in which Rabc could be read as "the point c lies between the points a and b" without its then automatically being the case that Rabb. Compare the problem of the "fat" definition of "collinearity" above. Also, the idea that R can be read as the totally symmetric relation of collinearity could never have occurred until the study of the strange hybrid **KR**.] It is puzzling that the geometrical insight was not exploited until over a decade had expired. Unfortunately, the penetrating remarks of Toohey 197+ were not followed up, perhaps because of Toohey's obscure expository style. We can, however, give a reason in our own case which has to do with the vagaries of geometrical terminology. The postulate we have been calling "Pasch's postulate" is in fact due to Peano 1889. The original Pasch axiom on which Peano improved says that if a line passes through one side of a triangle it passes either through another side or through a vertex. A large number of geometry books attribute the Peano axiom to Pasch. When we went to look up Pasch's postulate in a textbook, however, we found the original Pasch axiom, which looks so little like the Peano version that we immediately abandoned geometrical interpretations as hopeless. With only a little more effort we would have discovered that the bloodthirsty troll barring our way could be overcome with the greatest of ease.

There are lots of other possibilities for constructing interesting model structures out of geometries. For example, by using two copies of a geometry, we can construct rms's that are not krms's. In another direction, we can construct rms's from geometrical spaces satisfying the classical axioms of betweenness.

We hope, however, that by this time the reader is inspired to explore these possibilities independently and hence discover some of the wide unexplored territory lying between relevance logics and classical geometry.

§65.2. The undecidability of entailment and relevant implication. In this section we outline detailed arguments showing that the propositional logics **E** of entailment, **R** of relevant implication, and **T** of ticket entailment are undecidable. The decision problem is also shown to be unsolvable in an extensive class of related logics. As we indicated in §65.1.5., the main tool used in establishing these results is an adaptation of the von Neumann coordinatization theorem for modular lattices.

§65.2.1. Introduction. The results of this section have some general methodological interest since they furnish the first known examples of undecidable "natural" sentential logics. It has been known since the work of Linial and Post in the 1940s that undecidable subsystems of classical sentential logic exist (see Linial and Post 1949; for an exposition see Davis 1958). All previous examples, however, were systems constructed for the purpose of exhibiting an undecidable logic. The nearest previous approach to a natural undecidable logic is the relevance logic $\mathbf{Q}^+$ of Meyer and Routley 1973a.

Before proceeding to details, we outline the main ideas of the proof. John von Neumann 1960 shows how to coordinatize a complemented modular lattice L containing a coordinate frame. Given such a frame in L, he constructs an auxiliary ring relative to the frame in terms of pure lattice equations. The definition and proof of the ring properties use only the modularity of L. Any countable semigroup can be embedded in the multiplicative semigroup of the auxiliary ring of an appropriate modular lattice, so the word problem for modular lattices in unsolvable (Hutchinson 1973, Lipshitz 1974). In the logics **T**, **E**, and **R** we can define suitable lattice-like operations on a subset of the propositions. A difficulty arises, though, since we cannot prove the modular law in general. However, a careful examination of the von Neumann construction reveals that the definition of ring multiplication and the proof of its associativity use only modularity for elements of the frame. Now in **T** (and hence in **E** and **R**) it is possible, for any given proposition, to produce a finite set of formulas that guarantees its modularity. With this modification it is now possible to adapt the Hutchinson/Lipshitz proof, thus showing undecidability for a wide family of relevance logics.

For general background on the coordinatization theorem, the reader is referred to von Neumann 1960 and Lipshitz 1974.

§65.2.2. Coordinate frames in ordered monoids. In this section we define a type of ordered monoid, which we call a "t-monoid." The definition has no independent significance; it simply represents a useful codification of the algebraic properties needed for the undecidability proof.

DEFINITION 2.1. An algebra $\langle S; \wedge, +, \cup, \perp, 0\rangle$ is a *t-monoid* if:

(1) $\langle S; \wedge, \perp\rangle$ is a meet semilattice with least element $\perp$ ($a \leq b$ is defined as $a \wedge b = a$);
(2) $\langle S; +, 0\rangle$ forms a commutative monoid;
(3) $x \wedge (x+y) = x$, provided that $y \geq 0$;
(4) if $x \leq y$ and $z \leq w$, then $x+z \leq y+w$ and $x \cup z \leq y \cup w$;
(5) $x+(y \cup z) = (x+y) \cup (x+z)$;
(6) $x \cup \perp = x = \perp \cup x$.

In the sequel we shall use juxtaposition for the meet operation to reduce the apparent complexity of terms. We use ΣX for $x_1 + \cdots + x_n$, where $X = \{x_1, \ldots, x_n\}$.

DEFINITION 2.2. An element a in a t-monoid is *modular* if it satisfies the condition:

$$(\forall b)(\forall c) \text{ (if } a \geq c \text{ then } a(b+c) = ab+c)$$

DEFINITION 2.3. Let M be a t-monoid in which 0 is modular. A family of elements $\{a_1, \dots, a_n\} \cup \{c_{ij}: i \neq j, 1 \leq i, j \leq n\}$ is a *modular n-frame* in M if:

(1) for G, H $\subseteq \{a_1, \dots, a_n\}$ we have the following:
$G \cap H \neq \varnothing$ implies $\Sigma G \Sigma H = \Sigma(G \cap H)$, and $G \cap H = \varnothing$ implies $(\Sigma G)(\Sigma H) = 0$;
(2) $a_i + a_i = a_i$
(3) $(c_{ij} + c_{jk})(a_i + a_k) = c_{ik}$, i, j, k distinct
(4) $c_{ij} = c_{ji}$
(5) $c_{ij} + a_j = a_i + a_j$; $c_{ij} \wedge a_j = 0$
(6) ΣG is modular for $G \subseteq \{a_1, \dots, a_n\}$.

Condition (1) of the definition says that $a_1, \dots, a_n$ (provided that they are distinct) generate a copy of the Boolean algebra $\mathbf{2}^n$ with the a_i as atoms (in von Neumann's terminology, the set $\{a_1, \dots, a_n\}$ is independent). The elements c_{ij} serve as centers of perspectivity with respect to the coordinate frame. For a detailed discussion of coordinate frames in modular lattices the reader is referred to von Neumann 1960.

In following the algebraic manipulations below, it is helpful to visualize the coordinate frame as a configuration in three-dimensional (projective) space. Figure 3 is a diagram of a 4-frame in real projective 3-spacc. The a_is form the corners of a tetrahedron, with $a_i + a_j$ the line joining a_i and a_j. The centers of perspectivity c_{ij} are lined up as shown (compare part (3) of the definition of an n-frame).

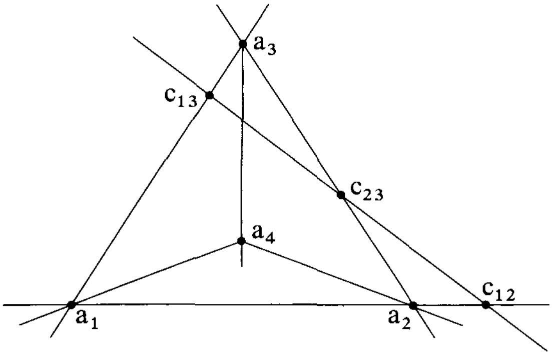


Figure 3

For the remainder of this section we assume that we are dealing with a fixed t-monoid M containing a modular n-frame with $n \geq 4$; subscripts i, j, k, etc. are assumed to range over $1, \dots, n$.

DEFINITION 2.4.

(a) For i, j distinct, $L_{ij} = \{x \in M: x + a_j = a_i + a_j \text{ and } x \wedge a_j = 0\}$;
(b) For i, j, k distinct, $b \in L_{ij}$, $d \in L_{jk}$, $b \otimes d = (b + d)(a_i + a_k)$;
(c) For $x, y \in L_{12}$, $x.y = (x \otimes c_{23}) \otimes (c_{31} \otimes y)$.

The definition of multiplication in part (c) of Definition 2.4, which differs inessentially from that of von Neumann 1960 (see pp. 117, 132), is the same as that of Freese 1980; it is closely related to the well-known definition of multiplication on a line in a projective space due to von Staudt (see Veblen and Young 1910, pp. 141–146; von Neumann 1960, pp. 133–135). That our definition is equivalent to von Neumann's follows easily from Lemma 6.1 of von Neumann 1960.

Figure 4 illustrates the general definition of multiplication. It should be compared carefully with the diagram of the von Staudt construction given earlier in §65.1.5. In reading the proofs, the reader is strongly urged to draw supplementary diagrams, as geometrical intuitions give life to what are otherwise unmotivated formal computations.

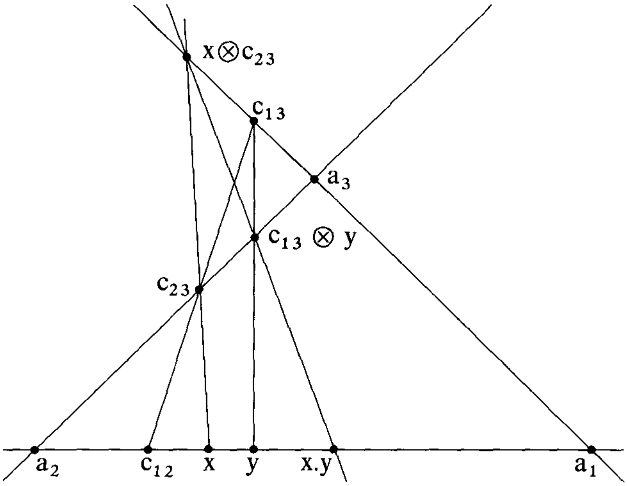


Figure 4

We now prove a series of lemmas leading to the proof that the operation $x.y$ defined on L_{12} is associative. The proofs are essentially those of von Neumann, though the details of the computations follow Lipshitz 1974. Because we are dealing with a generalization of the von Neumann proof, most steps are annotated; where the modularity of the frame element x is used, we indicate this by "Mod (x)."

Lemma 2.5. If $b \in L_{ij}$, $d \in L_{jk}$, with i, j, k distinct, then $b \otimes d \in L_{ik}$.

Proof.

$$\begin{aligned}
(b\otimes d)a_k &= (b+d)(a_i+a_k)a_k && \text{Def. 2.4}\\
&= (b+d)a_k && \text{Def 2.3, (1)}\\
&\leq (b+d)(a_j+a_k) && \text{Def. 2.1, (3), (4)}\\
&= (a_j+a_k)(b+d) && \text{Def. 2.1, (1)}\\
&= (a_j+a_k)b+d && \text{Mod } (a_j+a_k)
\end{aligned}$$

Now:

$$\begin{aligned}
(a_j+a_k)b &\leq (a_j+a_k)(b+a_j) && \text{Def. 2.1, (3)}\\
&= (a_j+a_k)(a_i+a_j) && b \in L_{ij}\\
&= a_j && \text{Def. 2.3, (1)}
\end{aligned}$$

Hence:

$$(a_j+a_k)b \leq b \wedge a_j = 0 \qquad b \in L_{ij}$$

So:

$$\begin{aligned}
(b\otimes d)a_k &\leq (0+d)a_k\\
&= 0+da_k = 0 && \text{since Mod } (a_k).
\end{aligned}$$

It follows that $(b\otimes d)a_k = 0$, because $b, d \geq 0$.

$$\begin{aligned}
(b\otimes d)+a_k &= (b+d)(a_i+a_k)+a_k && \text{Def. 2.4}\\
&= (a_i+a_k)(b+d)+a_k && \text{Def. 2.1, (1)}\\
&= (a_i+a_k)(b+d+a_k) && \text{Mod } (a_i+a_k)\\
&= (a_i+a_k)(b+a_j+a_k) && d \in L_{jk}\\
&= (a_i+a_k)(a_i+a_j+a_k) && b \in L_{ij}\\
&= a_i+a_k && \text{Def. 2.3, (1).}
\end{aligned}$$

Lemma 2.6. If $b \in L_{ij}$, $d \in L_{jk}$, $e \in L_{kl}$, with i, j, k, l distinct, then $(b\otimes d)\otimes e = b\otimes(d\otimes e)$.

Proof.

$$\begin{aligned}
(b\otimes d)\otimes e &= [(b+d)(a_i+a_k)+e](a_i+a_l)\\
&= [(b+d)(a_i+a_j+a_k)(a_i+a_k+a_l)+e](a_i+a_l)\\
&= [(b+d)(a_i+a_k+a_l)+e](a_i+a_l)\\
&= (b+d+e)(a_i+a_k+a_l)(a_i+a_l) && \text{Mod } (a_i+a_k+a_l)\\
&= (b+d+e)(a_i+a_j+a_l)(a_i+a_l)\\
&= [b+(d+e)(a_i+a_j+a_l)](a_i+a_l) && \text{Mod } (a_i+a_j+a_l)\\
&= [b+(d+e)(a_i+a_j+a_l)(a_j+a_k+a_l)](a_i+a_l)\\
&= [b+(d+e)(a_j+a_l)](a_i+a_l)\\
&= b\otimes(d\otimes e)
\end{aligned}$$

Lemma 2.7. For $x, y \in L_{ij}$, with i, j, k, l distinct,

$$(x\otimes c_{jk})\otimes(c_{ki}\otimes y) = (x\otimes c_{jl})\otimes(c_{li}\otimes y)$$

Proof.

$$\begin{aligned}(x\otimes c_{jk})\otimes(c_{ki}\otimes y) &= (x\otimes(c_{jl}\otimes c_{lk}))\otimes(c_{ki}\otimes y)\\ &= ((x\otimes c_{jl})\otimes c_{lk})\otimes(c_{ki}\otimes y)\\ &= (x\otimes c_{jl})\otimes(c_{lk}\otimes(c_{ki}\otimes y))\\ &= (x\otimes c_{jl})\otimes((c_{lk}\otimes c_{ki})\otimes y)\\ &= (x\otimes c_{jl})\otimes(c_{li}\otimes y)\end{aligned}$$

Lemma 2.8. For $x, y, z, \in L_{12}$, $(x.y).z = x.(y.z)$

Proof.

$$\begin{aligned}(x.y).z &= \{[(x\otimes c_{23})\otimes(c_{31}\otimes y)]\otimes c_{23}\}\otimes(c_{31}\otimes z)\\ &= \{[(x\otimes c_{24})\otimes(c_{41}\otimes y)]\otimes c_{23}\}\otimes(c_{31}\otimes z)\\ &= \{(x\otimes c_{24})\otimes[(c_{41}\otimes y)\otimes c_{23}]\}\otimes(c_{31}\otimes z)\\ &= (x\otimes c_{24})\otimes\{[(c_{41}\otimes y)\otimes c_{23}]\otimes(c_{31}\otimes z)\}\\ &= (x\otimes c_{24})\otimes\{[(c_{41}\otimes(y\otimes c_{23})]\otimes(c_{31}\otimes z)\}\\ &= (x\otimes c_{24})\otimes\{c_{41}\otimes[(y\otimes c_{23})\otimes(c_{31}\otimes z)]\}\\ &= (x\otimes c_{23})\otimes\{c_{31}\otimes[(y\otimes c_{23})\otimes(c_{31}\otimes z)]\}\\ &= x.(y.z)\end{aligned}$$

The final lemma in this section is important in enabling us to define modular elements in the context of relevance logic.

Lemma 2.9. Let a, k be elements of a t-monoid satisfying the conditions:

(a) $a+a = a, \quad k+a = k, \quad ak = \bot,$
(b) $\forall bc(a(b\cup c) = ab\cup ac),$
(c) $\forall b(b = ba\cup bk).$

Then a is modular.

Proof. Assume that $a \geq c$. Then:

$$\begin{aligned}a(b+c) &= a[(ba\cup bk)+c]\\ &= a[(ba+c)\cup(bk+c)]\\ &\leq a[(ba+c)\cup(k+a)]\\ &= a(ba+c)\cup a(k+a)\\ &\leq (ab+c)\cup\bot\\ &= ab+c.\end{aligned}$$

For the converse, $ab+c \leq a+a = a$ and $ab+c \leq b+c$; so $a(b+c) = ab+c$.

§65.2.3. The algebra of relevance logics. The weakest logic for which we prove an undecidability result is $\mathbf{T}_+ - \mathbf{W}$, which we formulate with propositional variables $p_1, \ldots, p_n, \ldots$, conjunction $A\&B$, disjunction $A \vee B$, entailment $A \to B$, and propositional constants $\boldsymbol{t}$, $\boldsymbol{f}$, $\boldsymbol{T}$, and $\boldsymbol{F}$. The axioms and rules of $\mathbf{T}_+ - \mathbf{W}$ follow (see also §R2):

A1 $A \to A$
A2 $A \to B \to .\ B \to C \to .A \to C$
A3 $A \to B \to .\ C \to A \to .C \to B$
A4 $A\&B \to A$
A5 $A\&B \to B$
A6 $(A \to B)\&(A \to C) \to .\ A \to B\&C$
A7 $A \to A \vee B$
A8 $B \to A \vee B$
A9 $(A \to C)\&(B \to C) \to .\ A \vee B \to C$
A10 $A\&(B \vee C) \to (A\&B) \vee C$
A11 $\boldsymbol{t}$
A12 $\boldsymbol{t} \to .\ A \to A$
A13 $A \to \boldsymbol{T}$
A14 $\boldsymbol{F} \to A$
R1 From $A \to B$ and A to infer B.
R2 From A and B to infer $A\&B$.

$\mathbf{T}_+ - \mathbf{W}$ has a natural deduction formulation. The natural deduction rules for $\mathbf{T}$ given in §§27.2 and R3 can be used, with the proviso that, in an application of $\to$E, when $B_{\mathrm{a} \cup \mathrm{b}}$ is inferred from $A \to B_{\mathrm{a}}$ and A_{b}, a and b must be disjoint.

Additional axioms can be added from the following list; $\bar{A}$ represents the negation of A, and $\Box A$ is defined as $\boldsymbol{t} \to A$.

A15 $((A \to B)\&A\&\boldsymbol{t}) \to B$
A16 $(A \to .A \to B) \to .\ A \to B$
A17 $A \to \bar{A} \to \bar{A}$
A18 $A \to \bar{B} \to .\ B \to \bar{A}$
A19 $\bar{\bar{A}} \to A$
A20 $\Box A \to A$
A21 $A \to \Box A$
A22 $(A\&\bar{A}) \to B$

Among the systems we consider are: $\mathbf{E}_+ - \mathbf{W} = \mathbf{T}_+ - \mathbf{W} + \mathrm{A}20$, $\mathbf{R}_+ - \mathbf{W} = \mathbf{E}_+ - \mathbf{W} + \mathrm{A}21$, $\mathbf{T}_+ = \mathbf{T}_+ - \mathbf{W} + \mathrm{A}16$, $\mathbf{E}_+ = \mathbf{E}_+ - \mathbf{W} + \mathrm{A}16$, $\mathbf{R}_+ = \mathbf{R}_+ - \mathbf{W} + \mathrm{A}16$, $\mathbf{T} = \mathbf{T}_+ + (\mathrm{A}17\text{–}19)$, $\mathbf{E} = \mathbf{E}_+ + (\mathrm{A}17\text{–}19)$, $\mathbf{R} = \mathbf{R}_+ + (\mathrm{A}17\text{–}19)$, $\mathbf{KR} = \mathbf{R} + \mathrm{A}22$.

If $\Gamma \cup \{A\}$ is a set of formulas in a logic L, we say that A is deducible from Γ in L if A can be derived from the axioms of L and the formulas in

Γ by the rules R1 and R2; we write $\Gamma \vdash_{\mathrm{L}} A$ if this holds. The deducibility problem for L is the problem of determining whether or not $\Gamma \vdash_{\mathrm{L}} A$ for a given A and finite Γ. The decision problem for L is the problem of determining for a given A whether or not $\vdash_{\mathrm{L}} A$.

In the remainder of this section, deducibility is understood to refer to the system $\mathbf{T}_{+}-\mathbf{W}$. We now embark on a series of lemmas which show that a t-monoid can be found in the algebra of propositions in $\mathbf{T}_{+}-\mathbf{W}$.

DEFINITION 3.1 We let $A \rightleftarrows B$ be $(A \to B) \& (B \to A)$ as usual, and furthermore define the following:

a $\neg A = A \to f$
b $\Box A = t \to A$
c $\mathrm{R}(A) = (\neg\neg A \rightleftarrows A)$ (A is regular)
d $\mathrm{N}(A) = (\Box A \rightleftarrows A)$ (A is a necessitive)
e $A \circ B = \neg(A \to \neg B)$ (read "A fuse B")
f $A^2 = A \circ A$
g $A \cup B = \neg\neg(A \vee B)$

In the remainder of the section we shall use the symbol Θ for the following set of variable-free formulas: $\{\mathrm{R}(t, f, T, F),\ \mathrm{N}(t, f, T, F),\ \neg T \rightleftarrows F\}$, where $\mathrm{R}(A_1, \ldots, A_n)$ stands for the conjunction of $\mathrm{R}(A_1), \ldots, \mathrm{R}(A_n)$, $\mathrm{N}(A_1, \ldots, A_n)$ for the conjunction of $\mathrm{N}(A_1), \ldots, \mathrm{N}(A_n)$. Later we will use $\mathrm{K}(A_1, \ldots, A_n)$ in the same way.

LEMMA 3.2.

(1) $\vdash A \rightleftarrows A$
(2) $A \rightleftarrows B \vdash B \rightleftarrows A$
(3) $A \rightleftarrows B,\ B \rightleftarrows C \vdash (A \rightleftarrows C)$
(4) $A \rightleftarrows B,\ C \rightleftarrows D \vdash (A \& C) \rightleftarrows (B \& D)$
(5) $A \rightleftarrows B,\ C \rightleftarrows D \vdash (A \to C) \rightleftarrows (B \to D)$
(6) $A \rightleftarrows B \vdash \neg A \rightleftarrows \neg B$
(7) $\mathrm{R}(A, B) \vdash A \to .A \to B \to B$
(8) $\mathrm{N}(A, B) \vdash A \to .A \to B \to B$
(9) $A \to .A \to C \to C \vdash (B \to .A \to C) \to (A \to .B \to C)$
(10) $\mathrm{R}(A) \vdash \mathrm{R}(\neg A)$
(11) $\Theta, \mathrm{R}(A) \vdash \mathrm{N}(A)$
(12) $\Theta, \mathrm{N}(A, B) \vdash \mathrm{N}(A \& B, A \to B, \neg A, A \circ B)$
(13) $\Theta, \mathrm{R}(A, B) \vdash \mathrm{R}(A \& B)$
(14) $\Theta, \mathrm{R}(A, B) \vdash \mathrm{R}(A \to B)$
(15) $\vdash (\neg A \& \neg B) \rightleftarrows \neg(A \vee B)$
(16) $\Theta, \mathrm{R}(A, B) \vdash \neg(A \& B) \rightleftarrows (\neg A \cup \neg B)$
(17) $\Theta, \mathrm{R}(A, B) \vdash \mathrm{R}(A \circ B, A \cup B)$.

Proof. Parts (1) to (6) are easily provable in the natural deduction system for $\mathbf{T}_+ - \mathbf{W}$. For (7) and (8), we have two theorems of $\mathbf{T}_+ - \mathbf{W}$: $\neg\neg A \to . A \to B \to \neg\neg B$ and $\Box A \to . A \to B \to \Box B$, from which the conclusion follows by (5). For (9), assuming $A \to . A \to C \to C$, we have $(A \to C \to C \to . B \to C) \to (A \to . B \to C)$, by A2. But then $(B \to . A \to C) \to (A \to . B \to C)$ follows by two uses of A2.

(10) follows trivially from (6). For (11), assuming R(A), Θ we have:

$$\begin{aligned}
\Box A &\rightleftarrows . t \to A && \text{Def. 3.1}\\
&\rightleftarrows . t \to . \neg A \to f && \mathrm{R}(A)\\
&\rightleftarrows . \neg A \to . t \to f && \mathrm{R}(\neg A, t, f)\\
&\rightleftarrows . \neg A \to f && \mathrm{N}(f)\\
&\rightleftarrows . A. && \mathrm{R}(A)
\end{aligned}$$

For (12), assuming N(A, B), N(A&B) follows by A4–6. For entailment:

$$\begin{aligned}
\Box(A \to B) &\rightleftarrows (t \to . A \to B) && \\
&\rightleftarrows (A \to . t \to B) && \mathrm{N}(A, B, t), (8), (9)\\
&\rightleftarrows . A \to B. && \mathrm{N}(B)
\end{aligned}$$

For (13), we have $(A \& B) \to A$, so $\neg\neg(A \& B) \to A$, and similarly $\neg\neg(A \& B) \to B$; hence, $\neg\neg(A \& B) \to (A \& B)$. Now assuming R($A$, B) and Θ, we have N(A, B), by (11); hence $A \& B \to \neg\neg(A \& B)$, by (8).

For (14), assume Θ and R(A, B). Then N(A, B), by (11); hence N($A \to B$) and $(A \to B) \to \neg\neg(A \to B)$, by (8). Since R($A$, B), $A \to . A \to B \to B$, by (7); so $A \to . \neg\neg(A \to B) \to \neg\neg B$. Now, by (12), N($\neg\neg(A \to B), \neg\neg B$); so $\neg\neg(A \to B) \to . A \to \neg\neg B$, by (9); hence $\neg\neg(A \to B) \to . A \to B$, since R($B$).

Parts (15) and (16) are immediate from A7–9. Part (17) follows from the definitions and (10), (13), (14), (15).

The preceding lemma simply serves to show that the regular elements are closed with respect to &, $\circ$, and $\cup$. Parts (1) to (6) show that $A \rightleftarrows B$ has the properties of an identity relation, so that propositions in $\mathbf{T}_+ - \mathbf{W}$ form an algebraic structure if we identify provably equivalent propositions. This algebra is naturally ordered by the implication relation. In the next lemmas we show that the regular propositions in $\mathbf{T}_+ - \mathbf{W}$ form a t-monoid, provided that we assume Θ.

Lemma 3.3.

(1) $\Theta, \mathrm{R}(A, B, C) \vdash (A \circ B) \circ C \rightleftarrows A \circ (B \circ C)$

(2) $A \to B, C \to D \vdash A \circ C \to B \circ D$

$A \to B, C \to D \vdash A \cup C \to B \cup D$

(3) $\Theta, \mathrm{R}(A, B, C) \vdash A \circ (B \cup C) \rightleftarrows (A \circ B) \cup (A \circ C)$

Proof. For part (1), by Lemma 3.2, (14), (17),

$$\begin{aligned}(A\circ B)\circ C &\rightleftarrows \neg(\neg(A\to\neg B)\to\neg C)\\ &\rightleftarrows \neg(C\to .A\to\neg B)\\ &\rightleftarrows \neg(A\to .C\to\neg B)\\ &\rightleftarrows \neg(A\to .B\to\neg C)\\ &\rightleftarrows \neg(A\to .\neg\neg(B\to\neg C))\\ &\rightleftarrows A\circ(B\circ C).\end{aligned}$$

For part (2), if $A\to B$, then $(B\to\neg D)\to .\ A\to\neg D$; if $C\to D$, then $(A\to\neg D)\to(A\to\neg C)$, so $(B\to\neg D)\to .\ A\to\neg C$, hence $(A\circ C)\to(B\circ D)$. For the second part, $A\vee C\to B\vee D$ so $A\cup C\to B\cup D$.

For part (3),

$$\begin{array}{rlll}A\circ(B\cup C) & \rightleftarrows \neg(A\to\neg\neg\neg(B\vee C)) & & \\ & \rightleftarrows \neg(A\to\neg(B\vee C)) & & 3.2, (15), \mathrm{R}(B, C)\\ & \rightleftarrows \neg(A\to(\neg B\wedge\neg C)) & & 3.2, (15)\\ & \rightleftarrows \neg((A\to\neg B)\wedge(A\to\neg C)) & & \text{A4–6}\\ & \rightleftarrows \neg(A\to\neg B)\cup\neg(A\to\neg C) & & 3.2, (16)\\ & \rightleftarrows (A\circ B)\cup(A\circ C). & & \end{array}$$

We define a mapping φ^{t} from the terms φ of the language of t-monoids into the formulas of $\mathbf{T}_{+}-\mathbf{W}$ by the translation:

$$\begin{aligned}(\mathrm{x}_i)^{\mathrm{t}} &= p_i\\ (\varphi\wedge\psi)^{\mathrm{t}} &= (\varphi^{\mathrm{t}}\&\psi^{\mathrm{t}})\\ (\varphi+\psi)^{\mathrm{t}} &= (\varphi^{\mathrm{t}}\circ\psi^{\mathrm{t}})\\ (\varphi\cup\psi)^{\mathrm{t}} &= (\varphi^{\mathrm{t}}\cup\psi^{\mathrm{t}})\\ (\bot)^{\mathrm{t}} &= F\\ (0)^{\mathrm{t}} &= \boldsymbol{t}\\ (\varphi=\psi)^{\mathrm{t}} &= (\varphi^{\mathrm{t}}\rightleftarrows\psi^{\mathrm{t}})\end{aligned}$$

Lemma 3.4. Let φ_i, ψ_i, $1\le i\le n$, ϵ, δ be terms of the t-monoid language in the variables $\mathrm{x}_1,\ldots,\mathrm{x}_k$. If $\epsilon=\delta$ is deducible from $\varphi_1=\psi_1,\ldots,\varphi_n=\psi_n$ in the theory of t-monoids, $(\epsilon=\delta)^{\mathrm{t}}$ is deducible from Θ, $\mathrm{R}(p_1,\ldots,p_k)$, and $(\varphi_1=\psi_1)^{\mathrm{t}},\ldots,(\varphi_n=\psi_n)^{\mathrm{t}}$ in $\mathbf{T}_{+}-\mathbf{W}$.

Proof: By Lemma 3.2 (1)–(6), $\rightleftarrows$ has the properties of the identity relation in the algebra of propositions; so it suffices to show that the translations of the axioms for a t-monoid are deducible from the assumptions Θ and $\mathrm{R}(p_1,\ldots,p_k)$.

It is immediate that the regular propositions form a semilattice, with F the least element.

It follows from Lemma 3.3 (1) that the regular propositions form a commutative semigroup, and

$$\begin{array}{lll} (A\circ \mathbf{t}) & \rightleftarrows \neg(A\rightarrow\neg \mathbf{t}) & \\ & \rightleftarrows \neg(A\rightarrow \mathbf{f}) & \mathrm{N}(\mathbf{f}) \\ & \rightleftarrows A & \mathrm{R}(A) \end{array}$$

so $\mathbf{t}$ forms the semigroup identity.

For part (3) of Definition 2.1, we note that if $\mathbf{t}\rightarrow B$, then B by $\rightarrow$E; so $A\rightarrow .A\circ B$, by 3.2, (7).

The translation of part (4) of Definition 2.1 is provable by Lemma 3.3 (2), and the translation of part (5) by Lemma 3.3 (3). Finally, the translation of part (6) follows by A14.

Definition 3.5.

$$\mathrm{I}(A) = (A^2 \rightleftarrows A) \qquad (A \text{ is an idempotent})$$

$$\mathrm{K}(A) = (\mathrm{R}(A)\&(A\&\neg A\rightleftarrows F)\wedge(A\vee\neg A\rightleftarrows T)\wedge(A\circ\neg A\rightleftarrows\neg A)\wedge \mathrm{I}(A))$$

(A is a classical idempotent).

Lemma 3.6.

(1) $\Theta, \mathrm{K}(A), \mathrm{R}(B) \vdash B\rightleftarrows(B\&A)\cup(B\&\neg A)$

(2) $\mathrm{K}(A)\vdash\neg(A\&B)\rightleftarrows\neg A\vee\neg B$

(3) $\mathrm{K}(A)\vdash A\&\neg\neg B\rightarrow\neg\neg(A\&B)$

(4) $\mathrm{K}(A), \mathrm{R}(B, C)\vdash A\&(B\cup C)\rightleftarrows(A\&B)\cup(A\&C)$.

Proof. For part (1):

$$\begin{array}{lll} B & \rightleftarrows B\&T & \\ & \rightleftarrows B\&(A\vee\neg A) & \\ & \rightleftarrows (B\&A)\vee(B\&\neg A) & \\ & \rightleftarrows (B\&A)\cup(B\&\neg A) & \text{since } \mathrm{R}(B). \end{array}$$

For part (2),

$$\begin{array}{ll} A\&\neg(A\&B) & \rightarrow \neg\neg A\&\neg(A\&B) \\ & \rightarrow \neg(\neg A\vee(A\&B)) \\ & \rightarrow \neg((\neg A\vee A)\&(\neg A\vee B)) \\ & \rightarrow \neg(\neg A\vee B) \\ & \rightarrow\neg B; \end{array}$$

hence,

$$\begin{array}{ll} \neg(A\&B) & \rightleftarrows (A\vee\neg A)\&\neg(A\&B) \\ & \rightleftarrows [A\&\neg(A\&B)]\vee[\neg A\&\neg(A\&B)] \\ & \rightleftarrows (\neg B\vee\neg A). \end{array}$$

For part (3),

$$\begin{aligned} A\&\neg\neg B &\to \neg\neg A\&\neg\neg B\\ &\to \neg(\neg A\vee\neg B)\\ &\to \neg\neg(A\&B) \qquad \text{by (2).}\end{aligned}$$

For part (4), assuming the antecedent, we have:

$$\begin{aligned} A\&(B\cup C) &\rightleftarrows A\&\neg\neg(B\vee C)\\ &\to \neg\neg(A\&(B\vee C))\\ &\to \neg\neg((A\&B)\vee(A\&C))\\ &\to (A\&B)\cup(A\&C).\end{aligned}$$

For the converse, $(A\&B)\vee(A\&C) \to A\&(B\vee C)$; hence

$(A\&B)\cup(A\&C) \to A\&(B\cup C)$, since $\mathrm{R}(A)$.

§65.2.4. De Morgan monoids and vector spaces. We now show that a very extensive class of models for relevance logics can be constructed from vector spaces.

DEFINITION 4.1. A De Morgan monoid (§28.2.1) is an algebra

$$\langle \mathrm{S}, \circ, \vee, \neg, t\rangle$$

where we define:

$$\begin{aligned} &\mathrm{a}\wedge\mathrm{b} = \neg(\neg\mathrm{a}\vee\neg\mathrm{b})\\ &\mathrm{a}\to\mathrm{b} = \neg(\mathrm{a}\circ\neg\mathrm{b})\\ &\mathrm{a}\leq\mathrm{b} \text{ iff } \mathrm{a}\vee\mathrm{b} = \mathrm{b},\end{aligned}$$

and the algebra satisfies the postulates:

$$\begin{aligned} (\mathrm{a}\circ\mathrm{b})\circ\mathrm{c} &= \mathrm{a}\circ(\mathrm{b}\circ\mathrm{c})\\ \mathrm{a}\circ\mathrm{b} &= \mathrm{b}\circ\mathrm{a}\\ t\circ\mathrm{a} &= \mathrm{a}\\ \mathrm{a}\vee\mathrm{b} &= \mathrm{b}\vee\mathrm{a}\\ (\mathrm{a}\vee\mathrm{b})\vee\mathrm{c} &= \mathrm{a}\vee(\mathrm{b}\vee\mathrm{c})\\ \mathrm{a} &= \mathrm{a}\vee(\mathrm{a}\wedge\mathrm{b})\\ \mathrm{a}\wedge(\mathrm{b}\vee\mathrm{c}) &= (\mathrm{a}\wedge\mathrm{b})\vee(\mathrm{a}\wedge\mathrm{c})\\ \neg\neg\mathrm{a} &= \mathrm{a}\\ \mathrm{a}\circ(\mathrm{b}\vee\mathrm{c}) &= (\mathrm{a}\circ\mathrm{b})\vee(\mathrm{a}\circ\mathrm{c})\\ \mathrm{a}\circ(\mathrm{a}\to\mathrm{b}) &\leq \mathrm{b}\\ \mathrm{a} &\leq \mathrm{a}\circ\mathrm{a}.\end{aligned}$$

DEFINITION 4.2. Let A be a formula of **R**, and D a De Morgan monoid. An interpretation φ of A in D is a function mapping the subformulas of A into D which satisfies the conditions: $\varphi(A\to B) = \varphi(A)\to\varphi(B)$, $\varphi(\bar{A}) = \neg\varphi(A)$,

$\varphi(A \vee B) = \varphi(A) \vee \varphi(B)$, $\varphi(\boldsymbol{t}) = \boldsymbol{t}$. A formula of **R** is said to be valid if $\varphi(A) \geq \boldsymbol{t}$ for all interpretations φ in a De Morgan monoid.

For the notion of validity just defined we have the following fundamental completeness theorem, due to Dunn 1966 (see §28.2):

Theorem 4.3. A formula of **R** is provable in **R** if and only if it is valid with respect to the class of all De Morgan monoids.

Lemma 4.4. Let R be a division ring of characteristic $\neq 2$, and V a vector space over R. Define a subset A of V to be *m-closed* if it satisfies the condition:

if $a \in R$, $a \neq 0$, and $x \in A$, then $ax \in A$.

For A, B m-closed, define operations as follows: $A \vee B = A \cup B$, $\neg A = V - A$, $A \circ B = A + B = \{x + y \colon x \in A \text{ and } y \in B\}$, and let $\boldsymbol{t} = \{\boldsymbol{0}\}$.

If M is the class of all m-closed sets in V then $\langle M, \circ, \vee, \neg, \boldsymbol{t} \rangle$ is a De Morgan monoid, which we denote by "M(V)."

Proof. First, we have to show that M is closed with respect to the operations. Assume $A, B \in M$. If $x \in \neg A$, $a \in R$, and $a \neq 0$ then a^{-1} exists and $a^{-1} \neq 0$; thus if $ax \in A$ then $a^{-1}ax = x \in A$, a contradiction; so $ax \in \neg A$. If $x \in A \circ B$, $a \neq 0$, then $x = y + z$, $y \in A$, $z \in B$; so $ax = a(y+z) = ay + az \in A \circ B$.

That the axioms and definitions involving only $\wedge$, $\vee$, and $\neg$ hold in the defined algebra is clear from the fact that it is a Boolean algebra.

The operation $A \circ B$ is obviously associative and commutative, because addition in any vector space is associative and commutative. In addition,

$$\begin{aligned} A \circ (B \vee C) &= \{x + y \colon x \in A \text{ and } y \in B \vee C\} \\ &= \{x + y \colon x \in A \text{ and } y \in B\} \cup \{x + y \colon x \in A \text{ and } y \in C\} \\ &= (A \circ B) \vee (A \circ C), \end{aligned}$$

and

$$\boldsymbol{t} \circ A = \{\boldsymbol{0} + x \colon x \in A\} = A.$$

Let us assume that $x \in A \circ (A \to B)$, $x \notin B$. Then for some y, z, $x = y + z$, $y \in A$, $z \in A \to B$; so $z \notin A \circ \neg B$. Now $z = (-1)y + x$; since $0 \neq -1$, $-y \in A$; so $z \in A \circ \neg B$, a contradiction.

Finally, if $x \in A$, then, since $2 \neq 0$, $x = \frac{1}{2}x + \frac{1}{2}x \in A \circ A$.

If M is a De Morgan monoid then the idempotent elements (that is, the elements a such that $a \circ a = a$) which are $\geq \boldsymbol{t}$ form a lattice where the lattice meet is $a \wedge b$ and the lattice join is $a \circ b$. This follows from the fact that if $b \geq \boldsymbol{t}$, then $a \leq a \circ b$.

The construction given in the preceding lemma is a special case of the construction used in Lemma 4.1 of §65.1.4. If we start with a vector space V over a division ring, as in the above lemma, then we can construct a projective space by the standard method given at the end of §65.1.3. If we then apply the construction method of §65.1.4 we obtain a **KR** model structure M constructed from the vector space. The family of all subsets of this model structure forms a Boolean algebra on which we can define a fusion operation by using the truth definition for fusion with respect to M. The result is a De Morgan monoid which is identical with the algebra constructed in the preceding lemma. The only difference between the two methods is that in the foregoing lemma the intermediate step of constructing the projective space is omitted. It is still heuristically useful to think of the m-closed subsets as sets of points in a projective space. The earlier construction is in fact the more general, because there are projective planes that cannot be coordinatized by a division ring (because they do not satisfy the Desargues law).

LEMMA 4.5. Let R and M(V) be as in Lemma 4.4. Then the lattice of idempotents $\geq \boldsymbol{t}$ in M(V) is identical with the lattice of subspaces of V.

PROOF. If A is a subspace then $0 \in A$, and $A \circ A = A$; so A satisfies the conditions $A \circ A = A$ and $A \geq \boldsymbol{t}$. Conversely, if A satisfies the conditions then, since $\boldsymbol{t} \leq A$, $x \in A$ implies $ax \in A$ for any element a of R, and x, $y \in A$ implies $x + y \in A \circ A = A$, showing that A is a subspace.

LEMMA 4.6. Let R be a division ring of characteristic $\neq 2$, and let V be an infinite-dimensional vector space over R. Then:

(1) The lattice L of subspaces of V forms a complemented modular lattice.
(2) L contains a 4-frame.
(3) Any countable semigroup is a subsemigroup of the multiplicative semigroup of the auxiliary ring associated with this 4-frame in L.

PROOF. See Lipshitz 1974, Lemma 3.1.

§65.2.5. Undecidability. We are now in a position to bring together our previous results to prove the main undecidability results. If M is a De Morgan monoid, the logic L(M) determined by M is defined as the set of all formulas in a given language which are valid in M (that is, formulas *A* such that $\varphi(A) \geq \boldsymbol{t}$ for any interpretation φ of *A* in M). If R, V are as in Lemma 4.6, then we define L(V) to be the logic L(M(V)) in the language of $\mathrm{T}_+ - \mathbf{W}$, interpreting $\varphi(\boldsymbol{T}) = \mathrm{V}$, $\varphi(\boldsymbol{F}) = \varnothing$, $\varphi(\boldsymbol{f}) = \mathrm{V} - \{0\}$. Then:

THEOREM 5.1. (1) If L is a logic intermediate between $\mathbf{T}_+ - \mathbf{W}$ and L(V) then the deducibility problem for L is unsolvable.

(2) If L is a logic intermediate between $\mathbf{T}_+ - \mathbf{W} + \mathrm{A15}$ and L(V) then the decision problem for L is unsolvable.

PROOF. Let P be a finitely presented semigroup with undecidable word problem. We show how to reduce the word problem for P to the deducibility problem for $\mathbf{T}_+ - \mathbf{W}$. Accordingly, let $\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n$ be the equations constituting the presentation of P, and let $\epsilon = \delta$ be a given equation in the language of semigroups. Let these equations be expressed in the variables $\mathrm{x}_1, \ldots, \mathrm{x}_k$. Let Π be the translation into $\mathbf{T}_+ - \mathbf{W}$ of the equations stating that $\mathrm{x}_{k+1}, \ldots, \mathrm{x}_{k+16}$ form a 4-frame (as in Definition 2.3, omitting modularity). Let Δ be the translation into $\mathbf{T}_+ - \mathbf{W}$ of the semigroup equations (using the semigroup operation defined in the 4-frame as in Definition 2.4 (c)) together with the translations of the equations stating that $\mathrm{x}_1, \ldots, \mathrm{x}_k \in \mathrm{L}_{12}$. Then we claim that $\epsilon = \delta$ is deducible from $\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n$ if and only if $\epsilon^{\mathrm{t}} \rightleftarrows \delta^{\mathrm{t}}$ is deducible in $\mathbf{T}_+ - \mathbf{W}$ from Θ, Π, Δ, $\mathrm{R}(p_1, \ldots, p_k)$, $\mathrm{K}(\boldsymbol{t}, Q_1, \ldots, Q_{15})$, where the Q_is are the translations of all expressions of the form $\Sigma \mathrm{G}$, where $\mathrm{G} \subseteq \{\mathrm{a}_1, \ldots, \mathrm{a}_4\}$, $\mathrm{G} \neq \varnothing$. By Lemma 3.6, from $\mathrm{K}(A)$ it can be deduced that A satisfies all the conditions of Lemma 2.9; so the modularity of A can be deduced in $\mathbf{T}_+ - \mathbf{W}$. The proofs of Lemmas 2.5–2.8 can therefore be carried through in $\mathbf{T}_+ - \mathbf{W}$, by Lemma 3.4, under the assumptions Θ, $\mathrm{R}(p_1, \ldots, p_k)$, Π, Δ. It follows by Lemma 2.8 that if $\epsilon = \delta$ is deducible from $\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n$ then $(\epsilon = \delta)^{\mathrm{t}}$ is deducible from Θ, Π, Δ, $\mathrm{R}(p_1, \ldots, p_k)$, $\mathrm{K}(\boldsymbol{t}, Q_1, \ldots, Q_{15})$.

For the converse, assume that $\epsilon = \delta$ is not deducible from $\varphi_1 = \psi_1, \ldots, \varphi_n = \psi_n$. Let R be any division ring of characteristic $\neq 2$ (say, the ring of all real numbers), and V an infinite-dimensional vector space over R. Then, by Lemma 4.6, there is a 4-frame in the lattice L of all subspaces of V, and values $\mathrm{v}(\mathrm{x}_i)$ in L can be assigned to the variables $\mathrm{x}_1, \ldots, \mathrm{x}_{k+16}$ so that, if the semigroup operation is interpreted as multiplication in the auxiliary ring associated with the 4-frame, then (1) $\varphi_i = \psi_i$ holds for $1 \leq i \leq n$; (2) $\epsilon = \delta$ does not hold; (3) $\mathrm{x}_{k+1}, \ldots, \mathrm{x}_{k+15}$ forms a 4-frame in L (note that all the elements in L are modular). By Lemma 4.5, the lattice of idempotents $\geq \boldsymbol{t}$ in M(V) is identical with the lattice L. Since, for any x, y in a De Morgan monoid, $\boldsymbol{t} \leq \mathrm{x} \rightarrow \mathrm{y}$ if and only if $\mathrm{x} \leq \mathrm{y}$, it follows that if we assign to the variables $p_1, \ldots, p_{k+16}$ the values $\varphi(p_i) = \mathrm{v}(\mathrm{x}_i)$ then $\boldsymbol{t} \leq \varphi(A)$ for any formula A in Δ, and $\boldsymbol{t} \not\leq (\epsilon = \delta)^{\mathrm{t}}$. It remains to show that $\boldsymbol{t} \leq \varphi(B)$ for any $B \in \Theta \cup \Pi \cup \{\mathrm{R}(p_1, \ldots, p_k), \mathrm{K}(\boldsymbol{t}, Q_1, \ldots, Q_{16})\}$. Now, since $\varphi(f) = \mathrm{V} - \{\mathbf{0}\}$, $\varphi(\neg A) = \mathrm{V} - \varphi(A)$. It follows that all the elements A in M(V) satisfy the equations $\mathrm{A} \wedge \neg \mathrm{A} = \varnothing$, $\mathrm{A} \vee \neg \mathrm{A} = \mathrm{V}$, and $\neg\neg \mathrm{A} = \mathrm{A}$. For any idempotent $\mathrm{a} \geq \boldsymbol{t}$, $\mathrm{a} = \mathrm{a} \rightarrow \mathrm{a}$; so $\mathrm{a} \circ \neg \mathrm{a} = \neg \mathrm{a}$; hence $\boldsymbol{t} \leq \varphi(\mathrm{K}(\boldsymbol{t}, Q_1, \ldots, Q_{15}))$. We have thus shown that $(\epsilon = \delta)^{\mathrm{t}}$ is not deducible from Θ, Π, Δ, $\mathrm{R}(p_1, \ldots, p_k)$ and $\mathrm{K}(\boldsymbol{t}, Q_1, \ldots, Q_{15})$, which completes the proof of part (1).

Part (2) of the theorem follows from the fact that, if A15 is a theorem of L, then, if $\boldsymbol{t} \in \Gamma$, $\Gamma \vdash_{\mathrm{L}} A$ holds if and only if $\vdash_{\mathrm{L}} B \rightarrow A$, where B is the conjunction of all formulas in Γ.

Corollary 5.2. Let L be a logic intermediate between $\mathbf{T}_{+}-\mathbf{W}$ and **KR**. Then the deducibility problem for L is unsolvable. If L contains A15 then the decision problem for L is unsolvable.

Corollary 5.3. (a) The deducibility problem for $\mathbf{E}_{+}-\mathbf{W}$ and $\mathbf{R}_{+}-\mathbf{W}$ is unsolvable. (b) The decision problem for $\mathbf{T}_{+}$, $\mathbf{E}_{+}$, and $\mathbf{R}_{+}$ is unsolvable.

[Note by principal authors: it is shown in §67 that the decision problems for $\mathbf{T}_{+}-\mathbf{W}$ and $\mathbf{R}_{+}-\mathbf{W}$ are solvable. The decision problem for $\mathbf{E}_{+}-\mathbf{W}$ apparently remains open.]

It should be pointed out that Corollary 5.2 only partially represents the content of Theorem 5.1. The logic L(V) is always a proper extension of **KR**. Since Desargues' law holds in a projective space over a division ring, L(V) contains as a theorem a formulation of the Arguesian identity (see Grätzer 1978, pp. 198–215). This formula is not a theorem of **KR** because of the existence of non-Desarguesian projective planes. In general, L(V) is a strong extension of **KR** which contains theorems reflecting the structure of the ring R.

The formulation of relevance logics in this section uses the propositional constants $\boldsymbol{t}$, $\boldsymbol{f}$, $\boldsymbol{T}$, and $\boldsymbol{F}$. However, the standard formulation of relevance logics does not use these constants (see §R1). Accordingly, to show undecidability for the standard formulation of $\mathbf{T}_{+}$, $\mathbf{E}_{+}$, $\mathbf{R}_{+}$, etc., we must show how to dispense with these constants. Let X be a finite set of propositional variables, and define t(X) to be the conjunction of all formulas of the form $p \rightarrow p$ for $p \in \mathrm{X}$. Then, for any formula A of $\mathbf{T}_{+}-\mathbf{W}$ whose variables are contained in X, $\mathrm{t(X)} \rightarrow . A \rightarrow A$ is provable in the standard formulation of $\mathbf{T}_{+}-\mathbf{W}$ and so, in contexts involving only formulas constructed from variables in X, t(X) can replace $\boldsymbol{t}$. To dispense with $\boldsymbol{T}$ and $\boldsymbol{F}$, add two new propositional variables r and s, and assume $r \rightarrow . s \rightarrow r$, $r \rightarrow s \rightarrow s$, and $(r \rightarrow p)$ and $(p \rightarrow s)$ for any propositional variable p in $\mathrm{X} \cup \{r, s\}$. From these assumptions we can deduce in the standard formulation of $\mathbf{T}_{+}-\mathbf{W}$ that $r \rightarrow A$ and $A \rightarrow s$ for any formula of $\mathbf{T}_{+}-\mathbf{W}$ in the variables in $\mathrm{X} \cup \{r, s\}$. The undecidability results of Theorem 5.1 and its corollaries, therefore, carry over unchanged to the formulation without sentential constants, since no special axioms were assumed for the constant $\boldsymbol{f}$.

In the proof of Theorem 5.1 we did not attempt to minimize the complexity of the formulas involved. However, it follows from Meyer's reduction described in §64.1 that the second degree fragments of all the logics in part (2) of Theorem 5.1 are undecidable. In another direction, we can show that, for extensions of $\mathbf{T}-\mathbf{W}$ + A15, the decision problem for formulas in five variables is unsolvable. Let $\mathrm{a}_1, \ldots, \mathrm{a}_4$, $\mathrm{c}_{12}, \ldots, \mathrm{c}_{34}$ be the 4-frame in M(V)

in Lemma 4.6. Let $p = a_1 \vee a_2, q = a_1 \vee a_3$, $c = c_{12} \vee c_{13} \vee \ldots \vee c_{34}$. Then $p \wedge c = \boldsymbol{t}$; so we can define $\sim x = \neg x \vee \boldsymbol{t}$. Then elementary linear algebra shows that $a_1 = p \wedge q$, $a_2 = p \wedge \sim q$, $a_3 = \sim p \wedge q$, $a_4 = \sim p \wedge \sim q$; we can then define $c_{ij} = c \wedge (a_i \circ a_j)$. Since there is a finitely presented semigroup with two generators with unsolvable word problem, it follows that we can prove undecidability for formulas in five variables. A similar argument shows undecidability for formulas in six variables for the positive systems.

The coding of the semigroup equations which is accomplished by Definition 2.4 uses only the connectives of conjunction and implication (given that the propositional constants are eliminated as above). In order to carry through the undecidability result it is necessary only to be able to prove that the frame elements are modular. Now in **KR** any idempotent element A such that $\boldsymbol{t} \rightarrow A$ satisfies the condition $K(A)$, that is, all propositions in **KR** are classical. Hence, by Lemmas 2.9 and 3.6, any such element is modular. For the extensions of **KR** we therefore have the following strong undecidability result:

THEOREM 5.4. Let L be a logic expressed solely in terms of & and → which is intermediate between $\mathbf{KR}_{\&\rightarrow}$ and $L(V)_{\&\rightarrow}$. Then L is undecidable.

§65.2.6. Further undecidability results. It is immediate from Theorem 5.1 that the logics **R**, **E**, and **T** do not have the finite model property. In fact, we can prove something stronger. In the case of $\mathbf{T}_+ - \mathbf{W}$ and its extensions, model theories have been developed by Routley and Meyer 1972a, Routley and Meyer 1972b, Meyer and Routley 1973, using both algebraic models and relational model structures, as partly described in §48. In the following theorems, the algebraic and relational models are understood to be formulated as in the work of Meyer and Routley. We say that a model structure is *suitable* for a logic L if it validates all the theorems of L.

THEOREM 6.1. Let F_p be the prime field of characteristic p, where p is greater than 2. Let C be a class of finite models which includes all the De Morgan monoids M(V), where V is a finite-dimensional vector space over F_p; let L(C) be the logic determined by C; and let L(C) be an extension of $\mathbf{T}_+ - \mathbf{W}$. Then: (1) the deducibility problem for L(C) is unsolvable; (2) if L(C) contains A15, the decision problem for L(C) is unsolvable.

PROOF. Theorem 5.1 exploits Post's result that the word problem for semigroups is unsolvable. In this case, we make use of the result of Gurevich that the word problem for finite semigroups is unsolvable. Let Γ be a finite set of semigroup equations, and FS the theory of finite semigroups. We can translate the equations in Γ into the language of $\mathbf{T}_+$ as in Section 5. We claim that an equation $\epsilon = \delta$ is deducible from Γ in FS if and only if its translation is deducible in L(C) from the translation of Γ, together with the auxiliary hypotheses as in §65.2.5. First, if M is a finite model then the semigroup defined relative to a 4-frame in the model is finite; so if $\epsilon = \delta$ is deducible from Γ then the translation of $\epsilon = \delta$ can be deduced from the assumptions. Second, we note that any finite semigroup can be embedded in the multiplicative group of the auxiliary ring associated with a 4-frame in the lattice of subspaces of a finite-dimensional vector space V over F_p; for the proof see Lipshitz 1974, 3.4–3.9. It follows that the assumptions can be verified and the equivalence $\epsilon^t \rightleftarrows \delta^t$ falsified in M(V).

COROLLARY 6.2. Let L be a finitely axiomatized logic between $\mathbf{T}_+ - \mathbf{W} +$ A15 and **KR**. Then the logic determined by the class of all finite model structures suitable for L is not recursively enumerable.

The generalization of the von Neumann construction employed in this section has applications in other areas, such as the theory of ordered semigroups.

THEOREM 6.3. The word problem for distributive lattice-ordered commutative semigroups is unsolvable.

PROOF (outline). The definition of a modular n-frame, the definition of multiplication, and the proof of its associativity can be carried out in the theory of distributive lattice-ordered semigroups. In this theory, the equations $a+k = k$, $a \vee k = \bot$, and $a \wedge k = \bot$ imply that a is modular, by the proof of Lemma 2.9. We can therefore express the condition that the n-frame is modular by a finite number of equations. Undecidability therefore follows by the arguments of the previous section.

Similar results can be proved in varieties of residuated semigroups analogous to relevance logics.

PROBLEM. The undecidability results of §65.2.5 cover an extensive class of logics, but omit one notable case. This is the logic consisting of all formulas valid in the semilattice semantics given in §47. This logic includes $((A \rightarrow .B \vee C) \& B \rightarrow D) \rightarrow (A \rightarrow .D \vee C)$, which is not valid in the De Morgan monoids M(V). The decision problem for this system is still open.

§66. Minimal logic again (by Errol P. Martin).

In §8.11 a logic is said to be "minimal" when there are no distinct formulas A and B such that both $A \rightarrow B$ and $B \rightarrow A$ are theorems of this logic. The question from §8.11 was whether the logic $\mathbf{T}_{\rightarrow} - \mathbf{W}$, with only *identity*, *prefixing*, and *suffixing* as axioms, has this property. It does. A proof is given here, using material from Martin 1978 and Martin and Meyer 1982. The proof is necessarily rather condensed; so these papers should be consulted for further details.

Terminology, systems, and notation are taken over from §8.11, with two important exceptions. The first is that full advantage is taken of the reduction of the problem due to L. Powers, mentioned in §8.11. What is proved here (Powers's conjecture) is that no instance of $A\to A$ can be proved from just the two transitivity axioms (and *modus ponens*). This subsystem of $\mathbf{T}_{\to}-\mathbf{W}$ is given the name S (for *s*yllogism—the transitivity axioms express in pure "$\to$" talk the syllogism in *Barbara*). Note that this is not the system **S** (for *S*miley) briefly considered in §20.1.1.

The second modification of the problem from §8.11 is that, following a result of R. Dwyer (reported in Meyer 1976c), the systems are formulated with the rule *modus ponens* replaced by the rules of *prefixing*: from $B\to C$ to infer $A\to B\to .A\to C$; *suffixing*: from $A\to B$ to infer $B\to C\to .A\to C$; and *transitivity*: from $A\to B$ and $B\to C$ to infer $A\to C$.

The general plan for the proof of Powers's conjecture is as follows: A class of models for **S** (and $\mathbf{T}_{\to}-\mathbf{W}$) is described, with a sketch of the details of soundness and completeness arguments. After some further analysis of the canonical model used for completeness, the main proof shows by induction on the length of a formula A that there exists a countermodel to $A\to A$.

The proof is carried out for a language L of formulas constructed from just one propositional variable p, which suffices for the conjecture. However, a development of the semantic theory for the general case is entirely straightforward.

§66.1. Three-valued metalogic. The models for **S** and $\mathbf{T}_{\to}-\mathbf{W}$ considered here are of the operational type, like those in Urquhart 1972 (see §47), Fine 1974 (see §51), and Routley with Plumwood, Meyer, and Brady 1982. The same class of models, called S-*models*, suffices for **S** and $\mathbf{T}_{\to}-\mathbf{W}$, the systems being distinguished by differing conditions of validity.

The main new point is that three-valued metalogical operations are employed to evaluate formulas in a model. Relations and operations on the three truth values $-\mathbf{1}$, $\mathbf{0}$, $+\mathbf{1}$ will be familiar from the analysis of the logic **RM3** (see §26.9 and §29.12). The usual ordering of the truth values (which is also the arithmetic ordering) is indicated by $\leq$, with $\geq$ for the converse. The relation $<$ is introduced with the special meaning that $a < b$ holds provided that $a \leq b$ but not both $a = \mathbf{0}$ and $b = \mathbf{0}$. The operations $\Rightarrow$ and $\wedge$ are defined by means of the tables

$\Rightarrow$	$+1$	0	-1
$+1$	$+1$	-1	-1
0	$+1$	0	-1
-1	$+1$	$+1$	$+1$

$\wedge$	$+1$	0	-1
$+1$	$+1$	$+1$	-1
0	$+1$	0	-1
-1	-1	-1	-1

Operations $(\forall x)$ and $(\exists x)$ corresponding to quantifiers are also used, with the meaning of greatest lower bound and least upper bound (of sets of truth values), respectively.

It would be possible to develop the metalogic in a suitably formal manner so that **RM3** (or rather, a quantified extension of it) played the role usually played by classical logic in formalized metatheory. But we prefer instead to write this section in English, leaving the operations of the metatheory, for the present purposes, as an uninterpreted algebra over $\{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$.

§66.2. S-models. By an S-*model* is meant a structure $\langle \mathrm{K}, \circ, \mathrm{R}, \mathrm{v}\rangle$, where (i) K is a set (of "worlds") closed under a binary operation $\circ$; (ii) there is a binary relation R over worlds; (iii) v: $(\mathrm{K} \times \{p\}) \rightarrow \{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$ is a *valuation* function; (iv) the structure $\langle \mathrm{K}, \circ, \mathrm{R}\rangle$, called an S-*model structure*, is subject to the following conditions on R (where $\mathrm{a}, \mathrm{b} \in \mathrm{K}$, we write "ab" for "$\mathrm{a}\circ\mathrm{b}$", and assume association to the left where parentheses are omitted): For all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{K}$.

(B) abc R a(bc)
(B′) abc R b(ac)
(μ) a R b implies ca R cb
(ν) a R b implies ac R bc
(τ) a R b and b R c imply a R c

(these principles are called the BB′-*conditions*; (B) and (B′) are named after the combinators of Curry and Feys 1958, which have similar reduction schemes); and (v) the valuation function satisfies a *hereditary condition*: For all $\mathrm{a}, \mathrm{b} \in \mathrm{K}$,

(Hp) a R b implies $\mathrm{v}(\mathrm{b}, p) < \mathrm{v}(\mathrm{a}, p)$.

S-models suffice to characterize the class of theorems of both $\mathbf{T}_{\rightarrow}\mathbf{-W}$ and S, on the following definitions: with each S-model $\mathrm{M} = \langle \mathrm{K}, \circ, \mathrm{R}, \mathrm{v}\rangle$ is associated an *interpretation function* I_{M} over formulas, defined recursively so that $\mathrm{I}_{\mathrm{M}}(\mathrm{a}, p) = \mathrm{v}(\mathrm{a}, p)$ for all $\mathrm{a}\in\mathrm{K}$, and, for $\mathrm{a}\in\mathrm{K}$, $A, B \in \mathrm{L}$,

(T) $\mathrm{I}_{\mathrm{M}}(\mathrm{a}, A\rightarrow B) = (\forall \mathrm{x})(\mathrm{I}_{\mathrm{M}}(\mathrm{x}, A) \Rightarrow \mathrm{I}_{\mathrm{M}}(\mathrm{ax}, B))$.

We say that *A entails B* in model M provided that $\mathrm{I}_{\mathrm{M}}(\mathrm{a}, A) \leq \mathrm{I}_{\mathrm{M}}(\mathrm{a}, B)$ for all $\mathrm{a}\in\mathrm{K}$ and that *A strictly entails B* when $\mathrm{I}_{\mathrm{M}}(\mathrm{a}, \mathrm{A}) < \mathrm{I}_{\mathrm{M}}(\mathrm{a}, \mathrm{B})$ for all $\mathrm{a}\in\mathrm{K}$. $A\rightarrow B$ is *valid* when A entails B in every S-model, and is *strictly valid* when A strictly entails B in every S-model.

For soundness of the logics in these models, use induction on length of proof to establish that $\vdash_{\mathbf{T}_{\rightarrow}\mathbf{-W}} A\rightarrow B$ implies that $A\rightarrow B$ is valid, and that $\vdash_{\mathbf{S}} A\rightarrow B$ implies that $A\rightarrow B$ is strictly valid. The proof requires as a lemma a generalized version of the hereditary condition: For $\mathrm{M} = \langle \mathrm{K}, \circ, \mathrm{R}, \mathrm{v}\rangle$, a, b $\in$ K, and $A \in$L,

(H) a R b implies $I_M(b, A) < I_M(a, A)$.

The soundness theorem suffices in principle for the proof of Powers' conjecture: since each formula A entails itself in every S-model, the formula $A \rightarrow A$ is valid. But, for given A, there may be some S-model in which A does not strictly entail A; so $A \rightarrow A$ is not strictly valid, hence not a theorem of S. In particular, it suffices for this result that $I_M(a, A) = \mathbf{0}$ for some world a in some S-model M; for then $I_M(a, A) < I_M(a, A)$ fails.

The aim of the argument from now on is to exhibit models that show in this manner that formulas of the form $A \rightarrow A$ are not theorems of S. For this purpose a canonical S-model is introduced, which as usual allows a proof of completeness, but which can then be adapted as required to provide countermodels for the proof of Powers's conjecture.

Let W be a set that consists of the formulas in L and is otherwise closed under a free binary operation $\circ$. Members of W are the *canonical worlds*, or just *worlds* in context. Let $\triangleright$ be the least binary relation over W that satisfies the BB′-conditions. $\langle W, \circ, \triangleright \rangle$ is called the *canonical model structure* (CMS).

Formulas are interpreted in CMS by means of a canonical interpretation function CI, recursively specified on the structure of worlds as follows (writing "ab" for "a$\circ$b," and "(a, A)" rather than "CI(a, A)"):

(i) For formulas A in W, and all $B \in$L,

$$(A, B) = \begin{cases} +\mathbf{1} \text{ if } \vdash_S A \rightarrow B \\ \mathbf{0} \text{ if } \dashv_S A \rightarrow B \text{ and } A = B, \\ -\mathbf{1} \text{ otherwise.} \end{cases}$$

(ii) For complex worlds ab, and all $B \in L$,

$$(ab, B) = (\exists Y)((a, Y \rightarrow B) \wedge (b, Y)).$$

A valuation cv is now defined by restricting CI appropriately to p. The structure CM = $\langle W, \circ, \triangleright, cv \rangle$ that results is called the *canonical model*. It is evident from the definitions that CM is an S-model provided only that conditions (T) and (H)—hence also (Hp)—hold for CI. Assuming this for the moment, a completeness argument is available: Suppose $\dashv_S A \rightarrow B$. Then $(A, B) \leq \mathbf{0}$ by definition of CI. Also $(A, A) \geq \mathbf{0}$; so $(A, A) < (A, B)$ fails. Thus A does not strictly entail B in CM; so $A \rightarrow B$ is not strictly valid.

That (H) and (T) hold in CM is straightforward. The following preliminary lemma is useful: $\vdash_S B \rightarrow C$ implies (a, B) < (a, C) for all a$\in$W. This lemma is proved by induction on the complexity of a$\in$W. For the proof of (H), that a $\triangleright$ b implies (b, A) < (a, A), for all a, b $\in$ W, $A \in$L, use induction on the length of proof of a $\triangleright$ b (by definition, a $\triangleright$ b holds only when it has a proof from the BB′-conditions). The proof that (T) holds in CM, i.e., that $(a, A \to B) = (\forall x)((x, A) \Rightarrow (ax, B))$, for all $a \in W$, $A, B \in L$, is immediate from the inequalities:

$$\begin{array}{ll} \text{(i)} & (a, A \to B) \leq (\forall x)((x, A) \Rightarrow (ax, B)) \\ \text{(ii)} & (\forall x)((x, A) \Rightarrow (ax, B)) \leq (aA, B) \\ \text{(iii)} & (aA, B) \leq (a, A \to B), \end{array}$$

which are proved from definitions and the preliminary lemma.

The inequalities (i)–(iii) are also used to prove an important result:

Lemma. $(a, A \to B) = (aA, B)$, for all $a \in W$, $A, B \in L$.

This lemma is particularly useful when taken together with the syntactic fact that every formula A in the language L can be expressed in terms of its nested antecedents, i.e., in the form $A_1 \to . \ldots \to . A_n \to p$, for some formulas $A_1, \ldots, A_n$, $n \geq 0$. Then n iterations of the lemma yield as a corollary (assuming association to the left in writing worlds):

Coding fact. $(a, A_1 \to . \ldots \to . A_n \to p) = (aA_1 \ldots A_n, p)$ for all $a \in W$ and formulas $A_1, \ldots, A_n$.

Thus the canonical valuation directly codes the value assigned by CI to complex formulas.

Condition (T) may also be modified to apply to the nested antecedent structure of formulas. Where A is as before, then (after some algebraic manipulation),

$$(a, A) = (\forall x_1 \ldots \forall x_n)((x_1, A_1) \wedge \ldots \wedge (x_n, A_n) \Rightarrow (ax_1 \ldots x_n, p))$$

holds in the canonical model. A similar transformation applies to other S-models.

§66.3. Reduced valuations. We wish to construct models over CMS valuation functions other than the valuation cv used for completeness. Such a valuation v is said to be *reducing* provided that $v(a, p) \leq (a, p)$ for all $a \in W$. In certain circumstances the value of complex formulas is also reduced in the resulting S-model:

Lemma (Reduced Valuations fact). Suppose v is a reducing valuation, and I the associated interpretation. Then $I(a, A) \leq (a, A)$ for all $a \in W$ and for each $A = A_1 \to . \ldots \to . A_n \to p$ such that $I(A_i, A_i) \geq \mathit{0}$, $1 \leq i \leq n$.

Proof. $I(a, A) \leq (I(A_1, A_1) \wedge \ldots \wedge I(A_n, A_n) \Rightarrow I(aA_1 \ldots A_n, p))$, by the generalized form of (T); so $I(a, A) \leq I(aA_1 \ldots A_n, p)$, given the hypotheses $I(A_i, A_i) \geq \mathit{0}$. Then use the fact that v is reducing, and the Coding fact.

For each formula $A = A_1 \to . \ldots \to . A_n \to p$, a reducing valuation v_A (the *A-valuation*) is defined over the CMS by: (i) $v_A(AA_1 \ldots A_n, p) = \mathit{0}$, (ii) $v_A(b, p) = -\mathit{1}$ for each $b \in W$ such that $AA_1 \ldots A_n \rhd b$, and (iii) v_A copies the canonical valuation otherwise. That v_A is well-defined from these specifications is not quite immediate—it is required that $AA_1 \ldots A_n \rhd AA_1 \ldots A_n$ does *not* hold. But this is a consequence of some facts about $\rhd$ set out in the next section.

It is, however, easy to check that v_A satisfies (H), so that $\langle W, \circ, \rhd, v_A \rangle$ for each formula A, is an S-model. Observing that, by definition, $(AA_1 \ldots A_n, p) = (A, A)$ is always $\geq \mathit{0}$, it is also easy to check that v_A is reducing. The worlds where v_A possibly differs from cv, viz. $AA_1 \ldots A_n$ and its BB′-relatives, are called *altered* worlds.

§66.4. The guarded merge theorem. In this section some results are set out concerning the relation $\rhd$, especially in connection with the altered worlds of the models just defined.

Worlds like $AA_1 \ldots A_n$ are said to be *lefted* (the term is borrowed from Powers 1976), meaning that they are uniformly constructed by association to the left. More precisely: all formulas are lefted, and, where $a \in W$ is lefted, so is aA for any formula A. Lefted worlds are treated here interchangeably with sequences of formulas.

It was stated above that $AA_1 \ldots A_n \rhd AA_1 \ldots A_n$ does not hold. This is an instance of the general point that $b \in W$ in $a \rhd b$ cannot be lefted. Proof is by induction on the length of proof of $a \rhd b$ from the BB′-conditions.

Thus $\rhd$ is not reflexive. (In fact, $\rhd$ can be shown to be irreflexive, but this goes beyond our present concerns.) Let $\unrhd$ be the reflexive relation generated by $\rhd$. The BB′-conditions continue to hold for $\unrhd$.

We now associate with each world $a \in W$ a set $[a]$ of lefted worlds, by means of the notion of a *guarded merge* of two lefted worlds. In §7.3 a Gentzen formulation of $\mathbf{T}_{\to}-\mathbf{W}$ (among other logics) is given which employs the notion of a merge of two sequences of formulas. The basic idea is to interweave the two sequences in such a way as to preserve the order of the component sequences in the resultant. In brief, no permutation in the component sequences is allowed. For a precise definition, see §7.3.

The idea of merging two sequences extends naturally to merging two lefted worlds. We add one further restriction, and say that a lefted world $b = B_1 \ldots B_n$ is a *guarded merge* of $c = C_1 \ldots C_k$ and $d = D_1 \ldots D_m$ provided first that b is a merge of c and d, and secondly that $B_n = D_m$, i.e., that the last formula in d acts as a kind of buffer on the merge. The idea is suggested by the role of the "turnstile guard" in the merge formulations of §7.3.

Now $[a]$ can be defined: For a formula B, $[B] = \{B\}$, and, for $ab \in W$, let $[ab]$ be the set of lefted worlds w where w is a guarded merge of some

$\mathrm{u} \in [\mathrm{a}]$ and $\mathrm{v} \in [\mathrm{b}]$. The following is then proved by induction on the length of proof of $\mathrm{a} \trianglerighteq \mathrm{b}$:

LEMMA. $\mathrm{a} \trianglerighteq \mathrm{b}$ implies $[\mathrm{a}] \subseteq [\mathrm{b}]$, for $\mathrm{a}, \mathrm{b} \in \mathrm{W}$.

This lemma cannot be strengthened to strict inclusion, even for $\mathrm{a} \triangleright \mathrm{b}$. Nor does the converse hold (consider $[AAA]$ and $[A(AA)]$). However, a specialized form of the lemma and its converse can be proved.

GUARDED MERGE THEOREM. For $A_1 \ldots A_n$, $\mathrm{b} \in \mathrm{W}$, $A_1 \ldots A_n \trianglerighteq \mathrm{b}$ if and only if $A_1 \ldots A_n \in [\mathrm{b}]$.

PROOF. Noting that $[A_1 \ldots A_n] = \{A_1 \ldots A_n\}$, the theorem from left to right is a consequence of the preceding lemma.

The proof from right to left is by induction on the length of b. Where b is a formula, say B, obviously $n = 1$ and $A_1 = B$. For the inductive case, suppose that $\mathrm{b} = \mathrm{cd}$. Now $A_1 \ldots A_n \in [\mathrm{cd}]$ implies that $A_1 \ldots A_n$ is a merge of lefted worlds $C_1 \ldots C_k \in [\mathrm{c}]$ and $D_1 \ldots D_m \in [\mathrm{d}]$, and that $A_n = D_m$. On inductive hypothesis, $C_1 \ldots C_k \trianglerighteq \mathrm{c}$ and $D_1 \ldots D_m \trianglerighteq \mathrm{d}$, from which $C_1 \ldots C_k(D_1 \ldots D_m) \trianglerighteq \mathrm{cd}$ follows by the BB′-conditions. Completion of the proof requires only $A_1 \ldots A_n \trianglerighteq C_1 \ldots C_k(D_1 \ldots D_m)$, which is proved by a subsidiary induction on $n = k+m$. The cases $m = 1$, which includes the base case $n = 2$, are immediate. For $m \geq 2$, consider the sequence $A_1 \ldots A_{n-1}$ which is a merge of $C_1 \ldots C_k$ and $D_1 \ldots D_{m-1}$. Here A_{n-1} must be either C_k or D_{m-1}; so either (i) $A_1 \ldots A_n \in [D_1 \ldots D_{m-1}(C_1 \ldots C_k)]$ or (ii) $A_1 \ldots A_n \in [C_1 \ldots C_k(D_1 \ldots D_{m-1})]$. In case (i), argue as follows:

1	$A_1 \ldots A_{n-1} \trianglerighteq (D_1 \ldots D_{m-1})(C_1 \ldots C_k)$	ex. ind. hyp.
2	$A_1 \ldots A_{n-1}A_n \trianglerighteq (D_1 \ldots D_{m-1})(C_1 \ldots C_k)D_m$	1, (ν)
3	$(D_1 \ldots D_{m-1})(C_1 \ldots C_k)D_m \trianglerighteq (C_1 \ldots C_k)(D_1 \ldots D_{m-1}D_m)$	(B′)
4	$A_1 \ldots A_n \trianglerighteq (C_1 \ldots C_k)(D_1 \ldots D_m)$	2, 3, (τ)

Case (ii) is similar, using (B) instead of (B′).

This theorem is applied in the proof of Powers's conjecture via two corollaries, which are stated without proof:

MEDIATING COROLLARY. Suppose $A_1 \ldots A_n \trianglerighteq \mathrm{b}_1 \ldots \mathrm{b}_m$ for some $A_1 \ldots A_n, \mathrm{b}_1, \ldots, \mathrm{b}_m \in \mathrm{W}$. Then there are lefted worlds $\mathrm{b}_1^*, \ldots, \mathrm{b}_m^*$ such that

(i) $\mathrm{b}_i^* \in [\mathrm{b}_i]$ $(1 \leq i \leq m)$
(ii) b_i^* is a subsequence of $A_1 \ldots A_n$ $(1 \leq i \leq m)$
(iii) $A_1 \ldots A_n \trianglerighteq \mathrm{b}_1^* \ldots \mathrm{b}_m^*$
(iv) $\mathrm{b}_1^* \ldots \mathrm{b}_m^* \trianglerighteq \mathrm{b}_1 \ldots \mathrm{b}_m$.

ALPHABETICAL COROLLARY. Suppose $A_1 \dots A_n \trianglerighteq \mathrm{b}_1 \dots \mathrm{b}_m$, and that $\mathrm{b}_i = B$ and $\mathrm{b}_j = C$ for some formulas B, C, where $1 \leq i \leq j \leq m$ and $i \neq j$. Then BC is a subsequence of $A_1 \dots A_n$.

§66.5. Powers's conjecture. The main theorem follows:

THEOREM. $\dashv_{\mathrm{S}} A \to A$ for every formula A.

PROOF. By induction on the length of A. *Base case.* It is well known (see, e.g., §8.11 and §8.12) that $\dashv_{\mathrm{S}} p \to p$. *Inductive case.* Let $A = A_1 \to . \dots \to . A_n \to p$, $n \geq 1$, and suppose on inductive hypothesis that $\dashv_{\mathrm{S}} B \to B$ for each formula B shorter than A, i.e., that $(B, B) = \boldsymbol{0}$.

Consider the S-model $\mathrm{M} = \langle \mathrm{W}, \circ, \rhd, \mathrm{v}_A \rangle$ constructed from the A-valuation. Let $'$ be the associated interpretation. The object of the proof is to show $(A, A)' = \boldsymbol{0}$, which suffices for $\dashv_{\mathrm{S}} A \to A$. The first step is to establish that $(A_i, A_i)' = \boldsymbol{0}$ for the nested antecedents A_i of A. This is a consequence of the next lemma, the proof of which will occupy us until further notice.

PRESERVATION LEMMA. $(B, B)' = \boldsymbol{0}$ for every proper subformula B of A.

PROOF. The lemma is proved by induction on the length of B, and under the hypotheses of the main theorem. *Base case.* $p \in \mathrm{W}$ is not an altered world, because (i) $n \geq 1$ and hence $p \neq AA_1 \dots A_n$, and (ii) p is lefted, hence $AA_1 \dots A_n \rhd p$ cannot occur. Thus $(p, p)' = (p, p) = \boldsymbol{0}$. *Inductive case.* Let $B = B_1 \to . \dots \to . B_m \to p$, $m \geq 1$, be a proper subformula of A. Since $(B_i, B_i)' = \boldsymbol{0}$, for $1 \leq i \leq m$, on inductive hypothesis, the Reduced Valuations fact applies to B; so $(B, B)' \leq (B, B)$. Now $(B, B) = \boldsymbol{0}$ on inductive hypothesis of the main theorem; so the proof of the lemma may be completed by showing that $(B, B)' \neq \boldsymbol{-1}$.

Suppose for reductio that $(B, B)' = \boldsymbol{-1}$. Then there are $\mathrm{b}_1, \dots, \mathrm{b}_m \in \mathrm{W}$ such that

1. $((\mathrm{b}_1, B_1)' \wedge \dots \wedge (\mathrm{b}_m, B_m)' \Rightarrow (B\mathrm{b}_1 \dots \mathrm{b}_m, p)') = \boldsymbol{-1}$.

Consequently, by $\wedge$ and $\Rightarrow$ properties,

2. $(\mathrm{b}_i, B_i)' \geq \boldsymbol{0}$, for $1 \leq i \leq m$.

The antecedents B_i of B, $1 \leq i \leq m$, also satisfy the Reduced Valuations fact; so

3. $(\mathrm{b}_i, B_i)' \leq (\mathrm{b}_i, B_i)$, for $1 \leq i \leq m$.

We observe that, because $(B, B) = \boldsymbol{0}$,

4. $((\mathrm{b}_1, B_1) \wedge \dots \wedge (\mathrm{b}_m, B_m) \Rightarrow (B\mathrm{b}_1 \dots \mathrm{b}_m, p)) \geq \boldsymbol{0}$.

There are now two cases. First suppose that $B\mathrm{b}_1 \ldots \mathrm{b}_m$ is not an altered world, so that $(Bb_1 \ldots \mathrm{b}_m, p)' = (B\mathrm{b}_1 \ldots \mathrm{b}_m, p)$. Together with 3 and 4, this implies that

$$((\mathrm{b}_1, B_1)' \wedge \ldots \wedge (\mathrm{b}_m, B_m)' \Rightarrow (B\mathrm{b}_1 \ldots \mathrm{b}_m, p)') \geq \mathit{0},$$

which contradicts 1.

The case remains in which $B\mathrm{b}_1 \ldots \mathrm{b}_m$ is an altered world, that is, $AA_1 \ldots A_n \trianglerighteq B\mathrm{b}_1 \ldots \mathrm{b}_m$. Applying the Mediating corollary of the Guarded Merge theorem, there are lefted worlds B^*, $\mathrm{b}_1^*, \ldots, \mathrm{b}_m^*$, each a subsequence of $AA_1 \ldots A_n$, such that

5. $\mathrm{b}_i^* \trianglerighteq b_i$, for $1 \leq i \leq m$,

and $AA_1 \ldots A_n \trianglerighteq \mathrm{B}^*\mathrm{b}_1^* \ldots \mathrm{b}_m^*$. Clearly B^* is B. Given that $A \neq B$, it follows that

6. $AA_1 \ldots A_n \vartriangleright B\mathrm{b}_1^* \ldots \mathrm{b}_m^*$.

From 2 and 3 we have $(\mathrm{b}_i, B_i) \geq \mathit{0}$, $1 \leq i \leq m$; then, using 5 and (H) if necessary, we obtain

7. $(\mathrm{b}_i^*, B_i) \geq \mathit{0}$, $1 \leq i \leq m$.

Now A occurs in $AA_1 \ldots A_n$ just once, consequently also just once in $B\mathrm{b}_1^* \ldots \mathrm{b}_m^*$. Since $A \neq B$, A occurs in just one b_i^*, $1 \leq i \leq m$, say b_j^*. But A cannot occur alone in b_j^*, for then 6 contradicts the Alphabetical corollary of the Guarded Merge theorem. Thus b_j^* has the form $AC_1 \ldots C_k$, $k \geq 1$, and from 7 and the Coding fact we conclude

8. $(A, C_1 \rightarrow. \ldots \rightarrow. C_k \rightarrow B_j) \geq \mathit{0}$.

Now the formula $C_1 \rightarrow. \ldots \rightarrow. C_k \rightarrow B_j$, call it C, is easily seen to be shorter than A. Thus it follows from 8 and the definition of CI that $(A, C) = +\mathit{1}$, hence that $\vdash_{\mathbf{S}} A \rightarrow C$.

We now seek to prove $\vdash_{\mathbf{S}} C \rightarrow A$ also, under the present hypotheses, so that $\vdash_{\mathbf{S}} C \rightarrow C$ also, for C shorter than A. This will contradict the inductive hypothesis of the main theorem, and complete the reductio argument that $(B, B) \neq -\mathit{1}$.

Statement 6 holds in virtue of some proof from the BB′-conditions. Consider the result of replacing A by C throughout this proof. Such a substitution evidently preserves $\vartriangleright$, so that

9. $CA_1 \ldots A_n \vartriangleright B\mathrm{b}_1^* \ldots \mathrm{b}_j^*[C/A] \ldots \mathrm{b}_m^*$.

We compute the canonical evaluation of p at $B\mathrm{b}_1^* \ldots \mathrm{b}_j^*[C/A] \ldots \mathrm{b}_m^*$. Applying definitions,

$$(B\mathrm{b}_1^* \ldots \mathrm{b}_j^*[C/A] \ldots \mathrm{b}_m^*, p) = (\exists Y_1 \ldots \exists Y_m)((B, Y_1 \rightarrow. \ldots \rightarrow. Y_m \rightarrow p) \wedge$$
$$(\mathrm{b}_1^*, Y_1) \wedge \ldots \wedge (\mathrm{b}_j^*[C/A], Y_j) \wedge \ldots \wedge (\mathrm{b}_m^*, Y_m)),$$

which is ≥ the value of

$$(B, B_1 \to . \ldots . \to . B_m \to p) \wedge (\mathrm{b}_1^*, B_1) \wedge \ldots \wedge (\mathrm{b}_j^*[C/A], B_j) \wedge \ldots \wedge (\mathrm{b}_m^*, \mathrm{B}_m).$$

Apart from $(\mathrm{b}_j^*[C/A], B_j)$, each of these conjuncts has been proved $\geq \boldsymbol{0}$ already, and $(\mathrm{b}_j^*[C/A], B_j) = (CC_1 \ldots C_k, B_j) = (C, C) = \boldsymbol{0}$ on inductive hypothesis of the main theorem. Thus $(B\mathrm{b}_1^* \ldots \mathrm{b}_j^*[C/A] \ldots \mathrm{b}_m^*, p) \geq \boldsymbol{0}$; so $(CA_1 \ldots A_n, p) = (C, A) = +\boldsymbol{1}$ by 9 and (H). Then $\vdash_{\mathbf{S}} C \to A$, which contradicts our hypotheses, completing the proof of the Preservation lemma.

The proof in the main theorem that $(A, A)' = \boldsymbol{0}$ can now be continued. We have

$$(A, A)' \leq ((A_1, A_1)' \wedge \ldots \wedge (A_n, A_n)' \Rightarrow (AA_1 \ldots A_n, p)').$$

The Preservation lemma gave as a first step that $(A_i, A_i)' = \boldsymbol{0}$; also $(AA_1 \ldots A_n, p)' = \boldsymbol{0}$ by construction. Thus $(A, A)' \leq \boldsymbol{0}$.

Suppose now for reductio that $(A, A)' = -\boldsymbol{1}$. Then there are $\mathrm{a}_1, \ldots, \mathrm{a}_n \in \mathrm{W}$ such that

$$((\mathrm{a}_1, A_1)' \wedge \ldots \wedge (\mathrm{a}_n, \mathrm{A}_n)' \Rightarrow (A\mathrm{a}_1 \ldots \mathrm{a}_n, \mathrm{p})') = -\boldsymbol{1}.$$

When $A\mathrm{a}_1 \ldots \mathrm{a}_n$ is not an altered world, it can be shown that this reductio assumption is untenable, by arguments similar to those used in the analogous case of the Preservation lemma. But also, if $A\mathrm{a}_1 \ldots \mathrm{a}_n$ is altered, then, by elementary properties of $\triangleright$, it follows that $A\mathrm{a}_1 \ldots \mathrm{a}_n$ is $AA_1 \ldots A_n$, i.e., $\mathrm{a}_i = A_i$, for $1 \leq i \leq m$, again contradicting the reductio assumption.

Thus $(A, A)' = \boldsymbol{0}$, which completes the proof of Powers's conjecture.

§66.6. Significance of all this. We have shown that $\mathbf{T}_{\to}-\mathbf{W}$ is minimal in the intended sense. The method of solution prompts this further observation.

The axiom of identity has turned out to be "both true and false," according to a familiar, albeit fairly loose, interpretation of the three-valued semantics. This was sufficient to prove the conjecture. But at another (deeper?) level it may be that this is As It Should Be. The law of identity, after all, is not only the "archetypal form of inference" (§1.3), but the "archetypal fallacy" as well, namely the fallacy of circular argument, of *begging the question.*

Thus, system **S** does not beg the question, and $\mathbf{T}_{\to}-\mathbf{W}$ takes this fallacy seriously enough that, though $A \to A$ is an axiom, this principle can never be used to prove anything other than more instances of the principle. Both systems conform to Aristotle's idea (*Analytica priora* 24b18–20 and *Topica* 100a25–27) that "reasoning is argument in which, certain things being laid down, something *other than these* necessarily comes about through them."

§67. Decision procedures for contractionless relevance logics (by Steve Giambrone). Here we make good on the promise of §63.3 to decide the positive fragments of the contraction-free subsystems of **R** and **T**. [Note by principal authors: the methods here give a decision procedure for theorems, but do not solve what §65 calls the "deducibility problem," which is there shown to be unsolvable for these systems—see §63.3.]

§67.1. Introduction. The reader will note from §61 that relevant consecution calculuses with the full power of & and $\vee$ are of a greater order of complexity than those for classical and intuitionistic logic, for example. Where the latter get by with simple sequences of formulas, the former require two different types of sequences—intensional sequences and extensional sequences—which must be allowed to be nested within one another to any arbitrary degree. Naturally this level of complexity makes such systems harder to use.

This section formulates consecution calculuses for $\mathbf{T}^{\circ}_{+}-\mathbf{W}$ and $\mathbf{R}^{\circ}_{+}-\mathbf{W}$ and for the first time puts such complex calculuses to one of their prime functions: answering the decision question for those systems. ($\mathbf{T}_{+}-\mathbf{W}$, is $\mathbf{T}_{+}$ of §R2 without A4—the contraction, or W axiom. Similarly for $\mathbf{R}_{+}-\mathbf{W}$ and $\mathbf{E}_{+}-\mathbf{W}$. These systems can be conservatively extended to include $\circ$ or $\boldsymbol{t}$ as stipulated for other systems in §R2.) We give here only the bare bones of the required arguments. For more detail see Giambrone 1985, and for most detail see Giambrone 1983.

The essence of the original argument for decidability in Gentzen 1934 lies in getting control over the length or complexity of sequences and, hence, over the number of consecutions that can occur in a proof-search tree for a given formula. Our method is analogous. However, since these relevant consecution calculuses have two types of sequences nested within each other, we must in a sense get simultaneous control over both the extensional and the intensional complexity of consecutions.

We begin by giving calculuses (containing $\boldsymbol{t}$) which are convenient for proving the Elimination and Equivalence theorems of §61. The formulations are then progressively refined (including getting rid of $\boldsymbol{t}$) into calculuses suitable for the decidability argument. For the sake of readability we use Slaney's conventions and write "$\mathbf{TW}_{+}$" for "$\mathbf{T}_{+}-\mathbf{W}$", "$\mathbf{RW}^{\circ}_{+}$" for "$\mathbf{R}^{\circ}_{+}-\mathbf{W}$", etc.

§67.2. $\mathbf{LTW}^{\circ t}_{+}$ and $\mathbf{LRW}^{\circ t}_{+}$. Notation and terminology are brought forward from §61.2. However, antecedents of the form $\mathsf{V}(\alpha)$ are disallowed (consecutions of these systems are said to be *denuded*), which forces a few more changes in the consecution calculus of that section, changes that would be wanted for the sake of the decidability argument in any event. (Of course, "$\Box$" is not in the language now.)

So $\mathbf{LRW}^{\circ t}_{+}$ is formulated by modifying $\mathbf{LR}^{\Box\circ t}_{+}$ as follows. (The reader is reminded of the "V" convention of §61.2; (WI$\vdash$) is half of (W$\mathsf{V}\vdash$).)

(1) Drop (WI$\vdash$).

(2) Drop the $\Box$ rules and the V_1 rules.

(3) Change (WE⊢) to

$$\frac{\Gamma_1\mathrm{E}(\alpha, \alpha)\Gamma_2 \vdash A}{\Gamma_1\alpha\Gamma_2 \vdash A}$$

(4) Replace the conjunction rules by

$$\frac{\Gamma_1 A\Gamma_2 \vdash C}{\Gamma_1 A\&B\Gamma_2 \vdash C} \qquad \frac{\Gamma_1 B\Gamma_2 \vdash C}{\Gamma_1 A\&B\Gamma_2 \vdash C}(\&\vdash)$$

$$\frac{\alpha \vdash A \quad \alpha \vdash B}{\alpha \vdash A\&B}(\vdash\&)$$

(5) Add

$$\frac{\Gamma_1\mathrm{I}(\boldsymbol{t}, \alpha)\Gamma_2 \vdash C}{\Gamma_1\alpha\Gamma_2 \vdash C}(\boldsymbol{t}-\vdash)$$

LTW$^{\circ\boldsymbol{t}}_{+}$ is formulated by further dropping (CI⊢) and the I_2 rules, and adding

$$\frac{\Gamma_1\mathrm{I}(\alpha, \mathrm{I}(\beta, \gamma))\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}(\mathrm{I}(\alpha, \beta), \gamma)\Gamma_2 \vdash C}(\mathrm{BI}\vdash) \qquad \frac{\Gamma_1\mathrm{I}(\alpha, \mathrm{I}(\beta, \gamma))\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}(\mathrm{I}(\beta, \alpha), \gamma)\Gamma_2 \vdash C}(\mathrm{B'I}\vdash)$$

The Elimination theorem can then be shown for these two systems as in §61, with the simplification of setting $n_0 = \mathrm{p}_0 = 1$. Then appropriate Equivalence theorems can be shown, as there. But note that ($\boldsymbol{t}-\vdash$) is used to show (→E) admissible in both systems and to show importation (∘E of §R2) admissible in LTW$^{\circ\boldsymbol{t}}_{+}$.

§67.3. Vanishing *t*. The rule ($\boldsymbol{t}-\vdash$) presents a problem for the coming decidability argument in that it is not degree preserving (see below). The easiest solution is to rid the systems of ***t*** and all its works. And the simplest method for getting rid of ***t*** is first to leave it in and make a few modifications to the original formulations (including being empty on the left), and then show that we no longer need ***t***.

So we keep the definition of structures as before. There will be *no* null or empty structure. We simply allow sequents to be entities either of the form $\alpha \vdash A$ or of the form $\vdash A$. To do otherwise is to introduce the ridiculous question of whether or not there are structures of the form $\mathrm{E}(\alpha_1, \ldots, \alpha_n)$, for instance, where each α_i is empty. Of course, the adopted policy is not without its own headache. Technically, whenever we want to say something about sequents in general we must speak double, once about sequents of the form $\alpha \vdash A$ and once about sequents of the form $\vdash A$.

Of course, when one has a headache, the sensible thing to do is to take aspirin. Our aspirin will be to use double-speak rather than speak double. We now allow structural variables to be existentialist variables; that is, they range over structures *and* the dreaded Nothingness. Otherwise, notation remains the same.

We must still occasionally restrict structural variables to ranging only over structures. But with a bit of good will (and common sense) on the part of the reader and a few conventions, this is not so cumbersome. In the first place, we insist that structural variables never range over Nothingness when used to represent an immediate constituent of an E-sequence. And likewise for structural variables that occur in the statement of structural rules.

Now let the L′-systems come from the L-systems by

I. adding $\vdash \boldsymbol{t}$ as an axiom;
II. leaving the structural rules as they are (but note the conventions on Nothingness);
III. for $\mathrm{L}'\mathbf{TW}^{\circ\boldsymbol{t}}_{+}$, insisting that (1) the left premiss of $(\rightarrow\vdash)$ is never empty on the left, and (2) the right premiss of $(\vdash\circ)$ is empty on the left only if the left premiss is;
IV. for $\mathrm{L}'\mathbf{RW}^{\circ\boldsymbol{t}}_{+}$, replacing $(\boldsymbol{t}\vdash)$ by the more general

$$\frac{\Gamma_1\alpha\Gamma_2 \vdash C}{\Gamma_1(\alpha;\beta)\Gamma_2 \vdash C} \qquad \frac{\Gamma_1\alpha\Gamma_2 \vdash C}{\Gamma_1(\beta;\alpha)\Gamma_2 \vdash C}\,(\boldsymbol{t}\#\vdash)$$

where β is a ***t***-structure, and a ***t***-structure of course is a structure in which the only formula that occurs is ***t***.

It is easy to show that $\mathrm{L}'\mathbf{TW}^{\circ\boldsymbol{t}}_{+}$ $\{\mathrm{L}'\mathbf{RW}^{\circ\boldsymbol{t}}_{+}\}$ is contained on translation in $\mathbf{TW}^{\circ\boldsymbol{t}}_{+}$ $\{\mathbf{RW}^{\circ\boldsymbol{t}}_{+}\}$, and that the L′-systems are supersystems of the L-systems. So, using the Equivalence theorems and $(\boldsymbol{t}-\vdash)$, $(\boldsymbol{t}\vdash)$, and $(\boldsymbol{t}\#\vdash)$, one can easily show that $\vdash A$ is provable in $\mathrm{L}'\mathbf{TW}^{\circ\boldsymbol{t}}_{+}$ $\{\mathrm{L}'\mathbf{RW}^{\circ\boldsymbol{t}}_{+}\}$ iff A is a theorem of $\mathbf{TW}^{\circ}_{+}$ $\{\mathbf{RW}^{\circ}_{+}\}$. However, there is as yet no guarantee that there is a ***t***-free proof of all provable ***t***-free formulas. To rectify this situation, one first shows

VANISHING-***t*** LEMMA. Let α be a ***t***-antecedent and let Σ be a consecution satisfying the following conditions:

(1) the consequent of Σ is ***t***-free;
(2) ***t*** is not a *proper* subformula of any formula occurring in the antecedent of Σ;
(3) Σ is not of the form $\Gamma_1\mathrm{E}(\beta_1,\ldots,\gamma,\ldots,\beta_n)\Gamma_2 \vdash C$, with γ a ***t***-antecedent and some β_i not a ***t***-antecedent.

If $\Sigma = \Gamma_1\mathrm{I}(\alpha,\beta)\Gamma_2 \vdash C$ is provable with weight n, then $\Gamma_1\beta\Gamma_2 \vdash C$ is provable with weight $\leq n$ where β is possibly empty if Γ_1 and Γ_2 are.

One can then show that a ***t***-free consecution is derivable iff it has a ***t***-free derivation. So, given known conservative-extension results, A is a theorem of $\mathbf{TW}^{\circ}_{+}$ $\{\mathbf{RW}^{\circ}_{+}\}$ iff there is a ***t***-free proof of $\vdash A$ in $\mathrm{L}'\mathbf{TW}^{\circ\boldsymbol{t}}$ $\{\mathrm{L}'\mathbf{RW}^{\circ\boldsymbol{t}}_{+}\}$.

So one can drop ***t*** from the language and let $\mathbf{LTW}^{\circ}_{+}$ and $\mathbf{LRW}^{\circ}_{+}$ come from the corresponding L′-systems above by dropping the ***t***-axioms and ***t***-rules. Obviously, *t*-FREE EQUIVALENCE THEOREM. $\vdash A$ is provable in $\mathbf{LTW}_+^\circ$ $\{\mathbf{LRW}_+^\circ\}$ iff A is a theorem of $\mathbf{TW}_+^\circ$ $\{\mathbf{RW}_+^\circ\}$.

§67.4. Denesting. A problem yet remains for decidability. Even if E-sequences were limited to reduced form, as they soon will be, there are still an infinite number of distinct E-sequences that can be built up even from a single formula, e.g.,

$$\mathrm{E}(p, p), \mathrm{E}(p, \mathrm{E}(p, p)), \mathrm{E}(p, \mathrm{E}(p, \mathrm{E}(p, p))), \ldots$$

This problem of "nested" E-sequences can be circumvented by adding the following rules to $\mathbf{LTW}_+^\circ$ $\{\mathbf{LRW}_+^\circ\}$:

$$\frac{\Gamma_1\mathrm{E}(\alpha_1, \ldots, \alpha_n)\Gamma_2 \vdash C}{\Gamma_1\mathrm{E}(\alpha_1, \ldots, \alpha_n, \beta)\Gamma_2 \vdash C}\,(\mathrm{K'E}\vdash)$$

$$\frac{\Gamma_1\mathrm{E}(\alpha_1, \ldots, \alpha_n, \beta, \beta)\Gamma_2 \vdash C}{\Gamma_1\mathrm{E}(\alpha_1, \ldots, \alpha_n, \beta)\Gamma_2 \vdash C}\; n \geq 1\; (\mathrm{W'E}\vdash)$$

Since these rules are obviously admissible in the original systems, the *t*-free Equivalence theorem still holds, and we now simply take those systems to be formulated with the additional rules as primitive.

Next, let us say that an antecedent is *denested* just in case it has no sub-antecedent (including itself) of the form $\mathrm{E}(\alpha_1, \ldots, \mathrm{E}(\beta_1, \ldots, \beta_m), \ldots, \alpha_n)$. We speak of denested consecutions and denested proofs in the obvious way. Then, for any antecedent γ, define the *denestation of* γ ($\mathrm{dn}(\gamma)$) as follows:

(1) $\mathrm{dn}(A) = A$, for any formula A;
(2) $\mathrm{dn}(\mathrm{I}(\alpha, \beta)) = \mathrm{I}(\mathrm{dn}(\alpha), \mathrm{dn}(\beta))$;
(3) $\mathrm{dn}(\mathrm{E}(\alpha_1, \ldots, \mathrm{E}(\beta_1, \ldots, \beta_m), \ldots, \alpha_n)) = \mathrm{dn}(\mathrm{E}\alpha_1, \ldots, \beta_1, \ldots, \beta_m, \ldots, \alpha_n))$;
(4) $\mathrm{dn}(\mathrm{E}(\alpha_1, \ldots, \alpha_n)) = \mathrm{E}(\mathrm{dn}(\alpha_1), \ldots, \mathrm{dn}(\alpha_n))$, where no α_i is an E-sequence.

(E_2elim) and (E_2int) of §61.2 guarantee that a consecution is provable iff its denestation is. And the new rules along with their original companions allow one to give a denested proof of any provable denested consecution. So

DENESTATION THEOREM. A consecution is provable in $\mathbf{LTW}_+^\circ$ $\{\mathbf{LRW}_+^\circ\}$ iff its denestation has a denested proof.

§67.5. Reduction. The Denestation theorem will allow E-sequences to be reduced more or less as in Gentzen 1934. So let us say that an antecedent is *reduced* just in case no antecedent occurs more than twice as an immediate constituent of any E-sequence occurring in it. Then an antecedent is

E-reduced iff it is denested and reduced. And extend this terminology to consecutions and to proofs in the obvious way.

Next, define an antecedent as *superreduced* just in case it contains no E-sequence with two distinct immediate constituents that are occurrences of the same antecedent. Again, the definition is extended to consecutions in the obvious way. Then define sr(γ), the *superreduct* of a *denested* antecedent γ, as follows:

(1) $\mathrm{sr}(A) = A$
(2) $\mathrm{sr}(\mathrm{I}(\alpha, \beta)) = \mathrm{I}(\mathrm{sr}(\alpha), \mathrm{sr}(\beta))$
(3) $\mathrm{sr}(\mathrm{E}(\alpha_1, \ldots, \alpha_n)) = \mathrm{sr}(\alpha_1)$ if $\alpha_i = \alpha_1$ for all $1 \leq i \leq n$; otherwise $\mathrm{sr}(\mathrm{E}(\alpha_1, \ldots, \alpha_n)) = \mathrm{E}(\beta_1, \ldots, \beta_m)$, where $\mathrm{E}(\beta_1, \ldots, \beta_m)$ is as follows: For each α_i, let k_i be the number of occurrences of $\mathrm{sr}(\alpha_i)$ in $\mathrm{E}(\mathrm{sr}(\alpha_1), \ldots, \mathrm{sr}(\alpha_n))$. Then $\mathrm{E}(\beta_1, \ldots, \beta_m)$ is the result of deleting the first $\mathrm{k}_i - 1$ occurrences of $\mathrm{sr}(\alpha_i)$ from $\mathrm{E}(\mathrm{sr}(\alpha_1), \ldots, \mathrm{sr}(\alpha_n))$.

Naturally, for any formula A and denested antecedent α, $\mathrm{sr}(\alpha \vdash A) = \mathrm{sr}(\alpha) \vdash \mathrm{sr}(A) = \mathrm{sr}(\alpha) \vdash A$; and $\mathrm{sr}(\vdash A) = \vdash \mathrm{sr}(A) = \vdash A$.

Extensional permutation, contraction, and weakening rules provide that a denested consecution is provable iff its superreduct is. But the real crux of the matter is the

REDUCTION THEOREM. A denested consecution is provable iff its superreduct has an E-reduced derivation.

§67.6. Degree and decidability. This theorem gives the desired control over the extensional complexity of consecutions that can occur in the proof-search tree for a given consecution, provided that control can be had over intensional complexity. So first define the *degree* (deg) of a formula as follows:

(1) $\deg(A) = 1$, if A is an atom;
(2) $\deg(B \& C) = \deg(B \vee C) = \deg(B) + \deg(C)$, for any formulas B and C; and
(3) $\deg(B \rightarrow C) = \deg(B \circ C) = \deg(B) + \deg(C) + 1$, for any formulas B and C.

Noting that degree is to be a measure of intensional complexity, the definition is obviously felicitous. And it is clear how the definition is to be extended to antecedents:

(1) $\deg(A)$ is the degree of A as above;
(2) $\deg(\mathrm{I}(\alpha_1, \ldots, \alpha_n)) = \deg(\alpha_1) + \cdots + \deg(\alpha_n) + n - 1$; and
(3) $\deg(\mathrm{E}(\alpha_1, \ldots, \alpha_n)) = \max\{\deg(\alpha_1), \ldots, \deg(\alpha_n)\}$.

Finally, $\deg(\alpha \vdash A) = \deg(\alpha) + \deg(A)$; and $\deg(\vdash A) = \deg(A)$. Note that the degree of a consecution is not raised by 1 by the $\vdash$.

Now let us say that a rule is *degree-preserving* just in case, for any instance of the rule, the degree of the conclusion is greater than or equal to that of any premiss. Then it is clear on inspection that we have the

DEGREE LEMMA. The rules of LTW$^{\circ}_{+}$ and LRW$^{\circ}_{+}$ are degree-preserving.

Now reduction and degree will work in tandem to deliver the virtual *coup de grace* to decidability:

COUNTING LEMMA. For any formula A and any $n \geq 0$, there are at most finitely many E-reduced antecedents of degree n built up from subformulas of A.

PROOF. By induction on n. The base step is trivial. So choose an arbitrary $m > 0$ and assume

Inductive hypothesis (IH). For any formula B and any $k < m$, there are at most finitely many E-reduced antecedents of degree k built up from subformulas of B.

Now choose an arbitrary formula A. It will then suffice to show that there are at most finitely many E-reduced antecedents of degree m built out of subformulas of A.

But any such antecedent is either

(1) a subformula of A, of which there are only finitely many; or

(2) an I-sequence of length $\leq m+1$, whose immediate constituents are of degree $< m$ (by definition of degree) and of course are built up out of subformulas of A. But, by IH, there are at most finitely many such antecedents to serve as immediate constituents. Hence there are but finitely many I-sequences of the required kind; or

(3) an E-sequence, each of whose immediate constituents is a *non-extensional* antecedent of degree $\leq m$ (by the definitions of "E-reduced" and "degree") and again built out of subformulas of A.

By induction hypothesis and (1) and (2) above, there are at most finitely many antecedents to serve as immediate constituents; and, by the definition of "E-reduced," none can occur more than twice as such. So there are at most finitely many E-sequences of the requisite sort.

And finitely many + finitely many + finitely many = finitely many. So we are finished. (Of course, the lemma holds equally well for E-reduced sequents built up from subformulas of any of a finite number of formulas.)

Decidability is now clearly in sight. To see whether a consecution is provable, the Denestation and Reduction theorems allow us simply to construct an E-reduced proof-search tree (defined as in §13.3) for the superreduct of that consecution. $\mathbf{LTW}^{\circ}_{+}$ and $\mathbf{LRW}^{\circ}_{+}$ can then be shown to be decidable by adapting the argument of §13.3, with the Degree and Counting lemmas (along with the obvious Subformula property) delivering the Finite Branch property. So by the $\boldsymbol{t}$-free Equivalence theorem: YES is the answer to the decision questions for $\mathbf{TW}^{\circ}_{+}$ and $\mathbf{RW}^{\circ}_{+}$. And, by known conservative-extension results, the answer is the same for $\mathbf{TW}_{+}$ and $\mathbf{RW}_{+}$, as well as for $\mathbf{TW}_{\rightarrow}$ and $\mathbf{RW}_{\rightarrow}$.

§67.7. $\mathbf{EW}^{\circ}_{+}$.

The reader will have noted the conspicuous absence of a result for $\mathbf{EW}^{\circ}_{+}$. The straightforward way to Gentzenize $\mathbf{EW}^{\circ t}_{+}$ is to add

$$\frac{\Gamma_1\mathrm{I}(\alpha;\, \boldsymbol{t})\Gamma_2 \vdash C}{\Gamma_1\mathrm{I}(\boldsymbol{t};\, \alpha)\Gamma_2 \vdash C}\ (\mathrm{CI}\boldsymbol{t}\vdash)$$

to the formulations for $\mathbf{TW}^{\circ t}_{+}$. The resulting systems are equivalent to $\mathbf{EW}^{\circ t}_{+}$. However, the proof of the Vanishing-$\boldsymbol{t}$ lemma breaks down. So $(\boldsymbol{t}-\vdash)$ cannot be removed from the system; hence the decidability argument will not go through, since that rule is not degree-preserving.

PROBLEM. So the decision question for $\mathbf{EW}^{\circ}_{+}$ remains open at this point.

CHAPTER XI

FUNCTIONS, ARITHMETIC, AND OTHER SPECIAL TOPICS

§70. Functions that really depend on their arguments. We begin with two queries.

QUERY 1. Why are we interested in the concept of functions that "really depend" on their arguments?

First, it is clear that this entire book is permeated with the notion of an implication such that its consequent should "really depend" on its antecedent. Now put this together with the notion, derivative from several sources, e.g., Läuchli 1970, that a useful analysis of an implication is as a family of functions that somehow take one from a proof or argument for the antecedent to a proof or argument for the consequent. Clearly what is wanted to represent *relevant* implications is a family of functions that "really depend on their arguments" so that they will not completely ignore information concerning the proofs of the antecedent. One would consequently expect, then, an interplay between the concepts of relevance logic and those of argument-dependent functions. The two families of ideas ought to cohere, to hang together—we ignore those who suggest they should hang separately. (For developments, see §71 and Helman 1977a, and, for related suggestions along these lines, we refer the reader to Myhill 1989, Urquhart 1989, and Pottinger 1979a.)

Second, with respect to property theory, there has been abroad in the land for a long time a desire to distinguish "real properties" from "phony properties," and the notion of argument dependence gives hope of making some sense of the distinction. For if we go with Russell in assuming that properties are propositional functions, then real properties ought really to depend on their arguments. See §74 for another approach to real properties via relevance logic and for a comparison of that approach to this.

Third, with respect to modal logic, it has been known for a long time that, if one is going to treat the higher-order logic of modalities, things don't begin to get very interesting until one allows the range of the property variables themselves to have a modal character, as in, for example, Bressan 1972 and Montague 1974. Analogously, it may turn out that something interesting happens when the property variables of higher-order relevance logic themselves somehow embody the notion of relevance, and, in particular, it seems to us well worth exploring what happens when one restricts the range of such property variables to properties that really depend on their arguments. Indeed, it may be sensible to look at what happens to first-order relevance logic when one allows only formulas that express such properties.

QUERY 2. What does it mean to say that a function really depends upon its arguments? We are not going to give a finally satisfactory answer to this question, but we shall at least try to make the question itself sound sane, and we shall make some tentative proposals.

The development will come in four steps: first we try to get at a *mathematical* notion of dependence, so that we may say when a function (an abstract entity) depends on its arguments; then we derive from this a *semantic* notion of dependence, applicable to formulas (pieces of language) that express functions; third, we offer a purely *syntactic* concept of dependence. And fourth, we deliver the matter to §71 for a more conclusive treatment.

§70.1. Mathematical concept of dependence. In the first place, let us agree to call a function that depends upon its arguments *strict*. That is, for a function to be strict, what its arguments are should be relevant to what its values for those arguments are. Thinking of functions on the input-output model, for a function to be strict, the inputs should make a difference to the outputs. No function can be strict that totally ignores its inputs.

To convince you that there really is an intuitive idea here, we propose some examples that are plausibly paradigmatic. First, any one-one function is a paradigm case of a function that takes its arguments into account, for whenever the arguments are different, so are the values; and no function could pay more attention than *that* to its arguments. One-one functions are, then, maximally strict. At any rate, if only "plain" functions of some basic domain are considered, that seems right, but O'Donnell 1985 points out that when we begin to consider a function f whose domain is primitive but whose range of values is itself composed of functions, there is room for an even stronger condition, namely, not only the one-one condition:

If $a \neq b$ then $fa \neq fb$,

but the more ferocious

If $a \neq b$ then, for all x in the domain of fa and all y in the domain of fb, $fax \neq fby$.

In contrast are the so-called "constant functions," that ignore their arguments entirely: the constant function 3 hands you back 3 no matter what argument you throw at it. For a more "normal" example of a constant function, try the operation of multiplying by zero: this operation always yields zero, no matter what the argument.

In the middle come functions like the squaring function, which are very far from constant, but also certainly not one-one: squaring does not pay as much attention as it should to such pairs of arguments as -2 and $+2$, since it does not discriminate between them, giving the same value for both. These considerations suggest that we should not merely sort functions into those which are strict and those which are not, but rather carry the notion of strictness as an ordering concept: define "f is at least as strict as g" by "whenever g differentiates between a pair of arguments, so does f" (i.e., $gx \neq gy$ implies $fx \neq fy$). But still it seems there are possibilities for a dichotomous concept that would simply divide the functions into those which are strict and those which are not, doubtless putting "most" functions—those we know and love—on the side of the strict.

One thing we might try has a topological flavor. Let us consider just functions on the reals. Then we might say that "f depends on its arguments around x" just in case it is not a constant function anywhere "around" x; that is, it would have to be that every open set containing x also contained a y such that $fx \neq fy$. Or, no matter how close you get to x, there is a y even closer such that $fy \neq fx$. Then we might define a function as strict if it depended on its arguments around x for each x in its range. Squaring would turn out to be strict in such a sense, whereas various "step" functions would not be strict. Given this account, one could tell a strict from a nonstrict function by its Cartesian picture: just look to see whether or not its curve has any flat places on it.

Another idea: say f depends on its argument at x if changing from x just a little changes the value a lot; that is, if f is discontinuous at x. Of course hardly any functions, surely not the usual ones we want to think of as depending on their arguments, have this featuure—only pathological curves are discontinuous at every point; so we drop this idea.

For still another idea, consult O'Donnell 1985, which defines a notion of "relevant function" along lines suggested by taking into account one-oneness of and disjointness of ranges of functions, as mentioned above; the intended application is to automated reasoning in the context of potential errors. (The latter topic is considered again in §§81 and 83 below.)

The foregoing effort to disentangle strict from nonstrict functions focused exclusively on their graphs and is accordingly purely extensional. We next take up an idea of Scott 1969 which is *almost* extensional, but which looks a little beyond what a function does to each of its usual arguments. (Our citation is to an unpublished paper, because Scott's 1969 notion of strictness appears not to have found its way into print. The line of research there initiated, however, is represented by a rich set of publications, including Scott 1970, Scott 1971, Scott 1972, Scott 1973a, and the *Compendium* Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott 1980, which points to numerous additional papers.) The idea of strictness in Scott 1969 depends in the first place on considering functions that may be only partial, that is, functions that may be undefined or have no value for certain of their arguments, as the division function is undefined when its second argument is zero. This is common talk. But further, in order to introduce Scott's idea, we need to have available the idea of an argument itself being undefined. Since that is not so common, let us pause a minute. If we start with the set of real numbers, we might wish to enrich it with something we could call "approximate real numbers," reified as certain sets of reals. One way of putting this idea is by saying that, when you ask a question the answer to which should be a real number, you may get only an approximate answer, that is, a set of reals. Now some of these approximate reals are better defined than others; the smaller the set, the better defined it is, with the unit sets representing the best defined of all. But at the other end is the set of all reals, which is useless as an answer to any question and which can accordingly be thought of as the completely undefined real number. As Scott has shown, it is possible to extend ordinary arithmetical operations on the reals to this family of approximate reals and to do this in an intuitively satisfying way. So we have a space of functions that are not everywhere defined and *also* such that it makes *sense* to say that one of the arguments is undefined.

For another example, also due to Scott, consider second-order partial functions whose arguments are themselves first-order partial functions. Now these first-order partial functions can be more or less well defined, with total functions being the most defined; but, again, at the other end of the spectrum it makes sense to talk about a partial function that is completely undefined: that function which is defined for none of its arguments. So again we have a case of functions (the second-order functions), which themselves may be undefined for some of their arguments, applying to a space of arguments (first-order partial functions) one of which might be thought to be completely undefined.

Now for the Scott notion of strictness (the word as well as the concept derives from him): a function is said to be *strict* in Scott's sense just in case the function is itself undefined at a completely undefined argument.

In contexts where the idea is applicable, this surely represents a minimum idea (we do not say "the" minimum idea) of dependence. A Scott strict function depends upon its arguments at least in the minimal sense that if its argument is wholly undefined, then so is its value. That is, a Scott strict function has to look at its argument at least a little bit in order to hand you back a value; it cannot *completely* ignore its argument, as could a function that was nonstrict in the sense of Scott. The latter sort of function could hand you back an answer even when its argument was completely undefined and hence gave it, in a sense, no input at all.

Something needs pointing out here that ought to come back to haunt us. In the space of the approximate reals there are *two* constant functions 3 (for example). There is the constant function 3 that hands back the value 3 for every argument, including the totally undefined argument. That function is not strict by our definition. On the other hand, there is the constant function 3 that hands back the value 3 for every argument *except* the totally undefined argument. The latter function is strict. So, for one thing, there is available a strict function that will do all the work that any sane mathematician would want done by the constant function 3; but, for another, the strictness of this function emphasizes how minimal is the concept of dependence we have articulated. The "strict constant function 3" has to look at its argument only enough to be sure it is defined; then quick as a wink it hands you back 3.

Aside from the domain of approximate reals and the domain of partial functions, there is another place where it makes sense and is useful to talk about undefined arguments and, hence, about Scott strictness: computer programs and procedures. Ignoring subtleties, a good thing to mean by saying that a computer program is undefined for certain arguments is that in response to those arguments it runs on forever, never terminating. (The sort of subtlety we are ignoring is illustrated by the computer that checks to see if you are trying to divide by zero, so that, if it catches you out, instead of running on forever it *halts* with a loud buzzing noise.) Now when one program receives as its input a second program, this running-on-forever of the second program would be the totally undefined input. If the first program printed you out an answer even though the second program never terminated then the first program would not be strict either in our technical sense or intuitively, since clearly in such circumstances it never looked at its input at all. On the other hand, the typical computer program is such that it at least minimally depends on its arguments in the sense that if a second program on which it relies for input does not terminate with supplying that input, then the first program will itself fail to hand back an output; so your average single-argument computer program is meaningfully strict in the sense of Scott.

There is no reason to be satisfied with Scott strictness as an explication of intuitive dependence, and no reason to be surprised to learn that an account that is nonextensional only to the obviously minimal extent of adding an undefined argument is not fully adequate. For example, suppose f is strict in Scott's sense, but otherwise constant 3. Suppose g is not strict, but is a step function, say constant 2 up to a certain point and then constant 3. The present analysis ranks f as more strict than g, but one may fairly ask why we are entitled to that judgment. If one thinks of the entire content of the analysis as contained in the enlarged functional graph obtained by adding the undefined argument to the list of ordinary arguments, then there is perhaps no answer; one needs in addition to be thinking of the new undefined argument as such, or perhaps to represent its special role by means of additional structure (as in Scott's own work). Nevertheless, as indicated to a certain extent in the immediately following material, but more adequately in §71, the explication of dependence as strictness has considerable theoretical power.

§70.2 Semantic and syntactic concepts of dependence. In any event, we proceed to use the nearly extensional notion of strictness as a basis for some semantic and syntactic ideas.

An appropriate semantic derivative from the mathematical notion of strictness of a function is that the value of a *formula* should depend on the value of one of its *variables.* That is, we explicate the notion that computing a value for a formula M depends upon computing a value for the variable x in Scott-ish terms by means of the following definition: a formula M *is (semantically) strict in the variable* x just in case, if the value of x is undefined, so is the value of M. We have in mind in this definition that, if there are any other variables free in x, their values are fixed by an appropriate assignment; accordingly, strictness of M in x is relative to such an assignment.

The next step is to use these intuitions as a basis for some stable syntactic ideas. Since it is functions we are about and since the theory of functions is, near enough, the theory of the lambda calculus (Church 1941; Curry and Feys 1958), what we are after is an image of the notion of strictness in the lambda calculus. In the remainder of this section, we explore in a tentative way how things go for the *untyped* λ-calculus; in §71 we present some sharp results for the case of the *typed* λ-calculus.

Because we are speaking of the untyped case, where there is no bar to writing such things as (MM)—self-application of a function M—it is better if we have in mind a definite model or family of models; we are thinking of those of Scott 1973a. These models naturally contain an entity to be thought of as the maximally undefined entity. Against this background, the key idea is that if M is semantically strict in x in the above sense then the expression $\lambda x.M$ will denote, on its usual reading, a function that is strict in the mathematical sense. This observation correctly forecasts that we are going to try to catch the notion of strictness in the λ-calculus by means of a restriction on the formation of λ-abstracts. What is wanted is a *syntactic* relation between formulas and variables such that a formula M will bear this relation to a variable x just in case M can be seen by *syntactic* inspection to be *semantically* strict in the variable x.

Let us carry the fundamental syntactic relation in the notation st(M), *the set of variables in which the formula M is strict*, so that $x \in \text{st}(M)$ just in case M is strict in x. How to define the set of variables st(M)? The definition will of course be inductive, with a base case for variables and inductive clauses for application (MN) and λ-abstraction $\lambda x.M$. The basis is easy:

$$x \in \text{st}(x).$$

So are the inductive clauses:

If $y \in \mathrm{st}(M)$ or $y \in \mathrm{st}(N)$ then $y \in \mathrm{st}(MN)$.
If $y \neq x$ and $y \in \mathrm{st}(M)$ then $y \in \mathrm{st}(\lambda x.M)$.

There is, however, a snappy noninductive way of putting all this: y belongs to $\mathrm{st}(M)$ just in case y has at least one free occurrence in M. (This makes the idea look somewhat more trivial than it really is; the nontriviality emerges only in the extensions mentioned toward the end of the section.)

Those familiar with the history of λ-conversion will be reminded of Church's 1941 λ-calculus, often now called the λ-I-calculus, which has the following definition of "well-formed formula":

1 Variables are well-formed.
2 If M and N are well-formed, so is (MN).
3 If M is well-formed, so is $\lambda x.M$, *provided* M contains at least one free occurrence of the variable x.

The third clause exactly answers to our notion of strictness: if we want our well-formed formulas each to denote a function that can be seen by syntactic inspection really to depend on its arguments, then we had better not allow $\lambda x.M$ to be well-formed unless $x \in \mathrm{st}(M)$. So we may conjecture that Church's λ-I-calculus is the right one to embody the notion of a strict function.

Before proceeding with a discussion of this conjecture, we pursue for a minute a direction suggested by Church's own work. Consider a closed term as expressing a *Church-undefined* entity just in case it has no normal form. This is in the spirit of Church, since he thinks of such formulas as "meaningless" (Church 1941, p. 59). Next, define a formula M as *Church-semantically strict in* a variable x just in case substituting a formula expressing a Church-undefined entity for x in M yields a formula that itself expresses a Church-undefined entity. This definition puts together a notion derived from Church as to what it is to express an undefined entity with a notion derived from Scott as to what it is for a formula to depend on or to be strict in one of its variables.

The upshot is that in this way we secure agreement between the semantic and the syntactic notions of strictness; this is an immediate consequence of Church's theorem 7 XXXII, p. 27. That is, one can calculate that if a formula M is syntactically strict in x then it is also Church-semantically strict in x. And the converse also holds, trivially, unless M itself has no normal form.

There are, however, a couple of difficulties with this line. In the first place, it is really proof-theoretical rather than semantical or mathematical, since the notion of having a normal form is purely proof-theoretical. In the second place, there are expressions which have no normal form but which nevertheless we—or some of us—want to think of as perfectly well-defined. The paradigm example is perhaps that of expressions involving the "fixpoint finder" *Y*, which when applied to any function f always finds a fixpoint of f; that is, *Y* satisfies the equation (see Curry and Feys 1958)

$$Y\mathrm{f} = \mathrm{f}(Y\mathrm{f}).$$

Clearly $Y\mathrm{f}$ has no normal form, and yet it has definite meaning as a fixpoint of f. Since this idea has lots of applications, we do not want to throw it out in a hurry.

§70.3. Church's λ-*I*-calculus and Scott's strictness. Returning to the concept of strictness of Scott 1969, recall the notion of the last section that this idea is somehow embodied in Church's λ-*I*-calculus, where vacuous variables are forbidden. Here we discuss some conjectures that arise from that notion in the context of Scott's models for the lambda calculus as accessibly described in Scott 1973a; although our deliberations concerning strictness and the untyped lambda calculus will not be conclusive, there is comfort in the prospect of §71, where switching attention to the *typed* lambda calculus generates some strong and pointed results.

Let us begin with a key-feature account of the Scott models. After a process of construction that involves a number of philosophically important auxiliary ideas, we are given the following. (1) A space D, to be the range of values of the variables of the lambda calculus. (2) A family D′ of ordinary garden-variety functions from D into D (but by no means all of them). (3) A one-one correspondence α between D and D′. Now define a translation function * from the untyped into the ordinary typed lambda calculus as follows (untyped inside the asterisks on the left, typed on the right):

$$\begin{aligned} x^* &= x \\ (MN)^* &= (\alpha M^*)N^* \\ (\lambda x.M)^* &= \alpha^{-1}(\lambda x.(M^*)) \end{aligned}$$

The variable x on the right is to range over D. Suppose lastly that (4) if M is a well-formed formula, and if the variables in M other than x are given values in D, then $\lambda x.(\mathrm{M}^*)$ is a function in the family D′.

In these circumstances we know that we have a model of the untyped λ-calculus; for (a) every well-formed formula denotes the α-mate of a function (in D′) and (b) λ-conversion is verified as an identity. Also (c) extensionality is satisfied. For the model to be interesting, D′ must be an interesting family of functions, as indeed it is in the Scott models, and the relation of the model to the calculus must also be subject to some tests of nontriviality, as is in part indicated by its ability to distinguish semantically between a number of important formulas distinguished proof-theoretically by the lambda calculus. (The Scott models admit the odd case in which it is impossible to distinguish semantically between closed formulas that can be distinguished proof-theoretically.)

We mentioned that the matched spaces D and D′ arise out of a process of construction. That process always starts with a basis, D_0, which is a lattice governed by a relation intuitively interpretable as "approximation" (see §81 for a little more use of this idea). The process then generates a sequence of function spaces $D_0, D_1, \ldots, D_n$, each of which is constituted by an interesting subset of the functions on the prior space and each of which is shown to "fit" with its predecessor and its successor by means of appropriate downward and upward cross-type transfer functions. The final space D is taken as a certain sort of "limit" of this sequence, and theorems are brought forth to guarantee that it all makes sense.

The lattice structure involved in the construction of D and D′ singles out an entity, $\perp$, which it is natural to think of as a maximally undefined entity and which accordingly supports the concept of strictness. Furthermore, the α-mate of $\perp$ is invariably the constant function $\perp$, and so, by interpretation, $\alpha(\perp)$ is the undefined function in D′, inasmuch as it is undefined at every point—everything fits. Let S be the set of strict elements of a model D (or strict functions in D′—given the hovering presence of the one-one function α, we can afford to be sloppy about this).

Here are some conjectures.

CONJECTURE 1. By definition, a *pure* lambda abstract is a closed expression looking like $\lambda x_1, \ldots \lambda x_n.M$, where M is built from variables only; we might reasonably conjecture that if a pure lambda abstract invariably denotes a strict function then it is syntactically strict, that is, well-formed in the λ-$\boldsymbol{I}$-calculus. Counterexample:

$$\boldsymbol{K} = \lambda x.\lambda y.x, \text{ which satisfies the equation } ((\boldsymbol{K}a)b) = a.$$

So $(\boldsymbol{K}a)$ is the constant function a. Indeed $\boldsymbol{K}$ is the strictest of the strict, being one-one: if $(\boldsymbol{K}x) = (\boldsymbol{K}y)$ then $x = y$. But of course $\boldsymbol{K}$ is a paradigm among those lambda abstracts which violate syntactic strictness, since y is not strict in x. Accordingly, the stronger conjecture, that every formula not in the λ-$\boldsymbol{I}$-calculus denotes a nonstrict function in some Scott model, is thoroughly false.

CONJECTURE 2. Each well-formed formula of the λ-$\boldsymbol{I}$-calculus denotes a strict entity whenever its free variables are assigned strict entities. Counterexample: (xy) already; for strictness in the Scott models is not closed under application. $\boldsymbol{K}$ is again the source of trouble, for, although $\boldsymbol{K}$ is strict (as noted above), $(\boldsymbol{KK})$ (for example) is not, since $((\boldsymbol{KK})\perp)$ evaluates to $\boldsymbol{K}$ instead of to $\perp$, as would be required if $(\boldsymbol{KK})$ were a strict function.

CONJECTURE 3. Begin by remarking that $\boldsymbol{K}$ is strict, but it does not invariably carry strict entities into strict entities, as we just saw, whereas $(\boldsymbol{KK})$

on the other hand always carries strict entities into strict entities (that is, it carries all arguments into $\boldsymbol{K}$, which is known to be strict), but $(\boldsymbol{KK})$, as we also just saw, is not itself strict. Suppose we let S_1 be the set of entities satisfying both requirements in order to start to build a kind of strainer. At least we would know that neither $\boldsymbol{K}$ nor $(\boldsymbol{KK})$ lies in S_1, but we nevertheless say "*start* to build" because S_1 itself is not closed under application and so is insufficient to be a target for any interesting collection of well-formed formulas. We therefore go on to consider elements that not only belong to S_1, but also carry S_1 into itself—let S_2 be this subset of S_1. Even before we consider whether S_2 is closed under application, there is trouble: some wonderful functions expressible in the notation of the λ-$\boldsymbol{I}$-calculus do not belong to S_2; so any conjecture along these lines is forbidden. For example,

$\boldsymbol{W} = \lambda f.\lambda x.((fx)x)$, satisfying the equation $((\boldsymbol{W}f)a) = ((fa)a)$

does not belong to S_2; choose f as the identity function $\boldsymbol{I}$, which clearly belongs to S_1. One can calculate that $(\boldsymbol{WI})$ does not belong to S_1 because, although $\boldsymbol{K}$ is in S, $((\boldsymbol{WI})\boldsymbol{K}) = (\boldsymbol{KK})$ is not.

CONJECTURE 4. Even though some open formulas of the λ-$\boldsymbol{I}$-calculus lead outside of S, as we saw in our discussion of Conjecture 2, it might be true that each *closed* formula of the λ-$\boldsymbol{I}$-calculus invariably denotes something in S, that is, a strict entity. Evidently if this is true it will not be true because S is closed under application, because S is not closed under application; but the conjecture might still be true. For difficulty in considering the conjecture, think about $((\boldsymbol{WW})\boldsymbol{W})$. For this to be strict we should need $(((\boldsymbol{WW})\boldsymbol{W})\bot) = \bot$, but all the lambda calculus delivers when using the properties of $\boldsymbol{W}$ is that the left side is identical to itself. It does not, however, follow that the Scott models leave the conjecture undetermined, and so the question is open.

All these conjectures involve the "strainer" tactic; that is, they make an effort to strain out strictness from the designedly unstrict Scott models. The tactic of §71 in connection with the *typed* λ-calculus is not of this kind; instead, there the concept of strictness is built into the hierarchy from the ground up. It turns out that something of the same procedure is available for the models of the *untyped* calculus, because Scott's procedure in constructing them is itself based on a hierarchy (the models are found as limits of an interesting hierarchy), but what is far from clear is the extent to which these procedures have intuitive significance in the untyped case. For that reason, we offer here only hints (and no proofs). First, confine the construction to structures having a bottom, $\bot$, that is not meet-reducible, and a top that is "isolated" in the sense that it is not the union of any directed set not including itself. Second, in ascending to function spaces, take only functions that are both continuous *and* strict. Third, the initial transfer functions required to tie together the elements of the hierarchy are given upwards by the "strict constant function" operator, and downwards by taking the value at next-to-bottom. By using these materials we may use Scott's construction to determine the following (compare the beginning of this section). (1) A space S, to be the range of values of the variables of the λ-$\boldsymbol{I}$-calculus, including an "undefined" entity, $\perp$. (2) A family S′ of strict ordinary garden-variety functions from S into S (but by no means all of them), including the totally undefined function (that is, the constant function $\perp$). (3) A strict one-one correspondence α between S and S′; strictness in effect means that α shall mate the undefined entity in S with the totally undefined function in S′. Now define a translation function * from the untyped into the ordinary typed λ-$\boldsymbol{I}$-calculus as follows (untyped inside the asterisks on the left, typed on the right):

$$\begin{aligned} x^* &= x \\ (MN)^* &= (\alpha M^*)N^* \\ (\lambda x.M)^* &= \alpha^{-1}(\lambda x.(M^*)). \end{aligned}$$

The variable x on the right is to range over S. Suppose lastly that (4) if M is a well-formed formula of the λ-$\boldsymbol{I}$-calculus and if $x \in \mathrm{st}(M)$ and if the variables in M other than x are given values in S, then $\lambda x.(M^*)$ is a function in the family S′.

In these circumstances we know that we have a model, based on strictness, of the untyped λ-$\boldsymbol{I}$-calculus; for (a) every well-formed formula denotes the α-mate of a strict function (in S′) and (b) λ-conversion is verified as an identify. Also (c) extensionality is satisfied. For the model to be interesting, S′ must be an interesting family of strict functions, as indeed it is, and the relation of the model to the calculus should pass some tests of nontriviality, as is indicated in part by its ability to distinguish semantically between a number of important formulas distinguished proof-theoretically by the λ-$\boldsymbol{I}$-calculus and in part by its lack of any entity $\boldsymbol{K}$ (here we are using $\boldsymbol{K}$ as a variable) satisfying the usual equation $((\boldsymbol{K}a)b) = a$ (with a and b ranging over S) or any other entity with a "similar" property.

But for straightforward results as to what is left out of a family of models embodying strictness, the reader is invited to turn to our treatment of the *typed* lambda calculus in the following section.

§71. Relevant implication and relevant functions (by Glen Helman). The study of relevance began in §3 with an account of proof from hypotheses which was designed to respect this relation. And perhaps our best understanding of relevance logics still comes through the notion of proof *from* hypotheses. The present section is devoted largely to a formal interpretation of $\mathbf{R}_{\rightarrow\&}$ along these lines.

Half the job of constructing such an interpretation comes in providing a formalization of the notion of proof from hypotheses that is appropriate to $\mathbf{R}_{\rightarrow\&}$. The other half is a formal interpretation of the notion of proof from hypotheses itself. In carrying out these tasks, we will draw on two sources: Curry's analogy between the provability of implicational formulas and the definability of functions of certain types, and the notion of §70 of a function that depends on its arguments. Curry's analogy is used to formalize the notion of proof from hypotheses by a system of typed lambda abstraction with a restriction on the formation of abstracts which plays the role of a restriction on →I ("→I" as in §1.3). This system of lambda abstraction has natural interpretations in hierarchies of set-theoretic functions. We adapt the notion of §70 of an argument-dependent function to distinguish a class of "relevant" functions in a hierarchy of functions individuated intensionally. In the end, a proof from hypotheses is understood as a proof that can serve to define a relevant function.

We shall present this interpretation of $\mathbf{R}_{\rightarrow\&}$ in two stages. §71.1 and §71.2 will develop a simplified form adequate for $\mathbf{R}_{\rightarrow}$, and §71.3 and §71.4 will provide the extensions necessary for conjunction.

§71.1 Terms and proofs. In this section we prepare the way for the interpretation of $\mathbf{R}_{\rightarrow}$ by discussing the use of terms formed by lambda abstraction to provide a formulation of $\mathbf{H}_{\rightarrow}$. "Lambda abstraction" is the now common name for functional abstraction, deriving from the notation introduced in Church 1932. Given a term t,

$$\lambda xt$$

denotes the function f whose value for any argument a is the denotation of t when the variable x is assigned a. This interpretation is embodied in an equational calculus to be described below. Adding the notation and its calculus to a language has the effect of enriching the language by a certain form of explicit definition. Specifically, when t is a term containing only x free, the formation of λxt has the same effect as the introduction of a constant f by means of the definition

$$fx = t.$$

Free variables in t beyond x would have the status of parameters; so, in the general case, the formation of λxt has the same effect as the introduction of a parameterized constant $f_{\bar{y}}$ by the definition

$$f_{\bar{y}}x = t,$$

where $\bar{y}$ is a perhaps empty sequence of variables consisting of the free variables of t other than x. The notation λxt has the advantage of providing a term that displays the defining expression t.

Lambda abstraction is usually studied in the context of a language otherwise containing only variables and notation for untyped functional application. For example, this is the language used to specify the "λ-definable functions," the formulation of effective computability for numerical functions given in Church 1936. Such a language is also closely related to Curry's program of combinatory logic. The standard reference for the language and for Curry's program generally is Curry and Feys 1958 and Curry, Hindley, and Seldin 1972. Hindley, Lercher, and Seldin 1972 and Barendregt 1977 are concise introductions; much further material about recent work can be found in Barendregt 1984. However, lambda abstraction also provides a convenient notation for type theory (see, for example, Church 1941), and it is in the context of typed languages that we will consider it here.

For this and the next section, a *type* will be any sentence formed from propositional variables using only the conditional. A conditional $A \rightarrow B$ is to be the type of a function that is defined on arguments of type A and yields values of type B. We will see below that this is more than a pun on the usual informal mathematical notation.

Our typed language contains infinitely many variables of each type and is closed under typed application and lambda abstraction. That is, it is the collection of *pure λ-terms* ("λ-terms" for short, in this and the next section), where

(1) if t and u are pure λ-terms of types $A \rightarrow B$ and A for some A and B, then (tu) is a pure λ-term of type B; and

(2) if t is a pure λ-term of type B and x is a variable of type A, then λxt is a pure λ-term of type $A \rightarrow B$.

(tu) is interpreted as the result of applying the function denoted by t to the object denoted by u. It is standard to manage parentheses by the conventions of §1.2, except that a left parenthesis is usually replaced by a dot only when it immediately follows an abstraction operator. The operator λx binds the free occurrences of x in its scope. When a term u and a variable x have the same type,

$$[u/x]t$$

is to be the result of substituting u for the free occurrences of x in the term t, rewriting bound variables of t as necessary to avoid capturing the free variables of u (for a precise definition, see Curry and Feys 1958). The variable for which substitution is made will often be fixed in a given context, and we will use the abbreviated form "$t[u]$".

A glance at the conditions (1) and (2) above will show that the operations on types introduced by the formation of the terms (tu) and λxt match those of →E ("→E" as in §1.3) and →I, respectively. Indeed, we can regard the terms of our typed language as natural deduction proofs for a pure implicational logic. A term proves the sentence that is its type, with the types of its free variables as undispatched hypotheses. The formation of lambda abstracts dispatches hypotheses by binding variables. Indeed, if we identify the λ-terms that differ only in their choice among bound variables of the same type, we have something quite close to the tree-form proofs of a Gentzen N system defined using the "different notion of a deduction" of Prawitz 1965 (pp. 29–31). The basis for this coincidence was noted in Curry and Feys 1958 (pp. 312–315), and it has been extended to richer logics by Howard 1969, Laüchli 1970, Martin-Löf 1972, and others.

The intuitive interpretation of a term λxt as notation for functional abstraction is given formal bite by providing a calculus that allows us to compute values of the function it denotes. It is standard to do this by defining a relation between terms. A number of relations have been studied, all of which imply identity of denotation for the terms related. Two of these relations are of interest here. The first is an equivalence relation, $=$, of *extensional equality* ("equality" for short), which is analogous to the "$\beta\eta$-conversion" for untyped terms of Curry and Feys 1958. It is the least relation between λ-terms that satisfies the following:

(1) $=$ is reflexive, symmetric, and transitive;
(2) if $u = v$ then $t[u] = t[v]$;
(3) if neither x nor y is free in t, $\lambda x.t[x] = \lambda y.t[y]$;
(4) $(\lambda x.t[x])u = t[u]$;
(5) if $t = u$ then $\lambda xt = \lambda xu$;
(6) if x is not free in t then $\lambda x.tx = t$.

The conditions (3)–(6) are roughly those which Curry labels "(α)", "(β)", "(ξ)", and "(η)". Conditions (1) and (2) justify us in calling the relation an equality. (3) licenses the rewriting of bound variables. The instances of (4) play the role of defining equations for the abstracts $\lambda x.t[x]$. (5) and (6) are principles of extensionality. (5) implies extensionality for abstracts, and the addition of (6) extends this to other terms denoting functions.

Conditions (β) and (η) provide ways of putting some terms into simpler equivalent forms. This sort of simplification motivates the relation, $\gtrsim$, of *strong reduction* ("reduction" for short), which is analogous to the "$\beta\eta$-reduction" of Curry and Feys 1958. It is the least reflexive and transitive relation between λ-terms that satisfies the conditions (2)–(6) above. With no requirement of symmetry, reduction holds in fewer cases than equality. If $t \gtrsim u$, then u is at least as simple in form as t and has the same denotation.

This idea of simplification also brings with it the idea of a term that cannot be further simplified. A λ-term is said to be in *normal form* if it contains no subterm either of the form $(\lambda xt)u$ or of the form $\lambda x.tx$ with x not free in t. These are the two sorts of terms to which the reductions of the conditions

(β) and (η) apply directly, and they are called *β-redexes* and *η-redexes*, respectively. Simplification consists in the elimination of these redexes, and they are completely eliminable. That is, each λ-term reduces to one in normal form. Indeed, any sequence of eliminations of β- and η-redexes from a λ-term must terminate (proofs may be found in Stenlund 1972, pp. 126–131, and Troelstra 1973, pp. 111–113). Similar notions of reduction and normal form for proofs, employing an analogue of β-reduction but not of η-reduction, have been investigated by Prawitz and others (see Prawitz 1965 and 1971). And the λ-notation itself has proved to be convenient for proof theory (see Pottinger 1976 and 1977a).

The full class of λ-terms serves as a system of proof for $\mathbf{H}_\to$, and it will be referred to as $\lambda\mathbf{H}_\to$, the *λ-formulation* of $\mathbf{H}_\to$. Specifically, A is a theorem of $\mathbf{H}_\to$ if and only if it is the type of a closed term of $\lambda\mathbf{H}_\to$. It is easy to see that all theorems are the types of closed λ-terms. Below we list a number of characteristic theorems along with closed terms that have them as types when their variables have the types indicated, using "t: A" to say that t has the type A:

$A \to A$	$\lambda xx \quad (x: A)$
$B \to .A \to B$	$\lambda x\lambda yx \quad (x: B,\ y: A)$
$B \to C \to .A \to B \to .A \to C$	$\lambda x\lambda y\lambda z.x(yz)$ $(x: B \to C,\ y: A \to B,\ z: A)$
$(A \to .B \to C) \to .B \to .A \to C$	$\lambda x\lambda y\lambda z.xzy$ $(x: A \to .B \to C,\ y: B,\ z: A)$
$(A \to .B \to C) \to .A \to B \to .A \to C$	$\lambda x\lambda y\lambda z.xz(yz)$ $(x: A \to .B \to C,\ y: A \to B,\ z: A)$

Theorems of these forms serve as a redundant set of axioms for $\mathbf{H}_\to$. So terms may be formed by application from those on the right which will have as their types all the further theorems of $\mathbf{H}_\to$. We could complete an argument for the adequacy of $\lambda\mathbf{H}_\to$ by using the deduction theorem for $\mathbf{H}_\to$ to show that, if B is the type of a term the types of whose free variables are among $A_1, \ldots, A_n$, then there is an axiomatic proof in $\mathbf{H}_\to$ of B from $A_1, \ldots, A_n$. Instead we will use an approach that is more convenient in proving the adequacy of the λ-formulations of other logics.

When we look at the proof of the deduction theorem itself, we see that it provides a technique for transforming a proof constructed using →E along with →I into one constructed using →E along with certain axioms. Our λ-terms represent proofs constructed using →E and →I, and there is a related collection of terms that can serve to represent proofs constructed using →E and axioms. Schönfinkel 1924 introduced function-denoting constants called *combinators* as a substitute for functional abstraction that avoided the use of bound variables. These have been studied extensively since, in both an untyped and a typed setting. This research was long under the leadership of Curry, who had hit upon combinators independently. The standard reference is Curry and Feys 1958, with Curry, Hindley, and Seldin 1972 and Barendregt 1984 providing further developments.

Our combinators will be constant terms grouped into families, each of which is indexed by a single type or a sequence of types. For all A, B, and C, we will employ the combinators listed below with their types indicated at the right:

$\boldsymbol{I}_A$	$A \to A$
$\boldsymbol{B}_{A,B,C}$	$B \to C \to .\ A \to B \to .A \to C$
$\boldsymbol{C}_{A,B,C}$	$(A \to .B \to C) \to .\ B \to .A \to C$
$\boldsymbol{S}_{A,B,C}$	$(A \to .B \to C) \to .\ A \to B \to .A \to C$
$\boldsymbol{K}_{A,B}$	$B \to .A \to B$

These are typed versions of Schönfinkel's original selection of combinators, most with names due to Curry. We will often suppress subscripts when these are recoverable from the context or when distinctions within the families are not at issue.

The *pure terms* will be those formed from combinators and variables using typed application and abstraction. The λ-terms are then the pure terms that contain no combinators. The *pure c-terms* are the pure terms that contain no abstraction. As with λ-terms, the qualification "pure" will be supressed in this and the next section.

The types of the combinators are all theorems of $\mathbf{H}_\to$ and form a sufficient set of axioms. C-terms are formed from combinators and variables alone; so they can serve to represent axiomatic proofs. As in the case of λ-terms, the conclusion of the proof is the type of the c-term, and the proof's hypotheses are the types of its free variables. Indeed, we can count the class of all c-terms as a *combinatory formulation* $\mathbf{CH}_\to$ of $\mathbf{H}_\to$, declaring its adequacy in the following.

THEOREM. A formula A is a theorem of $\mathbf{H}_\to$ if and only if it is the type of a closed c-term.

PROOF. To argue for this, we simply marshal the reasons already given for taking c-terms to represent axiomatic proofs. First, suppose that A is a theorem of $\mathbf{H}_\to$. Then there is a proof of A that employs axioms that we may assume to be among the types of our combinators. Each step of that proof can be assigned a closed c-term with the step as its type, assigning combinators to axioms and using application to form a term for a step obtained by →E. On the other hand, suppose that A is the type of a closed c-term t. Then there is a sequence of terms ending with t, each term of which either is a combinator or is formed from previous terms by application. The types of these terms in sequence form an axiomatic proof of A.

A deduction theorem shows that the power of →I can be obtained through the use of certain axioms. An analogous result, a "combinatory completeness" theorem, shows that the power of functional abstraction can be obtained through the use of certain combinators. In the case of c-terms and λ-terms, the power at issue is not just the power of provability under hypotheses—the existence of a term of type B the types of whose free variables are among $A_1, \ldots, A_n$—but the combinatory power of the systems. For c-terms, as for λ-terms, this power is embodied in an equational calculus. *Extensional equality* ("equality" for short) is the least relation between c-terms that satisfies the following:

(1) $=$ is reflexive, symmetric, and transitive;
(2) if $u = v$, then $t[u] = t[v]$;
(3) if $tx = ux$ and x is free in neither t nor u, then $t = u$;
(4) $\boldsymbol{I}t = t$
(5) $\boldsymbol{B}tuv = t(uv)$;
(6) $\boldsymbol{C}tuv = tvu$;
(7) $\boldsymbol{S}tuv = tv(uv)$;
(8) $\boldsymbol{K}tu = t$.

In addition to conditions (1) and (2), which provide for the basic properties of an equality, we have in condition (3) assurance of extensionality, and, in conditions (4)–(8), defining equations for the combinators.

Combinatory completeness is then the following property of the system of c-terms: for any c-term t of type B and variable x of type A, there is a c-term u of type $A \to B$ in which x does not occur free and which is such that, for any c-term v,

$$(uv) = [v/x]t,$$

where $[v/x]t$ is the result of replacing all occurrences of x in t by v. That is, given a c-term t, there is a c-term u with the properties of λxt. The existence of such a term is conveniently shown by defining an operation of abstraction on c-terms. There are a number of ways of doing this (see Curry and Feys 1958, pp. 190–194). The method we consider here is convenient for use with extensional equality and is easily modified for our later combinatory formulation of $\mathbf{R}_{\to}$. The *combinatory abstract*, $[x]t$, of a c-term t with respect to the variable x is defined as follows:

(1) $[x]x \equiv \boldsymbol{I}_{\mathrm{tp}(x)}$;
(2) if x is not free in t,
$[x].tx \equiv t$;
(3) if x is free in u but not in t and $u \not\equiv x$,
$[x].tu \equiv \boldsymbol{B}_{\mathrm{tp}(x),\mathrm{tp}(u),\mathrm{tp}(tu)}t([x]u)$;

(4) if x is free in t but not in u,
$[x].tu \equiv \boldsymbol{C}_{\mathrm{tp}(x),\mathrm{tp}(u),\mathrm{tp}(tu)}([x]t)u$;
(5) if x is free in both t and u,
$[x].tu \equiv \boldsymbol{S}_{\mathrm{tp}(x),\mathrm{tp}(u),\mathrm{tp}(tu)}([x]t)([x]u)$; and
(6) if x is not free in t,
$[x]t \equiv \boldsymbol{K}_{\mathrm{tp}(x),\mathrm{tp}(t)}t$.

Here $\mathrm{tp}(t)$ is the type of t, and "$\equiv$" stands for the relation of syntactic identity between terms. The reader should verify that the type of $[x]t$ is $\mathrm{tp}(x) \rightarrow \mathrm{tp}(t)$, that x does not occur free in t, and that $([x].t[x])u = t[u]$.

Combinatory abstraction enables us to define the *combinatory translation*, t^{c}, of a λ-term t by:

(1) $x^{\mathrm{c}} \equiv x$;
(2) $(tu)^{\mathrm{c}} \equiv (t^{\mathrm{c}}u^{\mathrm{c}})$;
(3) $(\lambda xt)^{\mathrm{c}} \equiv [x].t^{\mathrm{c}}$.

We can define a translation in the other direction by using the closed λ-terms offered earlier as proofs of theorems of $\mathbf{H}_{\rightarrow}$ to play the role of the corresponding combinators. The *lambda translation*, t^{λ}, of a c-term t is given by the following, where we assume in (2)–(6) that x, y, and z are distinct and are the least variables of the types indicated:

(1) $x^{\lambda} \equiv x$;
(2) $\boldsymbol{I}_A^{\lambda} \equiv \lambda xx \quad (x{:}\ A)$;
(3) $\boldsymbol{B}_{A,B,C}{}^{\lambda} \equiv \lambda x\lambda y\lambda z.x(yz) \quad (x{:}\ B\rightarrow C,\ y{:}\ A\rightarrow B,\ z{:}\ A)$;
(4) $\boldsymbol{C}_{A,B,C}{}^{\lambda} \equiv \lambda x\lambda y\lambda z.xzy \quad (x{:}\ A\rightarrow .B\rightarrow C,\ y{:}\ B,\ z{:}\ A)$;
(5) $\boldsymbol{S}_{A,B,C}{}^{\lambda} \equiv \lambda x\lambda y\lambda z.xz(yz) \quad (x{:}\ A\rightarrow .B\rightarrow C,\ y{:}\ A\rightarrow B,\ z{:}\ A)$;
(6) $\boldsymbol{K}_{A,B}{}^{\lambda} \equiv \lambda x\lambda yx \quad (x{:}\ B,\ y{:}\ A)$;
(7) $(tu)^{\lambda} \equiv (t^{\lambda}u^{\lambda})$.

It can be shown that these translations preserve extensional equality. An argument for this in the case of a slightly different definition of combinatory abstraction can be found in Hindley and Seldin 1986, and the argument for the case of the present definition is similar. But, for our purposes, it is the following two claims that are important:

FACT 1. t^{c} and t have the same types and the same free variables.

FACT 2. t^{λ} and t have the same types and the same free variables.

The first of these follows from our earlier comments on the properties of $[x]t$. For the second, we need only check that the lambda translations of the combinators are closed terms with the correct types. The two together show that the collection of types of the closed c-terms is identical with the collection of types of the closed λ-terms. Hence the formulations $\mathbf{CH}_{\rightarrow}$ and $\lambda\mathbf{H}_{\rightarrow}$

are equivalent. Our earlier theorem established the adequacy of $\mathbf{CH}_{\rightarrow}$. Adding these facts provides the promised argument for the adequacy of $\lambda\mathbf{H}_{\rightarrow}$.

§71.2. Relevant abstraction and monadic relevant functions. We wish to find a restriction on abstraction that can be used to provide a λ-formulation of $\mathbf{R}_{\rightarrow}$. The restriction on $\rightarrow$I that is used for $\mathbf{R}_{\rightarrow}$ is intended to allow $A \rightarrow B$ to be derived by $\rightarrow$I only when the hypothesis A is actually used in the proof of B. A parallel restriction would permit the formation of the abstract λxt only when the expression t uses the argument variable x in specifying the values of the function that λxt denotes. And a natural test for this use is the free occurrence of x in t. Let us call an abstract λxt *vacuous* when x is not free in t. Then our desired restriction seems to be the prohibition of vacuous abstracts.

But the free occurrence of a variable in a term is a satisfactory indication of its use only if we can be sure that the occurrence is not redundant. That is, free occurrence can serve as a general test for use only if the free variables of a term remain free in any simplification of it. Strong reduction is our formal account of simplification, and inspection of its properties will show that, although free variables can be lost in reduction, this can happen only when λ-reduction is applied to a term $(\lambda x.t[x])u$ where $\lambda x.t[x]$ is vacuous. So, when a prohibition of vacuous abstracts is in force, free occurrence is a sign of use and the prohibition has the effect we want. The prohibition of vacuous abstracts appears in Church's work on lambda abstraction. The central object of study in Church 1941 is a calculus for terms formed using untyped application and nonvacuous abstraction, which is now known as the "λ-$\boldsymbol{I}$-calculus."

We will call the class of λ-terms that contain no vacuous abstracts $\lambda\mathbf{R}_{\rightarrow}$. Our aim now is to show that this is indeed an adequate formulation of $\boldsymbol{R}_{\rightarrow}$. As with $\lambda\mathbf{H}_{\rightarrow}$, we first consider a combinatory formulation. Inspection will show that the types of the combinators $\boldsymbol{I}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{S}$ are all theorems of $\mathbf{R}_{\rightarrow}$ and form a sufficient set of axioms. Accordingly, we fix $\mathbf{CR}_{\rightarrow}$ as the set of c-terms that contain none of the combinators $\boldsymbol{K}$. An argument similar to that for $\mathbf{CH}_{\rightarrow}$ shows that $\mathbf{CR}$ is an adequate formulation of $\mathbf{R}_{\rightarrow}$ (as was noted, in effect, in Curry and Feys 1958, p. 315).

Given the adequacy of $\mathbf{CR}_{\rightarrow}$, we can prove the adequacy of $\lambda\mathbf{R}_{\rightarrow}$ by showing that every term in $\lambda\mathbf{R}_{\rightarrow}$ has a translation in $\mathbf{CR}_{\rightarrow}$ with the same type and the same free variables and that each term in $\mathbf{CR}_{\rightarrow}$ has a translation in $\lambda\mathbf{R}_{\rightarrow}$ with the same type and the same free variables. The translations of the last section suffice. The combinatory translation of λ-terms introduces a combinator $\boldsymbol{K}$ only in the case of vacuous abstracts, and the lambda translation of c-terms employs vacuous abstraction only in the translation of the combinators $\boldsymbol{K}$. So we may add one more fact to those of the last section and count the adequacy of $\lambda\mathbf{R}_{\rightarrow}$ as established.

Fact. If t is a term of $\lambda\mathbf{R}_{\to}$, then t^{c} is a term of $\mathbf{CR}_{\to}$; and, if t is a term of $\mathbf{CR}_{\to}$, then t^{λ} is a term of $\lambda\mathbf{R}_{\to}$.

In $\lambda\mathbf{R}_{\to}$ we have fixed a formal representation of proofs for $\mathbf{R}_{\to}$, and we now go on to interpret this system. Intuitively, λ-terms denote functions, and we can specify a natural interpretation for each λ-term in a hierarchy of set-theoretic functions. We use "$\mathrm{X}\to\mathrm{Y}$" for the set of all functions defined on the set X which yield values in the set Y. A *standard monadic hierarchy* is a set-valued function M defined on types which satisfies the following:

(1) M_p is nonempty for each atomic type p;
(2) $\mathrm{M}_{A\to B} = \mathrm{M}_A \to \mathrm{M}_B$.

We will use "M_t" to abbreviate "$\mathrm{M}_{\mathrm{tp}(t)}$", and a similar style of abbreviation will be used for other notation later. An *assignment* s over a hierarchy M is a function defined on variables where, for each variable x, $\mathrm{s}(x) \in \mathrm{M}_x$. For $\mathrm{a} \in \mathrm{M}_x$, $\mathrm{s}^{\mathrm{a}}\!/_{x}$ is the assignment that assigns a to x and is otherwise like s. Given a hierarchy M and an assignment s over M, we define the *denotation*, $t[\mathrm{s}]$, of a λ-term t with respect to s as follows:

(1) $x[\mathrm{s}] = \mathrm{s}(x)$;
(2) $(tu)[\mathrm{s}] = t[\mathrm{s}](u[\mathrm{s}])$;
(3) $\lambda xt[\mathrm{s}] = \lambda\mathrm{a}\colon \mathrm{M}_x . t[\mathrm{s}^{\mathrm{a}}\!/_{x}]$.

The last clause uses notation for typed functional abstraction due to Scott to define $\lambda xt[\mathrm{s}]$ as the function with domain M_x whose value for each $\mathrm{a} \in \mathrm{M}_x$ is $t[\mathrm{s}^{\mathrm{a}}\!/_{x}]$.

To interpret $\lambda\mathbf{R}_{\to}$, we will define a restricted monadic hierarchy consisting of *relevant* functions. Our relevant functions will be functions whose values depend on their arguments. §70 suggests the following explication of this idea: a function is argument-dependent if it determines no value without a well-defined argument. For example, the identity function denoted by λxx is relevant. It can determine no value without a well-defined argument, since the value determined *is* the argument. On the other hand, we can be sure that the function defined by $\lambda x\lambda yy$ will yield as its value the identity function of type $\mathrm{tp}(y)\to\mathrm{tp}(y)$ without any knowledge of its argument.

Relevance is an intensional property of functions. To see whether it holds of a function, we must look beyond the function's extension, or graph, to consider its intension. If we add λ-abstraction to the usual language of arithmetic then we may define a relevant function by $\lambda x.(x-x)+2$, provided the arithmetic operations used are themselves relevant. But $\lambda x2$ will not define a relevant function even though the two functions have the same graph, yielding the value 2 for every numerical argument.

The information about the intension of a function needed to judge its relevance is of a specific sort: its behavior in the absence of a well-defined argument. At the cost of hypostatization, the relevant functions can be characterized as those which yield an undefined value when applied to an undefined argument. The undefined objects we speak of here are a strange breed, but they may also be found at the bottom of Scott's approximation lattices (see Scott 1972 for an introduction). In that setting, relevant functions are what Scott has called *strict* functions, functions which carry a bottom to a bottom. §70 explores the use of strict functions as a mathematical representation of argument dependence.

We adapt this idea here to provide a partially intensional representation of functions. We adjoin an undefined object to the domain of each type and represent functions by graphs on these extended domains. These graphs individuate functions more finely than graphs on ordinary domains but not finely enough for them to be considered their intensions. They do provide enough information for judgments of relevance to be made. If two functions, perhaps differing in intension, have the same graph on such extended domains, then each is relevant if and only if the other is.

We use this representation to define a hierarchy restricted to relevant functions. To set the restriction, the domain of each type A is supplied with an *undefined object* $\boldsymbol{u}_A$. The *defined objects* of type A form the set D_A, and the full domain M_A is $\mathrm{D}_A \cup \{\boldsymbol{u}_A\}$. A *relevant monadic hierarchy* is then a pair, $\langle \mathrm{D}, \boldsymbol{u} \rangle$, of functions defined on types that satisfy the following:

(1) D_p is nonempty for each atomic type p;
(2) $\mathrm{D}_{A\to B} = \{\mathrm{f} \in \mathrm{M}_A \to \mathrm{M}_B\colon \mathrm{f}[\mathrm{D}_A] \subseteq \mathrm{D}_B \text{ and } \mathrm{f}(\boldsymbol{u}_A) = \boldsymbol{u}_B\}$;
(3) $\boldsymbol{u}_{A\to B} = \lambda \mathrm{a}\colon \mathrm{M}_A . \boldsymbol{u}_B$.

where $\mathrm{f}[\mathrm{X}] = \{\mathrm{f}(\mathrm{a})\colon \mathrm{a} \in \mathrm{X}\}$. In each case $\mathrm{D}_{A\to B}$ consists of those functions in $\mathrm{M}_A \to \mathrm{M}_B$ which yield defined values when applied to defined objects and an undefined value when applied to an undefined object. The undefined object of type $A \to B$ is the constant function whose value for each $\mathrm{a} \in \mathrm{M}_A$ is $\boldsymbol{u}_B$. $\mathrm{M}_{A\to B}$ then consists of a class of relevant total functions together with a totally undefined function. Assignments and denotations may be defined as before, except that we allow only denotations in the restricted hierarchy so that, in general, the denotation function will be properly partial. However, we can show that a denotation is defined for each term of $\lambda\mathbf{R}$.

LEMMA 1. Suppose t is a term of $\lambda\mathbf{R}_{\to}$. Then (i) $t[\mathrm{s}] \in \mathrm{D}_t$ if there is no x free in t such that $\mathrm{s}(x) = \boldsymbol{u}_x$, and (ii) $t[\mathrm{s}] = \boldsymbol{u}_t$ if $\mathrm{s}(x) = \boldsymbol{u}_x$ for some x free in t.

PROOF. We show this by induction on the structure of t. Both (i) and (ii) are immediate in the case of variables, as is (i) in the case of application. For (ii) in that case, suppose $\mathrm{s}(x) = \boldsymbol{u}_x$ for some x free in (tu). By inductive hypothesis, both $t[\mathrm{s}]$ and $u[\mathrm{s}]$ are defined, and either $t[\mathrm{s}] = \boldsymbol{u}_t$ or $u[\mathrm{s}] = \boldsymbol{u}_u$.

So $t[\mathrm{s}](u[\mathrm{s}]) = \boldsymbol{u}_{(tu)}$, in the first case by the definition of $\boldsymbol{u}_t$ and in the second because $t[\mathrm{s}]$ is relevant. In the case of an abstract λxt, suppose first that there is no y free in λxt such that $\mathrm{s}(y) = \boldsymbol{u}_y$. We must show that $t[\mathrm{s}^{\mathrm{a}}/_{x}] \in \mathrm{D}_t$ when $\mathrm{a} \in \mathrm{D}_x$ and that $t[\mathrm{s}^{\boldsymbol{u}_x}/_{x}] = \boldsymbol{u}_t$. The first claim follows immediately from the inductive hypothesis, and the second claim follows as well, once we note that x must be free in t since λxt is a term of $\lambda\mathbf{R}_{\rightarrow}$. We must also show that, if $\mathrm{s}(y) = \boldsymbol{u}_y$ for some y free in λxt, then $t[\mathrm{s}^{\mathrm{a}}/_{x}] = \boldsymbol{u}_t$ for every $\mathrm{a} \in \mathrm{M}_x$. But if y is free in λxt, then it is distinct from x and is free in t, so $\mathrm{s}^{\mathrm{a}}/_{x}$ also assigns $\boldsymbol{u}_y$ to y and $t[\mathrm{s}^{\mathrm{a}}/_{x}] = \boldsymbol{u}_t$, by inductive hypothesis.

Although each term of $\lambda\mathbf{R}_{\rightarrow}$ will always denote some object in the restricted hierarchy, this is not true for all terms of $\lambda\mathbf{H}_{\rightarrow}$. For example, the denotation of λxy for distinct x and y is $\lambda \mathrm{a}\colon \mathrm{M}_x.\mathrm{s}(y)$. This constant function will not be in $\mathrm{M}_{\lambda xy}$ except in the special case when $\mathrm{s}(y) = \boldsymbol{u}_y$. The function defined by λxy ignores its argument entirely and, consequently, ignores the difference between a defined and an undefined argument. In fact, only terms of $\lambda\mathbf{R}_{\rightarrow}$ will have denotations in a relevant monadic hierarchy under all assignments. To show this, we first prove another lemma.

Lemma 2. If $t[\mathrm{s}] = \boldsymbol{u}_t$ then there is a variable x free in t such that $\mathrm{s}(x) = \boldsymbol{u}_x$.

Proof. To carry through an induction, we need only two remarks. First, note that if $(tu)[\mathrm{s}] = \boldsymbol{u}_{(tu)}$ then either $t[\mathrm{s}] = \boldsymbol{u}_t$ or $u[\mathrm{s}] = \boldsymbol{u}_u$. Also, if $\lambda yt[\mathrm{s}] = \boldsymbol{u}_{\lambda yt}$ then $t[\mathrm{s}^{a}/_{y}] = \boldsymbol{u}_t$ for any $a \in \mathrm{D}_y$. But then, in this case, the variable promised by the inductive hypothesis must be distinct from y and, therefore, free in λyt.

We can now go on to argue that a term that has a denotation under all assignments is a term of $\lambda\mathbf{R}_{\rightarrow}$. If t has a denotation under all assignments, all proper subterms of t must, too. So suppose that $t[\mathrm{s}] \in \mathrm{M}_t$ for all s and that each proper subterm of t is a term of $\lambda\mathbf{R}_{\rightarrow}$. We must show that t is also a term of $\lambda\mathbf{R}_{\rightarrow}$. Clearly, this is so if t is either a variable or an application. If t is an abstract λxu, then u is a term of $\lambda\mathbf{R}_{\rightarrow}$, and $u[\mathrm{s}^{\boldsymbol{u}_x}/_{x}] = \boldsymbol{u}_u$ even when s itself does not assign an undefined object to any variable. So, by Lemma 2, x must be free in u, and the abstraction forming λxu is permitted in $\lambda\mathbf{R}_{\rightarrow}$. We may combine this argument with Lemma 1 to establish the connection between relevant abstraction and relevant functions.

Theorem. A λ-term is a term of $\lambda\mathbf{R}_{\rightarrow}$ if and only if it has a denotation under all assignments in any relevant monadic hierarchy.

Another sort of model for $\lambda\mathbf{R}_{\rightarrow}$ is possible. Instead of forming a hierarchy restricted to relevant functions, we could distinguish the relevant functions within a standard monadic hierarchy whose members are individuated intensionally. We will not elaborate this approach further here, but we will consider models of this sort in the case of relevant polyadic functions.

We have seen that the problems of the relevance of proofs in pure implicational systems and the relevance of λ-terms are not merely analogous but formally identical. And perhaps there is a fundamental identity also between the problems of the relevance of implication and the relevance of functions. The terms of $\lambda\mathbf{H}_{\rightarrow}$ provide a natural beginning for a formal rendering of the intuitionistic notion of proof or construction, particularly if we think of Heyting's intuitive accounts of the intuitionistic connectives, according to which a proof of an implication $A\rightarrow B$ is a function that applies to proofs of A to yield proofs of B (see Heyting 1956, pp. 98–99). Formal interpretations of intuitionistic logic have been given along this and similar lines by Kreisel 1962 and 1965, Goodman 1970, Läuchli 1970, Scott 1970, and Martin-Löf 1975.

Even without constructivist commitments, this sort of interpretation of logic is attractive. If we have a reason for accepting an implication $A\rightarrow B$, then we may use it, together with any reason we have for accepting A, to provide a reason for accepting B. We can ascribe the possibility of this use to the nature of the reason for $A\rightarrow B$ if we regard it as a function that applies to a reason for A to yield a reason for B. An implication $A\rightarrow B$ is valid if there is a reason for it provided by logic. It is then the task of a logical theory to specify a collection of functions that serve as "logical reasons." And the central problem of relevance is the specification of the reasons for implications that are relevant—the specification of the relevant functions.

We cannot advance this as a plausible conception of the business of logic unless it provides the basis for interpreting properties of implication besides relevance and for interpreting relevance in languages richer than the pure implication fragment we have been considering. The next two sections take one step in showing the latter. As to the former, interpretations along these lines of $\mathbf{S4}_{\rightarrow\&}$, $\mathbf{E}_{\rightarrow\&}$, $\mathbf{T}_{\rightarrow\&}$, and some logics without contraction may be found in Helman 1977.

§71.3. Pairing and conjunction. The pure λ-terms and pure c-terms all denote either monadic functions or objects of atomic type, and they represent proofs in a pure implicational logic. In this section, we will enrich these languages to provide formulations of the implication and conjunction fragments of **H** and **R**. The enrichment comes with the addition of apparatus for pairing and projection. This will provide us with terms that denote polyadic functions and represent proofs involving conjunction.

The types of these richer languages are formulas generated from propositional variables using implication and conjunction. We avoid retyping the terms discussed in the last two sections by taking ourselves to be now adopting a finer grammatical analysis of the types we have been using all along. More precisely, we fix a bijective mapping from the set of propositional variables onto the set consisting of the propositional variables together with the formulas $A\&B$. This can be extended to a bijective mapping from the full set of pure implicational formulas onto the set of implication and conjunction formulas which respects implicational structure. This mapping specifies the new grammatical analysis of the old pure implicational types.

We enlarge the class of pure terms to the full class of *terms* by adding operators for pairing and for typed left and right projection, along with certain new combinators. If t and u are terms with the types A and B, respectively, then their pair

$$(t, u)$$

is a term of type $A\&B$. And if t is a term of type $A\&B$, then its left and right projections,

$\mathrm{p}t$ and $\mathrm{q}t$,

are terms of types A and B, respectively. There are three new families of combinators with types as follows:

$\boldsymbol{P}_{A,B}$	$A\&B \rightarrow A$
$\boldsymbol{Q}_{A,B}$	$A\&B \rightarrow B$
$\boldsymbol{X}_{A,B,C}$	$(A\rightarrow B)\&(A\rightarrow C) \rightarrow .\ A\rightarrow(B\&C)$

The λ-terms are the terms that contain no combinators, and the *c-terms* are the terms that do not contain operators for either abstraction or projection. Substitution for both sorts of terms is defined as before.

The interpretations of these operators and constants are again fixed by defining relations of equality and reduction. *Extensional equality* for λ-terms is defined by adding the following conditions to those of §71.1:

(7) $\mathrm{p}(t, u) = t$;
(8) $\mathrm{q}(t, u) = u$;
(9) if t and u differ at most by change of bound variables, $(\mathrm{p}t, \mathrm{q}u) = t$.

Strong reduction is defined by the same conditions less the requirement of symmetry. The conditions (7) and (8) are analogous to the condition (β) for application and abstraction, providing a similar sort of simplification. The simplification in condition (9) is analogous to that provided by the condition (η). So we now include among the *redexes* terms of the forms $\mathrm{p}(t, u)$ and $\mathrm{q}(t, u)$ and of the form $(\mathrm{p}t, \mathrm{q}u)$ where t and u differ at most by the change of bound variables, and we redefine normal forms accordingly. A normal form theorem for this wider class of λ-terms can be proved along the lines of the proofs cited in §71.1.

To define *extensional equality* for c-terms, we add the following conditions to the original group in §71.1:

(9) $(\boldsymbol{P}t, \boldsymbol{Q}t) = t$;
(10) $\boldsymbol{P}(t, u) = t$;
(11) $\boldsymbol{Q}(t, u) = u$;
(12) $\boldsymbol{X}(t, u)v = (tv, uv)$.

(9) is a principle of extensionality for pairs, and (10)–(12) provide the combinatory properties of the new combinators. We must add to the definition of combinatory abstraction the following case for pairs:

(7) $[x](t, u) \equiv \boldsymbol{X}_{\mathrm{tp}(x),\mathrm{tp}(t),\mathrm{tp}(u)}([x]t, [x]u)$ if x is free in (t, u).

We can then adopt the three cases of the definition of *combinatory translation* given for pure λ-terms, and add:

(4) $(\mathrm{p}t)^{\mathrm{c}} \equiv \boldsymbol{P}t^{\mathrm{c}}$;
(5) $(\mathrm{q}t)^{\mathrm{c}} \equiv \boldsymbol{Q}t^{\mathrm{c}}$;
(6) $(t, u)^{\mathrm{c}} \equiv (t^{\mathrm{c}}, u^{\mathrm{c}})$.

Finally, the *lambda translations* of the new combinators are:

$\boldsymbol{P}^{\lambda}_{A,B} \equiv \lambda x \mathrm{p} x$ $(x\colon A\&B)$
$\boldsymbol{Q}^{\lambda}_{A,B} \equiv \lambda x \mathrm{q} x$ $(x\colon A\&B)$
$\boldsymbol{X}^{\lambda}_{A,B,C} \equiv \lambda x \lambda y(\mathrm{p}xy, \mathrm{q}xy)$ $(x\colon (A\to B)\&(A\to C),\ y\colon A)$,

where x, y, and z are distinct and the least variables of the types indicated. A pair (t, u) of c-terms is translated by the pair $(t^{\lambda}, u^{\lambda})$ of λ-terms.

These translations can be shown to preserve equality, but for our purposes we need only observe that

FACT 1. t^{c} and t have the same types and the same free variables.

FACT 2. t^{λ} and t have the same types and the same free variables.

We will label the full classes of λ-terms and c-terms $\lambda\mathbf{H}_{\to\&}$ and $\mathbf{CH}_{\to\&}$, respectively. To see the adequacy of the formulation $\mathbf{CH}_{\to\&}$, note that the types of all combinators are theorems of $\mathbf{H}_{\to\&}$, that the operation of pairing has the same effect on types as &I, and that the types of the combinators form a sufficient set of axioms for $\mathbf{H}_{\to\&}$, given the rules →E and &I. Adding the facts above establishes the adequacy of $\lambda\mathbf{H}_{\to\&}$.

Although our primitive operations of application and abstraction are both monadic, the presence of pairing allows us to define syntactic operations of polyadic application and abstraction. First of all, we define ordered n-tuples for $n \geq 2$ by iterated pairing:

$$\langle t_1, \ldots, t_{n+1}\rangle \equiv \langle\langle t_1, \ldots, t_n\rangle, t_{n+1}\rangle.$$

The corresponding projection operations are defined by:

$$\pi^n_1 t \equiv \underbrace{\mathrm{p}\ldots\mathrm{p}}_{n-1}t$$

$$\pi^n_m t \equiv \underbrace{\mathrm{qp}\ldots\mathrm{p}}_{n-m}t,$$

where $n \geq m \geq 2$. The interpretation of these operations is exhibited by the derivable equations for $n \geq 2$ and $m \geq 1$:

$$\pi^n_m\langle t_1, \ldots, t_n\rangle = t_m$$
$$\langle \pi^n_1 t, \ldots, \pi^n_n t\rangle = t.$$

We may then express the polyadic application of a term t of type $A_1\&\ldots\&A_n\rightarrow B$ (where $n \geq 2$) to terms $u_1, \ldots, u_n$ of types $A_1, \ldots, A_n$, respectively as

$$t\langle u_1, \ldots, u_n\rangle.$$

We define the abstraction of a term t with respect to the distinct variables $x_1, \ldots, x_n$ (where $n \geq 2$) by

$$\lambda x_1, \ldots, x_n.t \equiv \lambda x.[\pi^n_1 x/x_1]\ldots[\pi^n_n x/x_n]t,$$

where x is the least variable of type $\mathrm{tp}(x_1)\&\ldots\&\mathrm{tp}(x_n)$ distinct from all variables free and bound in t. Analogues of the conditions (β) and (η) are derivable for polyadic application and abstraction.

In the above treatment, polyadic functions are represented by certain monadic functions of pairs. An even simpler representation of polyadic functions is possible and was used by both Schönfinkel and Church. Polyadic application may be handled by successive monadic application, so that a function, defined for a pair of arguments of types A and B, respectively, which takes values of type C could be represented by a function of type $A\rightarrow.B\rightarrow C$. The effect of polyadic abstraction is then obtained by successive monadic abstraction.

This representation, however, makes it difficult to distinguish the terms that denote relevant polyadic functions. One of the relevant functions will presumably be a function f that applies to objects x and y of certain types to yield their pair f(x, y). Let g be a projection function that applies to ordered pairs of this type to yield the left member; that is,

$$\mathrm{g}(\mathrm{f}(\mathrm{x}, \mathrm{y})) = \mathrm{x}.$$

And let f′ be a function that applies to objects successively to collect them into a pair:

$$\mathrm{f}'(\mathrm{x})(\mathrm{y}) = \mathrm{f}(\mathrm{x}, \mathrm{y}).$$

Where x is a relevant object of the appropriate type, let h be the composition of g and f′x; so

$$h(y) = g(f'(x, y)) = x,$$

by definition. So defined, h is irrelevant. Therefore, we cannot accept both f′ and g as relevant if the class of relevant objects is to be closed under application and composition.

Closure under application is inviolable. But Myhill 1989 suggests that both f′ and g be accepted as relevant and that we give up closure under composition. We will not pursue his suggestion here; for abandoning closure under composition would force drastic revisions in the account of relevance for monadic functions we have already given.

We must then reject either f′ or g. Pairing in any ordinary sense must be provided with a projection function; so it is f′ that we reject as irrelevant. Relevant pairing cannot be managed by successive application, and instead we have used the dyadic pairing operator (—, —). These considerations also force us to revise our criteria for relevant abstraction. If the relevant terms were closed under nonvacuous abstraction, we would have a term

$$\lambda x \lambda y(x, y),$$

which denotes the rejected function f′. The variables x and y have been collected as a couple, and, on pain of irrelevance, they may not be abstracted individually.

The same point can be made in a slightly different way. With projection, free occurrences of a variable can be redundant even in the absence of vacuous abstraction; consider, for example, the simplification

$$\mathrm{p}(x, y) \gtrsim x.$$

Free occurrence can then no longer serve as a general test for use.

The obvious alternative is to discount any variable occurrence in one half of a pair which is not appropriately matched in the other half. But there are at least two different ways this general idea might be implemented. One is to take the recursive definition of the set of variables occurring free in a term and modify the clause for pairs. We define the set of variables *strict* in a term t, st(t), as follows:

(1) $\mathrm{st}(x) = \{x\}$;
(2) $\mathrm{st}((t, u)) = \mathrm{st}(t) \cap \mathrm{st}(u)$;
(3) $\mathrm{st}((tu)) = \mathrm{st}(t) \cup \mathrm{st}(u)$;
(4) $\mathrm{st}(\lambda x t) = \mathrm{st}(t) - \{x\}$.

An abstract λxt is *strict* when x is strict in t. Strict abstraction provides one account of relevant abstraction. A polyadic version was studied by Belnap 197+ in the context of an untyped language. The class of λ-terms containing no abstracts that are not strict serves as a λ-formulation of the logic $\mathbf{U}_{\to\&}$ studied in Chidgey 197+ and Pottinger 1972 and 1979a. A corresponding Fitch system can be obtained from $\mathbf{FR}_{\to\&}$ by adding subscript deletion (see §27.2) or by using the following form of &I:

From A_{a} and B_{b} to infer $A\&B_{\mathrm{a}\cap\mathrm{b}}$.

The connection with $\lambda\mathbf{U}_{\to\&}$ lies in clause (2) above; the set of variables strict in (t, u) is the intersection of the sets of variables strict in t and u, respectively.

To get a formulation of $\mathbf{R}_{\to\&}$ instead, we might make pairing mimic the usual restricted form of &I. Urquhart 1989 suggests this. Call a pair (t, u) *even* if t and u have the same free variables. Urquhart's restricted class of λ-terms then consists of those with no uneven pairs and no vacuous abstracts.

But we can get by with a restriction on abstraction alone. We say that x is *used evenly* in t if x is free in t and no free occurrence of x in t appears in one half of a subterm (u, v) of t which has no free occurrence of x in the other half. Then λxt is *relevant* if x is used evenly in t.

The class of λ-terms that contain no abstraction that is not relevant is slightly larger than the class of terms that satisfy Urquhart's restriction. His requirement of even pairing ensures that all nonvacuous abstraction is relevant, but, by requiring only relevant abstraction, we permit uneven pairs like (x, y) for distinct x and y. However, the two classes must agree on closed terms; for a closed term containing uneven pairs will contain irrelevant abstracts. So there is little to choose between the two classes of terms as formulations of logics. We will adopt the second as the more convenient account of the relevant λ-terms and fix the class of λ-terms that contain no irrelevant abstracts as our λ-formulation $\lambda\mathbf{R}_{\to\&}$. This is analogous to the treatment of relevant implication and conjunction in natural deduction systems of Prawitz 1965.

It is easy to see that the class of c-terms not containing the combinator $\boldsymbol{K}$ provides an adequate combinatory formulation, $\mathbf{CR}_{\to\&}$, of $\mathbf{R}_{\to\&}$. To establish the adequacy of $\lambda\mathbf{R}_{\to\&}$, we must show that the combinatory and lambda translations may be restricted to $\lambda\mathbf{R}_{\to\&}$ and $\mathbf{CR}_{\to\&}$.

FACT 3. If t is a term of $\lambda\mathbf{R}_{\to\&}$ then t^{c} is a term of $\mathbf{CR}_{\to\&}$, and if t is a term of $\mathbf{CR}_{\to\&}$, then t^{λ} is a term of $\lambda\mathbf{R}_{\to\&}$.

PROOF. For the lambda translation, it suffices to note that the translations of all combinators besides $\boldsymbol{K}$ employ only relevant abstraction. For the combinatory translation, we must first show that if x is used evenly in a term t of $\mathbf{CR}_{\to\&}$ then $[x]t$ is a term of $\mathbf{CR}_{\to\&}$ and that, if $y \not\equiv x$ is used evenly in t, it is also used evenly in $[x]t$. This can be verified by an induction on t, noting for the case of pairs that a variable is used evenly in (u, v) if and only if it is used evenly in both u and v. It then follows that if t is a term of $\lambda\mathbf{R}_{\to\&}$ then

t^c is a term of $\mathbf{CR}_{\to\&}$ and if x is used evenly in t then x is also used evenly in t^c.

§71.4. Polyadic relevant functions. In models for λ-terms, the domains of type $A\&B$ will consist of set-theoretic pairs. So, recalling that what we now count as conjunctions were among the propositional variables of the first two sections, a *standard polyadic hierarchy* may be defined as a standard monadic hierarchy that meets the addition requirement:

$$\mathrm{M}_{A\&B} = \mathrm{M}_A \times \mathrm{M}_B.$$

Denotation is defined by the conditions of §71.2 together with three new ones which serve to interpret projection and pairing by the corresponding set-theoretic operations:

(4) $\mathrm{p}t[\mathrm{s}] = (t[\mathrm{s}])_0$;
(5) $\mathrm{q}t[\mathrm{s}] = (t[\mathrm{s}])_1$;
(6) $(t, u)[\mathrm{s}] = \langle t[\mathrm{s}], u[\mathrm{s}]\rangle$.

A *relevant polyadic hierarchy* is a relevant monadic hierarchy that satisfies the conditions:

(1) $\mathrm{D}_{A\&B} = \mathrm{D}_A \times \mathrm{D}_B$;
(2) $\boldsymbol{u}_{A\&B} = \langle \boldsymbol{u}_A, \boldsymbol{u}_B\rangle$.

That is, $\mathrm{M}_{A\&B}$ is limited to the totally defined pairs of $\mathrm{M}_A \times \mathrm{M}_B$ together with the totally undefined pair.

It can be shown that $\lambda\mathbf{R}_{\to\&}$ consists of the terms t that receive a denotation in any relevant polyadic hierarchy for any assignment s such that $\mathrm{s}(x) \in \mathrm{D}_x$ for each variable x free in t. The terms of $\lambda\mathbf{R}_{\to\&}$ that contain no uneven pairs are the λ-terms that receive denotations under all assignments whatsoever. Similar models can be defined for $\lambda\mathbf{U}_{\to\&}$ by allowing partial functions and pairs (see Helman 1977a).

We will not consider the proofs of these claims, but instead turn our attention to models that distinguish the relevant objects within a standard hierarchy. We will say that *relevant objects are distinguished* in a standard polyadic hierarchy M when, for each A, M_A has distinguished nonempty subsets R_A and U_A of *relevant* and *relevantly undefined* objects, respectively, which are disjoint and satisfy the following:

(1) $\mathrm{R}_{A\&B} = \mathrm{R}_A \times \mathrm{R}_B$;
(2) $\mathrm{U}_{A\&B} = \mathrm{U}_A \times \mathrm{U}_B$;
(3) $\mathrm{R}_{A\to B} = \{\mathrm{f} \in \mathrm{M}_{A\to B}\colon \mathrm{f}[\mathrm{U}_A] \subseteq \mathrm{U}_B \text{ and } \mathrm{f}[\mathrm{R}_A] \subseteq \mathrm{R}_B\}$;
(4) $\mathrm{U}_{A\to B} = \{\mathrm{f} \in \mathrm{M}_{A\to B}\colon \mathrm{f}[\mathrm{R}_A \cup \mathrm{U}_A] \subseteq \mathrm{U}_B\}$.

$\mathrm{U}_{A\to B}$ consists of the functions of $\mathrm{M}_{A\to B}$ that yield values in U_B for both relevant objects and objects in U_A. Thus they need not be totally undefined, and $U_{A\to B}$ will in general be more than a singleton. Multiple members have also been allowed in U_p, in order not to place on domains of atomic type any conditions that are not met by all domains. Members of $R_{A\to B}$ are not required to yield totally undefined values for totally undefined arguments, but are instead required to yield relevantly undefined values for all relevantly undefined arguments.

We need a definition, some observations, and a bit of new notation for the argument that the terms of $\lambda\mathbf{R}_{\to\&}$ all denote relevant objects. A set X of variables is *used evenly* in a term t if and only if some member of X is free in t, and, if any member of X has an occurrence in (u, v) that is free in t, then there are occurrences free in t of members of X in both u and v. When X is used evenly in t, its members taken together are used evenly in t, but no individual member need be. Note that if X is used evenly in λxt then $X-\{x\}$ is used evenly in t, and if, in addition, x is used evenly in t then $X\cup\{x\}$ is used evenly in t. Note also that a set X is used evenly in t just in case the set of members of X that are free in t is used evenly in t. U has been defined as a set-valued function of types. We will also use "U" for the union of the range of this function, so that $s^{-1}[U]$ is the set of variables x such that $s(x)\in U_x$.

LEMMA 1. Suppose t is a term of $\lambda\mathbf{R}_{\to\&}$ and $s(x)\in R_x\cup U_x$ for each variable x free in t. Then (i) $t[s]\in R_t$ if $s^{-1}[U]$ contains no free variables of t, and (ii) $t[s]\in U_t$ if $s^{-1}[U]$ is used evenly in t.

PROOF. Only the cases for pairs, application, and abstraction are of much interest; and (i) is obvious for the first two of these. For (ii) in the case of pairs, note that X is used evenly in (t, u) only if it is used evenly in both t and u. And, for application, note that X is used evenly in (tu) only if X is used evenly in one of t and u and, for each of t and u, X either is used evenly in the term or contains none of its free variables. By inductive hypothesis, then, $t[s]\in R_t\cup U_t$, $u[s]\in R_u\cup U_u$, and either $t[s]\in U_t$ or $u[s]\in U_u$. Thus $(tu)[s]\in U_{(tu)}$. For abstraction, we first consider (i). We must show that, if $s^{-1}[U]$ contains no free variable of λxt, then $t[s^a/_x]\in R_t$ when $a\in R_x$ and $t[s^a/_x]\in U_t$ when $a\in U_x$. In the first case, $(s^a/_x)^{-1}[U]$ contains no free variable of t. In the second, $(s^a/_x)^{-1}[U]$ contains only x among the free variables of t and is therefore used evenly in t. The inductive hypothesis then establishes (i). For (ii), suppose $s^{-1}[U]$ is used evenly in λxt. Then, if $a\in R_x$, $(s^a/_x)^{-1}[U] = s^{-1}[U]-\{x\}$ is used evenly in t; and, if $a\in U_x$, then $(s^a/_x)^{-1}[U] = s^{-1}[U]\cup\{x\}$ is used evenly in t, because x is used evenly in t. So, in either case, $t[s^a/_x]\in U_t$ by inductive hypothesis, and $\lambda xt[s]\in U_{\lambda xt}$.

All terms will receive denotations in these models under all assignments. We wish to distinguish those terms which denote relevant objects whenever relevant objects are assigned to their free variables. These will not all be terms of $\lambda\mathbf{R}_{\to\&}$. Equal terms always receive the same denotations, and $\lambda\mathbf{R}_{\to\&}$

is not closed under equality. For example, $(\lambda xt)x = t$; so $(\lambda xt)x$ will receive the same denotation as t no matter whether x is used evenly in t or not. However, $\lambda\mathbf{R}_{\rightarrow\&}$ contains all the terms in normal form that always receive relevant denotations on relevant assignments. Indeed, $\lambda\mathbf{R}_{\rightarrow\&}$ is complete in this sense for all terms without redexes of the forms $(\lambda xt)u$, $\mathrm{p}(t, u)$, and $\mathrm{q}(t, u)$. But, since all terms have normal forms, terms in normal form are all we need consider.

To prove this form of completeness, we must consider special models and assignments. It is easiest to impose the needed conditions by distinguishing special objects to be used in the assignments. We assume we are given a standard polyadic hierarchy in which relevant objects are distinguished and $\mathrm{M}_p - (\mathrm{R}_p \cup \mathrm{U}_p)$ is nonempty for all p. We fix r_p, u_p, and m_p as distinguished members of R_p, U_p, and $\mathrm{M}_p - (\mathrm{R}_p \cup \mathrm{U}_p)$, respectively. Values of r, u, and m are then defined for all types by the following:

(1) $\mathrm{r}_{A\&B} = \langle \mathrm{r}_A, \mathrm{r}_B \rangle$;
(2) $\mathrm{u}_{A\&B} = \langle \mathrm{u}_A, \mathrm{u}_B \rangle$;
(3) $\mathrm{m}_{A\&B} = \langle \mathrm{m}_A, \mathrm{m}_B \rangle$;

(4) $\mathrm{r}_{A\rightarrow B}$, $\mathrm{u}_{A\rightarrow B}$, and $\mathrm{m}_{A\rightarrow B}$ are functions whose values on M_A are given by the table below.

	R_A	U_A	$\mathrm{M}_A - (\mathrm{R}_A \cup \mathrm{U}_A)$
$\mathrm{r}_{A\rightarrow B}$	r_B	u_B	m_B
$\mathrm{u}_{A\rightarrow B}$	u_B	u_B	m_B
$\mathrm{m}_{A\rightarrow B}$	m_B	m_B	m_B

Note that if t is in normal form and is neither a pair nor an abstract and if, for each x free in t, $\mathrm{s}(x)$ is either r_x, u_x, or m_x, then $t[\mathrm{s}]$ is either r_t, u_t, or m_t. We now prove converses of clauses (i) and (ii) of Lemma 1 for terms in normal form and assignments to these special objects.

LEMMA 2. Suppose t is in normal form and, for each variable x free in t, $\mathrm{s}(x)$ is r_x, u_x, or m_x. Then, if $t[\mathrm{s}] \in \mathrm{R}_t$, $\mathrm{s}^{-1}[\mathrm{U}]$ contains no free variables of t.

PROOF. We consider cases for application and abstraction. If (tu) is in normal form, then both t and u are in normal form and t is not an abstract. So, if $(tu)[\mathrm{s}]$ is in $\mathrm{R}_{(tu)}$, $t[\mathrm{s}] = \mathrm{r}_t$ and $u[\mathrm{s}] \in \mathrm{R}_u$. So $\mathrm{s}^{-1}[\mathrm{U}]$ contains no free variables of either t or u by inductive hypothesis. For abstraction, if $\lambda xt[\mathrm{s}] \in \mathrm{R}_{\lambda xt}$ then $t[\mathrm{s}^{\mathrm{r}_x}\!/\!_x] \in \mathrm{R}_t$. And t is in normal form if λxt is. So $(\mathrm{s}^{\mathrm{r}_x}\!/\!_x)^{-1}[\mathrm{U}] = \mathrm{s}^{-1}[\mathrm{U}] - \{x\}$ contains no free variables of t by inductive hypothesis, and $\mathrm{s}^{-1}[\mathrm{U}]$ contains no free variables of λxt.

LEMMA 3. Suppose t is in normal form, and, for each variable x free in t, $\mathrm{s}(x)$ is r_x, u_x, or m_x. Then, if $t[\mathrm{s}] \in \mathrm{U}_t$, $\mathrm{s}^{-1}[\mathrm{U}]$ is used evenly in t.

PROOF. Again we consider the cases for application and abstraction. Given an application (tu) meeting the hypotheses, $t[\mathrm{s}]$ must be r_t or u_t, $u[\mathrm{s}]$ must be in R_u or U_u, and one of $t[\mathrm{s}]$ and $u[\mathrm{s}]$ must be in U. So, by Lemma 2 and the inductive hypothesis, $\mathrm{s}^{-1}[\mathrm{U}]$ is used evenly in each of t and u if any of its variables appear free in that term, and it must be used evenly in one of the two. But then it is used evenly in (tu). In the case of an abstract λxt, we must have $t[\mathrm{s}^{\mathrm{r}_x}\!/_x] \in \mathrm{U}_t$. The inductive hypothesis then implies that the variables of $\mathrm{s}^{-1}[\mathrm{U}]$ other than x are together used evenly in t, so $\mathrm{s}^{-1}[\mathrm{U}]$ itself is used evenly in λxt.

We are now able to specify an assignment that singles out the terms $\lambda\mathbf{R}_{\rightarrow\&}$ from among the λ-terms in normal form.

LEMMA 4. Consider a standard polyadic hierarchy in which relevant objects are distinguished and $\mathrm{M}_p - (\mathrm{R}_p \cup \mathrm{U}_p)$ is nonempty for each p. There is an assignment s on this hierarchy such that $\mathrm{s}(x) \in \mathrm{R}_x$ for each variable x and, for each term t in normal form, $t[\mathrm{s}] \in \mathrm{R}_t$ only if t is a term of $\lambda\mathbf{R}_{\rightarrow\&}$.

PROOF. We define the objects r_A, u_A, and m_A and let s be the assignment of r_x to each variable x. It can then be shown that if t is in normal form and $t[\mathrm{s}] \in \mathrm{R}_t$, then t is a term of $\lambda\mathbf{R}_{\rightarrow\&}$. We consider only the case of abstracts. If $\lambda xu \in R_{\lambda xu}$, then $u[\mathrm{s}^{\mathrm{r}_x}\!/_x] \in \mathrm{R}_u$ and $u[\mathrm{s}^{\mathrm{u}_x}\!/_x] \in \mathrm{U}_x$. But then u is a term of $\lambda\mathbf{R}_{\rightarrow\&}$, by inductive hypothesis, and x is used evenly in u, by Lemma 3.

Lemmas 1 and 4 combine to provide the characterization of the terms of $\lambda\mathbf{R}_{\rightarrow\&}$.

THEOREM. Suppose t is in normal form. Then t is a term of $\lambda\mathbf{R}_{\rightarrow\&}$ if and only if, for each standard polyadic hierarchy that distinguishes relevant objects, $t[\mathrm{s}] \in \mathrm{R}_t$ whenever s assigns relevant objects to all free variables of t.

§72. Relevant Peano arithmetic. As Peirce, Russell, etc., pointed out, Aristotle's logic of the *Prior Analytic* is good for *sorting* or *classifying* things, but it was reserved for modern logic to give us a rigorous language for *relating* things. Ironically, though not surprisingly, the contemporary study of arithmetic is in this sense largely Aristotelian: we are given ways of *sorting* arithmetic statements into those which are provable and those which are not, while little attention is paid to how arithmetic statements are *related*. Relevance logic gives us the means to bring arithmetic up to date in this regard by permitting us to make statements about whether or not certain arithmetic propositions are relevantly connected.

It is chiefly Meyer who has pursued this possibility. His idea has been to axiomatize Peano arithmetic in the context of $\mathbf{R}^{\forall\exists x}$, putting the arrow of **R** where a real connection is wanted, and a mere truth-functional horseshoe where mere sorting suffices. It turns out that the decision as to where to put the arrows is far from arbitrary, and that there can be definite intuitions about the matter, as we shall see.

Meyer's work is described extensively in the unpublished papers 197+a, 197+b, and briefly in the published paper 1976e. Some joint work is reported in Meyer and Mortensen 1984 and Meyer and Urbas 1986. In this section we explore and comment on the Meyer development—all the technical results reported in this section are due to Meyer, in a few instances jointly with his collaborators—and in §73 we present the outline of a variant theory.

§72.1. Postulates for relevant Peano arithmetic. Peano arithmetic begins with Dedekind in 1888; it was put into new and elegant notation by Peano in 1889 and was cast into first-order form by Gödel 1931 (see Hodges 1983). These postulates are generally thought of as governing three ideas, *natural number*, *zero*, and the *successor* operation, and as articulating with some precision the fact that the natural numbers just are those things which can be generated from zero by the operation of taking the successor (that is, the next) number. Using "0" for zero and "x'" for the successor of x, but leaving "number" in place for the nonnegative integers 0, 1, . . . (Dedekind actually started with 1), we might put the postulates as follows:

> 0 is a number
>
> Successors of numbers are numbers: if x is a number, then x' is a number.
>
> The successor operation is one-one on the numbers: if x and y are numbers such that $x' = y'$, then $x = y$.
>
> 0 is not the successor of any number: it is false that, for some number x, $0 = x'$.
>
> Induction. Where "$A[x]$" occupies the place of a condition involving the given nonlogical vocabulary and identity and other notation of $\mathbf{TV}^{\forall\exists x}$, if (1) $A[0]$ and if also (2) for every number x, if $A[x]$ then $A[x']$, then (3) for every number x, $A[x]$.

Since these postulates use identity, something must be said about that topic. That is, as background one needs to assume not only $\mathbf{TV}^{\forall\exists x}$ but also some postulates for identity, say the principles of reflexivity and replacement:

> Reflexivity: $x = x$

> *Replacement: where* "$A[x]$" occupies the place of a condition involving the nonlogical vocabulary and identity and other notation of $\mathbf{TV}^{\forall\exists x}$: if $y=z$ and $A[y]$ then $A[z]$ (with the usual precautions about collision of variables).

There are also some axioms having the form of identities that explain addition and multiplication when they (that is, $x+y$ and $x\times y$) are added to the vocabulary: we won't list those axioms here, because they are exactly the same as those presented below.

Meyer's program begins not with these postulates themselves, but with a frequently made classical adaptation that relies on thinking of the domain of quantification as exhausted by the numbers. This adaptation is made in the classical context just to simplify the statement of the postulates, since then one does not have to worry about the work of "is a number." Another frequent adaptation is the explicit restriction of the nonlogical vocabulary to that of arithmetic; this permits the deduction of all the required properties of identity from a couple of specific axioms, namely, axioms that say that replacement of identicals works for the identity context itself and for the successor context.

When these two adaptations are combined, we obtain a particular formalization of Peano arithmetic, $\mathbf{P}^\sharp$, based on a two-place relation predicate, $x=y$, for identity; two-place operators $x+y$ and $x\times y$ for addition and multiplication; a one-place operator, x', for successor; and a constant, 0, for zero. This arithmetical equipment is to be added to the variables, quantifiers, and connectives of $\mathbf{TV}^{\forall\exists x}$.

For postulates carrying out the two adaptations of the original Dedekind idea, the following nine are frequent:

R♯1 $x=y \rightarrow x'=y'$
R♯2 $x=y \rightarrow. (x=y)\rightarrow(y=z)$
R♯3 $x+0=x$
R♯4 $x+y'=(x+y)'$
R♯5 $x\times 0=0$
R♯6 $x\times y'=(x\times y)+x$
R♯7 $x'=y' \rightarrow x=y$
R♯8 $\sim(x'=0)$
R♯9 $(A[0]\ \&\ \forall x(A[x]\rightarrow A[x'])) \rightarrow \forall x A[x]$.

The numbering system has to do with the fact that in a moment we are going to use R♯1–R♯9 as relevant postulates, but here we are writing $A\rightarrow B$ for the material "implication" $\sim A\vee B$. With that understanding, all but the first two, i.e., those for identity, are to be found in precisely this form in standard presentations such as that of Hodges 1983. The system $\mathbf{P}^\sharp$ is defined as the result of adding the universal closures of the specific axioms R♯1–R♯8, and also adding the universal closures of the instances of the schema R#9, to the axioms of $\mathbf{TV}^{\forall\exists x}$ (recalling that these are closed under universal quantifier introduction), and taking the rules as *modus ponens* for the arrow and, redundantly, conjunction introduction.

This preprocessing of a long and complicated history now permits us to give a simple definition of *relevant* Peano arithmetic: just take exactly these same axioms R#1–R#9—but now let the arrow be the primitive relevant connective of **R**—and add them to $\mathbf{R}^{\forall\exists x}$ (instead of to $\mathbf{TV}^{\forall\exists x}$) restricted to the given arithmetical idiom. The rules remain as *modus ponens* for the arrow and conjunction introduction, but of course the latter is not redundant in this context. This defines the system $\mathbf{R}^\sharp$ of *relevant Peano arithmetic.* (We have presented the axiomatization of $\mathbf{R}^\sharp$ of Meyer and Mortensen 1984.)

§72.2. Strength and weakness of the extensional fragment. Although we think that $\mathbf{R}^\sharp$ is most interesting for what it allows us to say (and not say) with arrows, it is proper to compare its deliverances in its zero degree or extensional fragment with those of $\mathbf{P}^\sharp$. We do that in this section by means of a somewhat disjointed series of facts and comments. The upshot is that $\mathbf{R}^\sharp$ is strong enough to prove a host of arithmetical truths formulated in the purely extensional language of $\mathbf{P}^\sharp$, but is provably not too strong in this regard.

Axioms of $\mathbf{P}^\sharp$. One can prove in $\mathbf{R}^\sharp$ all the standard *axioms* of $\mathbf{P}^\sharp$, including all instances (in the language of $\mathbf{P}^\sharp$) of the induction schema of $\mathbf{P}^\sharp$. From this it does not follow that all of $\mathbf{P}^\sharp$ is available in $\mathbf{R}^\sharp$, because one cannot use the standard *rule* of $\mathbf{P}^\sharp$, namely, detachment for material "implication," which, as we know, is nothing but a beastly *consequentia canina* (§25.1).

Kleene-completeness. The system is nevertheless strong enough to prove retail (one by one) all the elementary extensional arithmetic facts—facts in the language of $\mathbf{P}^\sharp$—that are explicitly established in Kleene 1952 (which are many). We may therefore say that the system is *Kleene-complete*; this is not, however, all there is to it: we can also be sure that $\mathbf{R}^\sharp$ has provided *relevant proofs* of these facts, proofs that do not rely on mechanisms in classical logic that ignore relevance in the way that some classical proofs do. It is obvious that our proofs in $\mathbf{R}^\sharp$, though of facts stated extensionally, must involve the arrow of relevant implication. Such proofs reveal for inspection the relevant structure of arithmetic, even when their conclusions are purely extensional; they give us information that we did not have before.

Limited Relative Completeness. There are some wholesale relative-completeness facts. Without making any effort to sort through the many results of Meyer 197+a and 197+b, we mention the following result, which we shall refer to later: if (where A, B, are in the language of $\mathbf{P}^\sharp$) A has the property that one can prove $\sim A\rightarrow(0=0)$ in $\mathbf{R}^\sharp$ and if $A\vee B$ is provable in $\mathbf{P}^\sharp$ then $A\vee B$ is also provable in $\mathbf{R}^\sharp$.

Problem: Does $\mathbf{R}^\sharp$ *contain* $\mathbf{P}^\sharp$ *in toto*? It is, however, not known whether all formulas in the language of $\mathbf{P}^\sharp$ are provable in $\mathbf{R}^\sharp$ if they are provable in $\mathbf{P}^\sharp$. (Certainly not every formula of $\mathbf{P}^\sharp$ is such that its negation provably implies that $0 = 0$.) Either answer would be interesting; in particular, the aptness or value of $\mathbf{R}^\sharp$ does not depend on its being as complete as $\mathbf{P}^\sharp$, any more than the value of $\mathbf{P}^\sharp$ depends on a completeness that it does not possess. In fact, our view is that the chief interest of $\mathbf{R}^\sharp$ lies not in new proofs for old (extensional) theorems, though that is interesting enough, but rather in its explicitly relevant-implicational part, where intensional relations between arithmetical propositions can be properly expressed. Nevertheless, the question of whether $\mathbf{R}^\sharp$ contains $\mathbf{P}^\sharp$ remains an interesting open problem.

$\mathbf{R}^\sharp$ *is not negation complete.* $\mathbf{R}^\sharp$ is obviously no more negation-complete than is $\mathbf{P}^\sharp$, by Gödel's incompleteness theorem.

For the next results, keep in mind the distinction between *absolute consistency* (that is, unprovability of some formula) and *negation-consistency* (that is, never both A and $\sim A$ provable); in two-valued logic these go together, but certainly not in relevance logic.

Negation-consistency of $\mathbf{R}^\sharp$ *by transfinite induction.* One already knows, by a nonelementary argument, that $\mathbf{R}^\sharp$ is negation-consistent; for we have the proof of Gentzen 1936, using transfinite ordinal induction, that $\mathbf{P}^\sharp$ is negation-consistent, and evidently $\mathbf{R}^\sharp$ is a subsystem of $\mathbf{P}^\sharp$ with the arrow taken as material "implication."

Absolute consistency of $\mathbf{R}^\sharp$ *by elementary argument* (I). But for $\mathbf{R}^\sharp$ there are in addition elementary, arithmetical proofs that, independent of its relation to $\mathbf{P}^\sharp$, the system is absolutely consistent even with respect to its extensional formulas; for instance, $0 = 1$ is not provable. The proof of this in Meyer 1976e and 197+a and 197+b involves combining (i) a three-valued propositional point of view with (ii) a two-element ontological point of view. For (ii), interpret the terms as denoting just 0 or 1, and take the arithmetic operators modulo 2 ("circle arithmetic" or "clock arithmetic"). For (i), observe that modular considerations plausibly give rise to just three different propositions: $0=1$ is a "pure falsehood," since it is not equivalent modulo 2 to anything but falsehood; $\sim(0=1)$ is a "pure truth," since it is not equivalent modulo 2 to anything but truths; but $0=0$ is of "mixed status," since it is equivalent modulo 2 to some truths (e.g., itself) and some falsehoods (e.g., $0=2$). (Observe that $\sim(0=0)$ has the same mixed status as $0=0$.) This approach justifies interpreting $\mathbf{R}^\sharp$ as propositionally three-valued; it turns out that the logic **RM3** ("three-valued mingle") defined in §29.12 hits the mark and that, if we count the values associated with $\sim(0=1)$ and $0=0$ (and so also $\sim(0=0)$) as designated, then an elementary argument verifies all the theorems of $\mathbf{R}^\sharp$ (as well, it turns out, as the negations of some of its theorems), while ruling out $0=1$ as taking an undesignated value. The general situation envisaged is discussed in Dunn 1979.

Absolute consistency of $\mathbf{R}^\#$ *by elementary argument* (II). The second proof of the same absolute consistency is essentially in Meyer and Urbas 1986. The underlying lemma, proved in that paper, is that $\mathbf{R}^\#$ is a conservative extension of its positive fragment. That positive fragment (which does not include axiom R#8) is evidently interpretable in (ordinary nonrelevant) modular arithmetic, for any modulus you like. Therefore, any positive statement that has a counterexample in some modular arithmetic, such as $0=1$, is not provable in $\mathbf{R}^\#$. There are some related results in Meyer and Mortensen 1984.

Consistency proofs of $\mathbf{R}^\#$ *vis-à-vis Gödel.* One should also observe that the elementary arguments indicated above have as their common conclusion only absolute consistency (the unprovability of something or other) and not negation consistency (the unprovability of contradictions); in relevance logic the other does not of course follow from the one. This explains the particular way in which these arguments fail to pose a counterexample to Gödel's work, which certainly implies that $\mathbf{R}^\#$ cannot prove its own negation consistency (if it is negation-consistent).

Disjunctive syllogism admissible in $\mathbf{R}^\#$? The question, posed above, of whether $\mathbf{R}^\#$ contains all of $\mathbf{P}^\#$ comes down to whether Ackermann's rule (γ), the disjunctive syllogism, about which we have had much to say elsewhere (especially in §25), is admissible in $\mathbf{R}^\#$. The fact that (γ) is analogous to cut or the Elimination rule (see §7.2) and the fact that cut-free formulations of arithmetic are well known to be impossible, do not jointly necessitate the inadmissibility of (γ). At least the usual proofs regarding the inadmissibility of cut, e.g., that if cut were admissible then the Ackermann function (see, e.g., Rogers 1967, p. 8) would be primitive recursive, seem to have no clear analogue in the relevant arithmetic. For, even without (γ), proofs can detour through more complex formulas by way of detachment for the *relevant* implication of $\mathbf{R}^\#$.

One point seems worth making explicit as a consequence of the following two items each previously noted. (1) We have a proof of the absolute consistency of $\mathbf{R}^\#$ by *elementary* means, that is, means that do not outrun arithmetic itself. (2) We cannot, by Gödel, have a proof of the negation-consistency of $\mathbf{R}^\#$ by elementary means. Therefore, since it is elementary that if a system admits (γ) and admits disjunction introduction (from A to infer $A \vee B$) and is absolutely consistent then it is negation-consistent, it must be that an argument for the admissibility of (γ), if there had been one, would have been nonelementary, i.e., an argument that relies on transfinite induction but is otherwise "constructive," like Gentzen's proof of the negation consistency of $\mathbf{P}^*$.

This section, both the part preceding and the part following this paragraph, was written before Meyer and Friedman showed that (γ) is not admissible for $\mathbf{R}^\#$: Meyer showed that if (γ) were admissible for $\mathbf{R}^\#$, then all negation-free theorems of $\mathbf{P}^\#$ would be provable in $\mathbf{P}^\#$ without using the axiom $\sim(x' = 0)$, and Friedman showed that there is a counterexample to this last. These results may be found in Meyer and Friedman 1988 and Friedman 1988. As a consequence, we have rewritten bits of this section, which we take still to serve some purpose, in the subjunctive mood. We remark in support of this that the open question concerning relevant arithmetic that now seems of fundamental interest is whether there are some natural axioms to add to $\mathbf{R}^\#$ that might then render (γ) admissible. Our preceding and following discussion, originally intended to pertain to $\mathbf{R}^\#$, would apply equally to such a system, should it exist.

What if there had been a proof of (γ) for $\mathbf{R}^\#$? We say something in §80 about the meaning of this situation for a "relevantist," as there described; here we think for a minute about the classical mathematician, and imagine convincing such a one of the wisdom of using $\mathbf{R}^\#$ in place of $\mathbf{P}^\#$ by arguing in the following "Pascal's wager" sort of way.

> Look. You have equally good reason to believe in the negation-consistency of $\mathbf{P}^\#$ and in the (relative) completeness of $\mathbf{R}^\#$. In both cases you have a nonelementary proof that secures your belief, but that might be mistaken. Consider the consequences in each case if it is mistaken. If you are using $\mathbf{P}^\#$, disaster! Since even one contradiction classically "implies" everything, it follows that, for each theorem you have proved, you might just as well have proved its negation. But, if you are using $\mathbf{R}^\#$, things are not so bad. For at least large classes of sentences, it can be shown by elementary methods due to Meyer 1976e that not both the sentences and their negations are theorems.

§72.3. Relevant implications or material "implications"? What expressive powers does the relevant arrow of $\mathbf{R}^\#$ add to the extensional vocabulary of arithmetic? Is there any sense to choosing between relevant implication and material "implication" in expressing propositions of arithmetic?

We begin by observing that the underlying logic of $\mathbf{R}^\#$ is **R** rather than **E** or **T**; this means that the modal distinctions available in **E** or the ticket-fact distinctions of **T** have as yet found no use in thinking about arithmetic, a situation that one may take to be entirely reasonable without having an opinion as to fruitful lines of future research. To proceed, it will be helpful to discuss the various axioms R#1–9 of $\mathbf{R}^\#$. We will assume for this discussion that the reader shares with us some intuitions about relevant implications, but that he or she does not yet have any views about what—in arithmetic—is relevant to what. We work backwards through the postulates of §72.1.

R#9. We comment on three variations. (1) If *both* arrows displayed in R#9 of §72.1 are converted to material "implications," the result remains true to our intentions. A proof of this version for the special case when $A[x]$ is in the language of $\mathbf{P}^\sharp$ can be obtained by way of the Limited Relative Completeness of §72.2, because either $\sim A[0]$ or $A[x]$ is bound to have the property there indicated. Even though this material "implication" version is at least partly available as a theorem of $\mathbf{R}^\sharp$, however, it would not be of much use as an *axiom* of some (other) relevant theory, because of course one could not, having established its material "antecedent," infer to its material "consequent." (2) If, instead, the inductive arrow is made a material "implication" while the main connective remains an arrow, the result would clearly look too much like the forbidden $(A\&(A\supset B))\rightarrow B$ to be plausible, as indicated by very little jotting. (3) More subtle and thereby more revealing of the stability of our intuitions is the question of the merit of the classically equivalent exported version:

$$(1)\qquad A[0] \rightarrow .\ \forall x(A[\mathrm{x}]\rightarrow A[x'])\rightarrow\forall x A[x],$$

or, what is relevantly equivalent to (1) by permutation,

$$(2)\qquad \forall x(A[\mathrm{x}]\rightarrow A[x']) \rightarrow .\ A[0]\rightarrow\forall x A[x].$$

If one imagines universal quantifications as large conjunctions then R♯9 falls apart from these exported versions (1) and (2), as revealed by the difference between the following:

$$(3)\qquad \mathrm{F0}\ \&\ (\mathrm{F0}\rightarrow\mathrm{F1}\ \&\ \mathrm{F1}\rightarrow\mathrm{F2}\ \&\ \mathrm{F2}\rightarrow\mathrm{F3}\ \&\ \ldots)\rightarrow (\mathrm{F0}\&\mathrm{F1}\&\mathrm{F3}\&\ldots)$$

$$(4)\qquad (\mathrm{F0}\rightarrow\mathrm{F1}\ \&\ \mathrm{F1}\rightarrow\mathrm{F2}\ \&\ \mathrm{F2}\rightarrow\mathrm{F3}\ \&\ \ldots)\rightarrow (\mathrm{F0}\rightarrow\mathrm{F0}\ \&\ \mathrm{F0}\rightarrow\mathrm{F1}\ \&\ \mathrm{F0}\rightarrow\mathrm{F3}\ \&\ \ldots)$$

It is (3) that corresponds to R♯9, and (3) holds up nicely under "propositional inspection," but (4), which corresponds to (2), fails with respect to just one tiny conjunct: F0→F0 is doubtless true, but it does not in general follow in any remotely relevant fashion from (F0→F1 & F1→F2 & F2→F3 & ...). The conclusion is that the choice of R♯9 for the inductive postulate of $\mathbf{R}^\sharp$ is *interesting* because (i) that choice is from among an array of competitors all "equivalent" to R♯9 on merely truth-functional grounds, and (ii) that choice can be seen to be not at all arbitrary but instead grounded in an appeal to relevance considerations that even the unsympathetic can see as stable—in just the way that even a classical mathematician can sometimes see the difference between a constructive and a nonconstructive argument.

R♯8 is the only axiom featuring negation explicitly (it is of course important that negation can occur in the instances of R♯9). Aside from that, because R♯8 lies wholly within the extensional vocabulary, its sole and adequate justification is that it is a familiar truth of arithmetic, and one of Peano's own postulates to boot. The right question to ask here is not "Is it true?" but "What follows from it?"

R♯7 is an especially creative postulate, affirming as it does that there is a tight relevant connection as one passes *down* the constructed hierarchy of integers—an affirmation that, at least so far, has not been based on relevance insights such as those justifying R♯9. This intensional statement of the one-one character of the successor function has hidden consequences for what is relevant to what, as we shall see. Because there are other settings in which the analogue to R♯7 stands out as a source of power (it is often an axiom of infinity), perhaps one should not be surprised here.

R♯♯3–6 merit the same remarks as R♯8, except, of course, that R♯3–6 are all positive.

R♯2 (with a little help) gives reflexivity and symmetry of identity (ho hum). One might well wish to investigate theories based on a weakening of this axiom, for example, its "imported" version, $((x=y)\&(x=z))\rightarrow(y=z)$, but it seems certain that the strength of R♯2 is essential to the development being presently reported. One needs to say that there is little of a theoretical nature guiding our current understanding of the interaction of relevance and identity; so multiple programs are doubtless called for.

R♯1 is perhaps more "arithmetical" and less "logical" than one might at first suppose. Certainly there are verbal formulas that one might express by means of one-place operators that would make an analogue to R♯1 look most peculiar; for example, read "x'" as "the proposition that $Fx\&p$," for suitably chosen predicate F and irrelevant sentence p. It looks as if an analogue to R♯1 might then lead us to say that

$$x=y \rightarrow (Fy\&p)\rightarrow p.$$

which, although it has a true consequent, would be a clear fallacy of relevance. That is, to say that R♯1 holds is or might be to say something special about the successor function, something true because of arithmetic and not just because of logic. R♯1 says that the generation of the integers by the successor function is relevant.

So much for the axioms; we briefly comment on a few of their arrow-containing consequences and nonconsequences just to engender a feel for the theory.

The identity axiom R♯2 instantiates to $x=y\rightarrow.x=y\rightarrow y=y$, and accordingly yields

$$(5) \qquad x=y \rightarrow y=y$$

by contraction. It is hard to know whether to think of (5) as a cogent proof of a surprising relevant connection or as casting doubt on R♯2. One needs some independent reflections on identity in order to increase clarity on the matter.

In contrast to (5), it takes nearly all the axioms together to give replacement properties of identity in the form of relevant implications. Of particular note is the following special and vacuous instance of replacement:

(6a) $y=y \rightarrow (0=0)$,

which is established *not* trivially through truth of consequent (observe that the main connective is relevant), but instead in an inductive way that depends essentially on the intended exhaustion of the domain by the integers and on how that domain is structured by the successor function. One easily has by another induction that

(6b) $0=0 \rightarrow z=z$.

Combining (5), (6a), and (6b) gives

(7) $x=y \rightarrow z=z$,

which is perhaps even more jarring to the untrained eye—and even more obviously *not* a truth about relevant connections between identities in general, but instead a special arithmetic fact.

An interesting distinction is that between

(8) $0=1 \rightarrow 0=2$

and its converse

(9) $0=2 \rightarrow 0=1$.

Here we use "1" for "$0'$" and "2" for "$0''$". Certainly (8) and (9) do not differ in the pattern of the truth values of their components or even in the modal pattern (so to speak) of their components, but it is not so difficult to see how to argue *relevantly* from the antecedent to the consequent of (8), using the intuitions codified in the relevant Peano axioms for the successor function: suppose $0=1$; then $1=2$, by applying the successor function to both sides (R#1); so, by symmetry and transitivity, $0=2$. In contrast, there is nowhere to go from the supposition that $0=2$. In an ordinary extensional context, one could perhaps argue by dividing both sides of $0=2$ by 1 in order to obtain $0=1$, but one rapidly sees that such an argument is much too fast in a relevant context, invoking as it does a division function, a function that might or might not exist. After all, one knows that even in extensional logic there is no "everywhere-defined division function" having the property that $(a/x)=b$ if and only if $a=(b\times x)$, because of the problem when $x=0$. One has only a conditionally defined division function, that is, *if* $x\neq 0$ *then* $(a/x)=b$ if and only if $a=(b\times x)$, and a little experimentation suggests that there is trouble in interpreting the "if," given that one wants the "if and only if" to be relevant; so it may not be taken for granted that there is a division function having the properties required to make the suggested argument for (9) go through. In general, one soon sees that one must pay attention to "functions that really depend on their arguments" in the sense of §§70–71.

Of course one has (9) as a material "implication," by falsity of antecedent; Meyer's work contains a number of such examples, that is, examples where the material "implication" version of an "if" is provable in R# while the relevant version is not (and arguably not wanted); here are just a few:

If $(x+y)=0$ then $(x=0)\&(y=0)$.
If $(x\times y)=0$ then $(x=0)\vee(y=0)$.
If $\exists y(x+y=z)\&\exists y(z+y=x)$ then $x=z$.
If $x\neq 0$ then $\exists y(x=y')$.

The last example is of special interest, because later we look carefully at what happens when we endeavor to develop an arithmetic on the basis of taking the "if" of this last example as relevant. It turns out in §73 that, by making appropriate adjustments, an alternative interesting theory of arithmetical relevant connections can be developed.

Particularly revealing is the implicational role of what to early formalizers of arithmetic was a paradigm of arithmetic falsehood, $0=1$. One can prove that $0=1$ serves as an "absurdity" amid the extensional fragment in the sense that it relevantly implies all formulas in that fragment:

$$0=1\rightarrow A,$$

where A is an extensional formula (there is also another formula that relevantly implies all arithmetic formulas, even those with arrows). The contrast with the treatment of $0=1$ by the intuitionists is striking; for one obtains the absurdity of $0=1$ in intuitionist arithmetic not by working it out on the basis of constructive insights, but instead by just postulating it.

§72.4. Oddments. Here we set forth a few odd bits concerning relevant Peano arithmetic.

1. Meyer's work defines not only $\mathbf{R}^\sharp$ as in §72.1, but also $\mathbf{R}^{\sharp\sharp}$, which replaces R#9, that is, finite induction, with a rule of infinite induction: from all the numerical instances $A[0]$, $A[0']$, . . . , of $A[x]$, to infer $\forall x A[x]$. There is substantial information about $\mathbf{R}^{\sharp\sharp}$, including the fact that its set of theorems is closed under the disjunctive syllogism (recall from §72.2 that this is in contrast to $\mathbf{R}^\sharp$).

2. $\mathbf{R}^\sharp$ is a relevant theory about 0 and the successor function under the assumption that the universe of quantification is exhausted by the numbers. In contrast, the historical Dedekind-Peano theory that we described in §71.1 is a theory about three concepts, 0, successor, and *number*, without any exhaustion assumption about the universe. For example, using "N[x]" for "x

is a nonnegative integer," two of Peano's postulates were

N[0]

and

If N[x] then N[x'].

The problem arises in articulating arithmetical propositions of the type "All numbers have such and such a property," since neither material "implication" nor relevant implication can be counted on to give just the right results in every circumstance. One does not encounter this problem if the universe is assumed to be exhausted by the numbers, but without some such assumption, the theory of $\mathbf{R}^{\#}$ does not appear automatically to suggest a uniquely plausible theory of arithmetic taken as the theory of a special subdomain of a larger domain of inquiry.

3. A restricted quantification based on the conditional-assertion connective of §75 may solve or assist in solving the problem just raised.

4. Essentially due to Frege and heavily used by Russell is the construal of nonnegative integers as properties of properties (so that 0 is the property of being an empty property, 1 is the property of being a unit property, etc.). In classical logic this property-of-properties construal at least partially verifies the Peano postulates, and the Peano postulates guide our working out of the construal. Further, Bressan 1972 shows how this construal becomes surprisingly interesting in the context of a well-designed quantified modal logic. Can there be any cross-fertilization between this or a similar property-of-properties construal on the one hand and something like $\mathbf{R}^{\#}$ on the other?

§73. Relevant Robinson arithmetic. Robinson's system **Q** (see, for example, Boolos and Jeffrey 1974) is a finitely axiomatized subsystem of Peano arithmetic, $\mathbf{P}^{\#}$, famous for the fact that all the recursive functions can be represented in it even though its axioms are so few. Its principal difference from Peano arithmetic, $\mathbf{P}^{\#}$, lies in the replacement of the infinitely many instances of the induction scheme by a single axiom that says that every nonzero natural number has a predecessor.

In §72 (which should be read before this section) we built a relevant arithmetic on the basis of Peano arithmetic. What happens when, in fashioning a relevant arithmetic, we start with Robinson's system **Q** instead of Peano's $\mathbf{P}^{\#}$? We shall see that an apparently sensible way of carrying on from this start causes all relevance distinctions to evaporate, but we shall also see that, if we make appropriate adjustments, we can find an interesting alternative to $\mathbf{R}^{\#}$ possessed of a considerable amount of internal consistency of motivation.

§73.1. Robinson's axioms. To be explicit, Robinson's axioms are the universal closures of the following:

I. 1. $x=x$ 2. $x+0=x$ 3. $x+y'=(x+y)'$
4. $x\times 0=0$ 5. $x\times y'=(x\times y)+x$
6. $x=y \rightarrow y=x$ 7. $x=y \rightarrow x'=y'$
II. 8. $x'=y' \rightarrow x=y$ 9. $x=y \rightarrow (x+z=y+z)\&(z+x=z+y)$
10. $x=y \rightarrow (x\times z=y\times z)\&(z\times x=z\times y)$
III. 11. $x=y \rightarrow .\ y=z\rightarrow x=z$
IV. 12. $0\neq x'$
V. 13. $x\neq 0 \rightarrow \exists y(x=y')$

Robinson's systems **Q** results from adding these axioms to some complete set of axioms for classical first-order logic (thinking of $A\rightarrow B$ as defined by $\sim A\vee B$) such as $\mathbf{TV}^{\forall\exists x}$. It is natural then to consider the relevant version $\mathbf{Q_R}$ of **Q**, which results from adding these axioms (but this time with the arrow as relevant implication) to $\mathbf{R}^{\forall\exists x}$—restricted to the given arithmetic idiom, with rules, as always, *modus ponens* for relevant implication and introduction for conjunction.

§73.2. $\mathbf{Q_R}$ = Q. For the most part the relevant mate $\mathbf{Q_R}$ of Robinson's **Q** is weaker than the relevant mate $\mathbf{R}^{\#}$ of Peano's $\mathbf{P}^{\#}$, but axiom 13 signifies a difference: as we noted in §72.3, one has in $\mathbf{R}^{\#}$ only the material "implication" version of axiom 13, not the relevant version; only an extensional disjunction

13′. $(x=0)\vee\exists y(x=y')$

but not a relevant connection is affirmed in $\mathbf{R}^{\#}$. One can marshall naive or not so naive intuitions against axiom 13, and in particular one can see that one should never try to erect a plausibility argument for it on the basis of 13′; relevance considerations assist one to a point of view from which evidence for 13′ can be seen not to count for the relevant connection affirmed for 13. One can nevertheless also appreciate the *virtue* that axiom 13 may acquire in going *further* than 13′ in the direction of saying that every number is "connected to 0," or "is obtained from 0 by applications of the successor function," or some such thing. Its equivalent contrapositive says that if x is distinct from every successor then, speaking relevantly, x must be 0—not because of 13′, which would be a bad reason for such a claim, but just because in fact there is a relevant connection between the distinctness of x from all successors and its identity to 0. For example, and speaking subjunctively, if 14 were distinct from every successor, then that would be a relevantly conclusive reason for concluding that 14 would be 0. We add that axiom 13 seems to play a critical role in "relevantly representing" recursive functions, a subject that we do not explore in this book, but mention again briefly at the end of this section.

But, virtue or not, $\mathbf{Q_R}$ as it stands is too strong to be interesting: $\mathbf{Q_R}$ does not permit a difference between relevant implication as expressed by the arrow and material "implication" as carried by $\sim A \vee B$ and so is useless for its intended purpose of permitting the expression of interesting nonextensional relevant connections between arithmetic propositions. What is useful, however, is to see exactly how relevant implication in $\mathbf{Q_R}$ collapses into material "implication," and we shall next establish the following.

THEOREM. In $\mathbf{Q_R}$ one has $A \rightarrow . \sim A \rightarrow B$ as an inescapable theorem.

From the theorem we easily derive by double negation, contraposition, and elementary properties of disjunction (all available in **R**) the corollary that one has in $\mathbf{Q_R}$ the collapse of relevant implication into material "implication:"

$$(\sim A \vee B) \rightleftarrows (A \rightarrow B).$$

The theorem itself we obtain by a series of little results concerning $\mathbf{Q_R}$. Then, after proving the theorem, we show in §73.3 how to compensate for the apparent extra strength of axiom 13 by making appropriate adjustments elsewhere, and we see how the motivation for this compensation depends on our understanding of the relevant connections among arithmetic propositions.

PROOF of the theorem occupies us for the remainder of this subsection. Schematically, our strategy is this. We find formulas $\boldsymbol{F}$ and $\boldsymbol{t}$ (with $\boldsymbol{f}$ defined as $\sim \boldsymbol{t}$; see §27.1.2) such that

(1) $\vdash_{\mathbf{Q_R}} A \rightarrow . \boldsymbol{t} \rightarrow A$

(2) $\vdash_{\mathbf{Q_R}} \boldsymbol{F} \rightarrow B$

(3) $\vdash_{\mathbf{Q_R}} \boldsymbol{f} \rightarrow \boldsymbol{F}.$

It follows from (1), by contraposition in the consequent, that

(4) $\vdash_{\mathbf{Q_R}} A \rightarrow . \sim A \rightarrow \boldsymbol{f}.$

Then, from (4), (3), and (2), two applications of transitivity in the consequent yield the Theorem.

We follow Meyer 197+a in our choice of formulas to play the role of $\boldsymbol{F}$ and $\boldsymbol{t}$, letting $\boldsymbol{F}$ be $0=1$ and $\boldsymbol{t}$ be $0=0$. It should be noted, however, that of course (3) does not hold for $\mathbf{R}^{\sharp}$. Proofs of (1) and (2) are similar to those of the analogous facts for $\mathbf{R}^{\sharp}$, but must diverge at certain critical points where those proofs use induction, a move not open to us in $\mathbf{Q_R}$.

We now proceed to develop the needed theorems of $\mathbf{Q_R}$. Thus (1) will be found as T5 below, (2) as T15, and (3) as T16. We shall be using subproofs with dependence numerals in the style of §1.3, §23, and §R3, though we do not bother to draw Fitch-style vertical lines to the left, since it is clear that these are otiose for the system **R** (they are needed for the system **E** to guard against nonstrict reiteration). Note that our dependency numerals are written to the left of lines so as to avoid confusing the eye by subscripting numerals with numerals.

T1. $\vdash_{\mathbf{Q_R}} x=y \to z=z.$

PROOF.	$\{1\}$	1	$x=y$	hyp
	$\{1\}$	2	$x\times 0 = y\times 0$	1, Ax. 10
		3	$y\times 0 = 0$	Ax. 6
		4	$x\times 0 = y\times 0 \to .\ y\times 0=0 \to x\times 0=0$	Ax. 11
	$\{1\}$	5	$x\times 0 = 0$	2, 3, 4
		6	$0 = x\times 0$	5, Ax. 6
		7	$0=x\times 0 \to .\ x\times 0=0 \to 0=0$	Ax. 11
	$\{1\}$	8	$0=0$	6, 5, 7
	$\{1\}$	9	$z+0 = z+0$	8, Ax. 9
		10	$z+0 = z$	Ax. 2
		11	$z+0 = z+0 \to .\ z+0=z \to z+0=z$	Ax. 11
	$\{1\}$	12	$z+0 = z$	9, 10, 11
	$\{1\}$	13	$z = z+0$	12, Ax. 6
		14	$z = z+0 \to .\ z+0=z \to z=z$	Ax. 11
	$\{1\}$	15	$z = z$	13, 12, 14
		16	$x=y \to z=z$	1–15, →I

T2. $\vdash_{\mathbf{Q_R}} x=y \to .t(x, x)=t(x, y),$

where $t(x, x)$ is any individual term *possibly* containing occurrences of x and $t(x, y)$ is the result of *possibly* rewriting some of those occurrences to y.

PROOF. Perfectly standard induction on the complexity of $t(x, x)$, with the somewhat surprising T1 taking care of the degenerate subcases of the base case (these subcases are the "*possibly not*"s suggested in the statement of T2).

T3. $\vdash_{\mathbf{Q_R}} x=y \to .A(x, x) \to A(x, y),$

where $A(x, x)$ is any formula *possibly* containing free occurrences of x and $A(x, y)$ is the result of *possibly* rewriting some of those occurrences of free occurrences of y.

Proof. Perfectly standard induction on the complexity of $A(x, x)$, with T2 taking care of the base case.

T4. $\vdash_{\mathbf{Q_R}} 0=0\to.A\to A.$

Proof. Immediate instance of T3.

T5. $\vdash_{\mathbf{Q_R}} A\to.0=0\to A.$

Proof.			
	1	$0=0\to.A\to A$	T4
	2	$A\to.0=0\to A$	1, *Permutation*

T6. $\vdash_{\mathbf{Q_R}} A\rightleftarrows(0=0\to A)$

Proof. Left-to-right is T5. Right-to-left follows from $0=0$ (a theorem by Ax. 1) by *Assertion.*

The next fact was communicated to JMD by Meyer.

T7. $\vdash_{\mathbf{Q_R}} 0\neq 0\to 0=0.$

Proof.				
	$\{1\}$	1	$0\neq 0$	hyp
	$\{1\}$	2	$\exists y(0=y')$	1, Ax. 13
	$\{2\}$	3	$0=y'$	hyp
	$\{2\}$	4	$y'=0$	3, Ax. 6
	$\{2\}$	5	$0=0$	3, 4, Ax. 11
	$\{1\}$	6	$0=0$	2–5, $\exists E$
		7	$0\neq 0\to 0=0$	1–6, $\to$I

T8. $\vdash_{\mathbf{Q_R}} A\to.A\to A$

Proof.			
	1	$\boldsymbol{f}\to\boldsymbol{t}$	T7, dfs. $\boldsymbol{f}$ and $\boldsymbol{t}$
	2	$\sim(\sim A\to\sim A)\to\boldsymbol{f}$	T4, $(\sim A/A)$, *Contraposition*, *df.* $\boldsymbol{f}$
	3	$\boldsymbol{t}\to.\ A\to A$	T4, df. $\boldsymbol{t}$
	4	$\sim(\sim A\to\sim A)\to.\ A\to A$	1, 2, 3, *Transitivity*
	5	$A\to.\ \sim(\sim A\to\sim A)\to A$	4, *Permutation*
	6	$A\to.\ \sim A\to.\sim A\to\sim A$	6, *Contraposition* in consequent
	7	$A\to.\ \sim A\to\sim A$	6, *Contraction* in consequent
	8	$A\to.\ A\to A$	7, *Contraposition* in consequent

T9. $\vdash_{\mathbf{Q_R}} \sim A\to.B\to.A\to B.$

Proof. T8 is the characteristic **RM** axiom; so it suffices to note that T9 is a theorem of **RM** (T6 of Dunn 1970; also, RM71 in §29.3.1 is a close cousin).

T10. $\vdash_{Q_R} x\neq 0\to 0=0$.

Proof.

$\{1\}$	1	$x\neq 0$	hyp
$\{1\}$	2	$\exists y(x=y')$	1, Ax. 13
$\{1\}$	3	$x=x$	2, like steps 2–5 in proof of T7
$\{1\}$	4	$x\times 0=x\times 0$	3, Ax. 10
$\{1\}$	5	$x\times 0=0$	Ax. 4
$\{1\}$	6	$0=x\times 0$	4, 5, Ax. 11
$\{1\}$	7	$0=0$	5, 6, Ax. 11
	8	$x\neq 0\to 0=0$	1–7, →I

T11. $\vdash_{Q_R} 0\neq 0\to x=y$.

Proof.

$\{1\}$	1	$0\neq 0$	hyp
$\{1\}$	2	$x=0$	T10, *Contraposition*, 1
$\{1\}$	3	$y=0$	T10 (y/x), *Contraposition*, 1
$\{1\}$	4	$x=y$	2, 3, Axs. 6 and 11
	5	$0\neq 0\to x=y$	1–4, →I

T12. $\vdash_{Q_R} 0=1\to 0\neq 0$.

Proof.

$\{1\}$	1	$0=1$	hyp
	2	$0=0\to.\,0=1\to 0=1$	Ax. 11.
$\{1\}$	3	$0=0\to 0=1$	2, *Permutation*, 1
$\{1\}$	4	$0\neq 1\to 0\neq 0$	3, *Contraposition*
	5	$0\neq 1$	Ax. 12, $1 =_{df} 0'$
$\{1\}$	6	$0\neq 0$	4.5
	7	$0=1\to 0\neq 0$	1–6, →I

T13. $\vdash_{Q_R} 0 = 1\to x = y$.

Proof. From T12 and T11 by *Transitivity*.

T14. $\vdash_{Q_R} 0=1\to x\neq y$.

Proof.

$\{1\}$	1	$0=1$	hyp
	2	$0\neq 1$	Ax. 12, $1 =_{df} 0'$
$\{1\}$	3	$x=0$	1, T13 $(0/y)$
$\{1\}$	4	$x\neq 1$	2, 3, T3
$\{1\}$	5	$y=1$	1, T13 $(y/x,\ 1/y)$
$\{1\}$	6	$x\neq y$	4, 5, T3
	7	$0=1\to x\neq y$	1–6, →I

T15. $\vdash_{Q_R} 0=1\to A$.

Proof. By an easy induction on the complexity of A, showing simultaneously that

$$(\#)\qquad \vdash_{\mathbf{Q_R}} 0=1\to A \quad\text{and}\quad \vdash_{\mathbf{Q_R}} 0=1\to{\sim}A.$$

T13 and T14 constitute the base case. The cases where A is complex all fall easily from the inductive hypotheses, using obvious theorems of $\mathbf{R}^{\forall\exists x}$, with the exception of showing $0=1\to.B\to C$. But this last follows from the inductive hypotheses $\vdash_{\mathbf{Q_R}} 0=1\to{\sim}B$ and $\vdash_{\mathbf{Q_R}} 0=1\to C$, using T9.

T16. $\vdash_{\mathbf{Q_R}} 0\neq 0\to 0=1.$

Proof.	$\{1\}$	1	$0\neq 1$	hyp
	$\{1\}$	2	$1\neq 0$	Ax. 6, *Contraposition*
	$\{1\}$	3	$\exists y(1=y')$	2, Ax. 13
	$\{1\}$	4	$1=1$	like steps 2–5 in proof of T7
	$\{1\}$	5	$0=0$	4, Ax. 8, df. of 1
		6	$0\neq 1\to 0=0$	1–5, $\to$I
		7	$0\neq 0\to 0=1$	6, *Contraposition*

§73.3. $\mathbf{Q_R(1)}\neq\mathbf{Q(1)}$. By "$\mathbf{Q(1)}$" we mean the (classical) Robinson's arithmetic of the *positive* integers (*excluding zero*), and by "$\mathbf{Q_R(1)}$" its relevant counterpart. These are formulated by changing Robinson's axioms in §73.1 so that 2 becomes

2(1). $x+1=x'$,

4 becomes

4(1). $x\times 1=x$,

and 12(1) and 13(1) are obtained by changing 0 to 1 in 12 and 13. For the sake of an explicit distinction, we henceforth refer to the systems $\mathbf{Q}$ and $\mathbf{Q_R}$ of the preceding sections by the labels "$\mathbf{Q(0)}$" and "$\mathbf{Q_R(0)}$".

Strangely enough, $\mathbf{Q_R(1)}$ does not collapse into its fraternal twin $\mathbf{Q(1)}$. This is easily seen from the following "three-valued" model on a two-element domain $\{\omega, 1\}$. $\omega=\omega$ is $\boldsymbol{T}$, $1=1$ is $\boldsymbol{N}$, and both $\omega=1$ and $1=\omega$ are $\boldsymbol{F}$. The $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{F}$ are the elements $+1, 0, -1$, respectively of the three-point Sugihara matrix, $\mathrm{S}_1(\mathit{0})$ of §29.4 ($=\mathrm{S}_{\{-1,0,+1\}}$ of §29.3.2). We do not repeat here the definitions of the matrix operations, but we do mention that both $\boldsymbol{T}$ and $\boldsymbol{N}$ are designated and that existential (universal) quantification is valued on the pattern of an "indefinite" disjunction (conjunction), here quite definite because of the extreme finiteness of the domain. The arithmetical operations are then defined by the following tables.

$'$	
ω	ω
1	1

$+$	ω	1
ω	ω	ω
1	ω	1

$\times$	ω	1
ω	ω	ω
1	ω	1

It is easy to check that Robinson's axioms always take a designated value in this model, and it is easy but tedious to check that the axioms of $\mathbf{R}^{\forall\exists x}$ do so as well.

(It is not a misprint that makes the tables for + and × coincide. Their coincidence can be "explained" by looking at this model as obtained by the method of Dunn 1979 as the "three-valued counterpart" of a homomorphic image ($\omega \rightarrow \omega$, $n \rightarrow 1$) of a certain classical model of **Q(1)**. The classical model is defined on the positive integers together with ω as follows:

$'$	
ω	ω
m	$m+1$

$+$	ω	n
ω	ω	ω
m	ω	$m+n$

$\times$	ω	n
ω	ω	ω
m	ω	$m \times n$

Note that in the classical model + and × do not quite coincide, but $m+n$ and $m \times n$ are both carried onto 1 by the homomorphism.)

Let us now use this model to examine the reasoning that led to the collapse of $\mathbf{Q_R(0)}$. We can no longer define $\boldsymbol{F}$ as $0=1$ and $\boldsymbol{t}$ as $0=0$. But the obvious analogue is to define $\boldsymbol{F}'$ as $1=2$ and $\boldsymbol{t}'$ as $1=1$ (with $\boldsymbol{f}' =_{\mathrm{df}} {\sim}\boldsymbol{t}'$).

On the strategy of §73.2 then, the question is whether all the following hold:

(1′) $\vdash_{\mathbf{Q_R(1)}} \boldsymbol{F}' \rightarrow B$
(2′) $\vdash_{\mathbf{Q_R(1)}} A \rightarrow . \boldsymbol{t}' \rightarrow A$
(3′) $\vdash_{\mathbf{Q_R(1)}} \boldsymbol{f}' \rightarrow \boldsymbol{F}'$.

First, the model shows that (1′) does not. Indeed, the instance $1=2 \rightarrow x=y$ (an analogue of T13) receives the value $\boldsymbol{F}$ when x is assigned 1 and y is assigned ω, noting that $(\boldsymbol{N} \rightarrow \boldsymbol{F}) = \boldsymbol{F}$.

Second, (2′) and (3′) do hold. Indeed, the proof of (3′) is a precise analogue of the proof of T16 in $\mathbf{Q_R(0)}$—just replace 0 by 1 and 1 by 2 uniformly throughout. The proof of (2′) is not in as close analogy.

Recall that the proof in §73.2 of (2) went through the "Replacement theorem" T3. The proof we give below of (2′) cannot go this way, because T1 (the base case for T3 in the degenerate subcase where no replacement is made) is not a theorem of $\mathbf{Q_R(1)}$. Thus T1 receives the value $\boldsymbol{F}$ for the assignment of ω to x and y, and of 1 to z. But it turns out that the full "Replacement theorem" is not needed just to get the particular degenerate case

(!) $1=1 \rightarrow . A \rightarrow A$, from which (2′) follows by *Permutation* and the definition of t'. Thus (!) can be proved by routine induction on the complexity of A (left to the reader), the base case of which is guaranteed by the following:

T17. $\vdash_{\mathbf{Q_R(1)}} 1=1 \to .\ x=y \to x=y.$

PROOF.

$\{1\}$	1	$1=1$	hyp
$\{1\}$	2	$x\times 1 = x\times 1$	1, Ax. 10
	3	$x\times 1 = x$	Ax. 4(1)
$\{1\}$	4	$x=x$	2, 3, Ax. 11, Ax. 6
$\{2\}$	5	$x=y$	hyp
$\{1,2\}$	6	$x=y$	4, 5, Ax. 11
$\{1\}$	7	$x=y \to x=y$	5–6, →I
	8	$1=1 \to .\ x=y\to x=y$	1–7, →I

§73.4. The relations among $\mathbf{R}^\sharp$, $\mathbf{Q_R(0)}$, and $\mathbf{Q_R(1)}$. It might be thought natural that $\mathbf{Q_R(1)} \subseteq \mathbf{Q_R(0)} \subseteq \mathbf{R}^\sharp$. However, as it turns out, neither of these subsystem relations holds.

Thus $\mathbf{Q_R(1)} \not\subseteq \mathbf{Q_R(0)}$ on the technicality that

(*) $\quad x\times 1 = x,$

Axiom 4(1) of $\mathbf{Q_R(1)}$ (and of $\mathbf{Q(1)}$), is not a theorem of $\mathbf{Q_R(0)}$ (indeed not even of $\mathbf{Q(0)}$). It obviously suffices to show that (*) is not a theorem of $\mathbf{Q(0)}$. We observe that, if it were, an obvious chain of reasoning would produce

$$x=x\times 1=x\times 0'=x\times 0+x=0+x.$$

Yet $x=0+x$ is well known not to be a theorem of $\mathbf{Q}$ (see, e.g., Boolos and Jeffrey 1974).

The reason why $\mathbf{Q_R(0)} \not\subseteq \mathbf{R}^\sharp$ is more profound: In the three-valued and two-element model of the first proof of the absolute consistency of $\mathbf{R}^\sharp$ as described in §72.2, Axiom 13 takes the value $\boldsymbol{F}$ since $(\boldsymbol{T}\to\boldsymbol{N})=\boldsymbol{F}$:

$$1\neq 0 \to \exists y(1=y').$$

Similar reasoning with a three-valued model constructed on the *positive* integers *modulo* 2 and the appropriate instance of Axiom 13(1) shows that $\mathbf{Q_R(1)} \not\subseteq \mathbf{R}^\sharp$.

Considering for the sake of completeness the converse relations, we note it is easy to see that $\mathbf{R}^\sharp \not\subseteq \mathbf{Q_R(0)}$, since induction in $\mathbf{R}^\sharp$ allows the proof of $0+x=x$, which we have already noted is a nontheorem of $\mathbf{Q_R(0)}$. Each of $\mathbf{R}^\sharp$, $\mathbf{Q_R(0)} \not\subseteq \mathbf{Q_R(1)}$, for the trivial reason that 0 is not even in the vocabulary of $\mathbf{Q_R(1)}$. However, even restricting attention to 0-free theorems, still the subsystem relations fail to hold. Thus, for $\mathbf{R}^\sharp$, take any of the well-known theorems needing induction, e.g., $x+y=y+x$ (see e.g., Boolos and Jeffrey 1974). And, for $\mathbf{Q_R(0)}$, take T1 (§73.3).

§73.5. Remarks and speculations. The summary of our discoveries in the preceding sections is that "naught matters," since $\mathbf{Q_R(0)}$ collapses into its classical fraternal twin, whereas $\mathbf{Q_R(1)}$ does not. We might be tempted to agree with Kronecker in his oft-quoted "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk," generously interpreting him to be excluding zero as a "whole number" (it being well-known that zero was an invention of the Hindus). Perhaps this throws new light on the comparison in Curry 1963 (p. 252) of the paradoxes of implication with the "invention of zero."

However, it is not zero itself that is at fault—it is rather *multiplication* by zero. Thus consider the system $\mathbf{Q_R(0)}'_+$, which results from $\mathbf{Q_R(0)}$ by dropping the multiplication sign × (and of course Axioms 4–5). The model of §73.3 may be straightforwardly modified so as to be defined on the domain $\{0,\omega\}$ instead of $\{1, \omega\}$ by replacing throughout every mention of "1" by "0". It is easy to see that this modified model (more technically a retract of it omitting ×) satisfies all the axioms of $\mathbf{Q_R(0)}'_+$ and yet fails to satisfy the theorem T13 of $\mathbf{Q_R(0)}$, $0=1\rightarrow x=y$.

Why cannot the model of §73.3 be modified so as to include multiplication? It is clear that the "straightforward" modification will not do, since $\omega\times 0=0$ (instead of ω as replacement of "1" by "0" in $\omega\times 1=\omega$ would require). Fooling around, however, one is led to consider the tables

′	
ω	ω
0	0

+	ω	0
ω	ω	ω
0	0	0

×	ω	0
ω	ω	0
0	ω	0

These do satisfy the axioms recursively characterizing addition and multiplication. The reason that the + table is changed so that $0+\omega=0$ (rather than ω, as in the straightforward modification) is so that $0=\omega\times 0=\omega\times 0'=\omega\times 0+\omega=0+\omega$. But problems are caused for Axioms 9 and 10, in the instances

$$\omega=\omega \rightarrow 0+\omega=0+\omega,$$
$$\omega=\omega \rightarrow \omega\times 0=\omega\times 0,$$

since these both boil down to

$$\begin{array}{ccc} \omega=\omega & \rightarrow & 0=0. \\ T & F & N \end{array}$$

One might challenge Axioms 9 and 10 as too strong. R. Sylvan (see Routley 1977) has criticized similar things in the context of $\mathbf{R}^\sharp$. Without imputation, it would be entirely coherent with the general thrust of those criticisms of orthodox relevant logics as "too strong" to prefer to weaken Axiom 9 to

$$9'.\ (x=y)\&(z=z) \rightarrow (x+z=y+z)\&(z+x=z+y),$$

and similarly for Axiom 10. It is interesting to see that Axioms 9′ and 10′ are satisfied by the model just discussed, so that one can even have multiplication by zero without collapse if one is prepared to weaken substitution principles for identity.

We are, however, *inclined* not to want to modify Axioms 9 and 10. There is the purely technical reason that we see no way of making a corresponding modification of our argument for (γ), although this argument can be applied to all of $\mathbf{Q_R(0)}$, $\mathbf{Q_R(1)}$, and $\mathbf{Q_R(0)}'_+$ (the first is uninteresting given that $\mathbf{Q_R(0)} = \mathbf{Q(0)}$) but we shall not describe these applications of our argument here.

We also have philosophical reasons for our inclination, although they are not decisive. It seems that the general principle

$$(\alpha) \qquad x_i=y_i \rightarrow f(x_1 \ldots x_i \ldots x_n)=f(x_1 \ldots y_i \ldots x_n)$$

becomes relevantly suspect only when one sees that it may have as instances things like

$$(\alpha') \qquad x=y \rightarrow x\times 0=y\times 0,$$

or, more abstractly,

$$(\alpha'') \qquad x=y \rightarrow K_a(x)=K_a(y),$$

where K is the constant function always taking the value a. One wanted $K_a(x)$ to "depend" on its argument.

Now a number of people have observed that the general principle

$$(\beta) \qquad x=y \rightarrow .\ A(x, x)\rightarrow A(x, y)$$

becomes suspect when $A(x, x)$ does not "depend" on x, because x either does not occur in it or does so only in a dummy way (by virtue, say, of the equivalence $p\leftrightarrow p\wedge(p\vee Fx)$). Taking for the moment $A(x, x)$ as a propositional function, we can see that the question about (β) might be whether $A(x, x)$ "depends" on x, and we thereby make contact with §§70, 71, and 74.

It would lengthen an already too long discussion to pursue this business of constant functions much further. Let us say that the idea of functions that *really* depend on their arguments has been articulated and investigated recently by many workers in somewhat differing ways, as indicated in the sections cited, and seems not yet to have found a finished form.

One motive for such investigations has been to do for relevance logics what Läuchli did for intuitionistic logic, namely to provide a "realizability" interpretation using functions (expressed by λ-terms or combinators)—the relevant trick being to look at only those functions which depend on their arguments. Adapting a wise saying, we might say that a constant function is no more a kind of function than a blunderbuss is a kind of buss.

But the original Läuchli realizability interpretation, which he called "abstract," was modeled on the prior *concrete* realizability interpretation(s) of Kleene for intuitionistic number theory using partial recursive functions. There seems then to be a "logical niche" waiting to be filled. What is needed is a "relevant Kleene," developing a theory of "relevant partial recursive functions" that really depend on their arguments, and using these to investigate "relevant realizability interpretations" of the relevant arithmetics.

$\mathbf{Q_R(1)}$ might seem to be the ideal receptacle for these "relevant recursive functions," whatever they might be. But a note of caution must intrude. Since $\mathbf{Q_R(1)}$ contains **Q(1)** *on the classical vocabulary*, and since **Q(1)** represents *all* of the classical recursive functions on the positive integers, then so does $\mathbf{Q_R(1)}$. (We are assuming something here that we have not actually worked through, namely, that the absence of 0 does not affect the representability proof—the β-function and all that. All textbook presentations always deal with **Q(0)** when representability is afoot, but we think this is a historical accident.) Anyway, the constant 1 function can be easily seen to be represented by the formula $x=x\&y=1$. Thus it is not the case that "representability in $\mathbf{Q_R(1)}$" coincides with "relevant recursiveness" (probably what is needed is some notion of "relevant representability"). It is interesting that ordinary representability does not lead to things like T1 and collapse.

§74. Relevant predication: The formal theory.

> *I don't have a relationship with Mr. Humphreys outside the fact that I'm the Premier and he was the executive director of the Rugby League.* Testimony of Neville Wran (Premier of New South Wales), at a 1983 Royal Commission concerning possible improper influence on judicial proceedings against Mr. Humphreys.

§74.1. Introduction.

There is an issue regarding predication that seems not to have been much addressed. Recent philosophical literature has stressed one distinction of "intimacy" among properties of a given object: the distinction between the properties that the given object has essentially (or according to its nature), and those it has accidentally. This distinction has been expressed using the language of modal logic as a contrast between those properties which the object has necessarily, and those which it does not so have.

But the distinction between necessary and nonnecessary properties is not the only way to sort out those properties which have an intimate life with an object from those which do not.

Thus consider the property often attributed in logic classes to all of us: the property that each and every one of us (ships and shoes and sealing wax, too) has by virtue of the fact that Socrates is wise (the tenseless sense of "is"—if you do not believe in such a sense, please substitute the past tense).

Or, to take a two-placed example, consider the relation that logic books allege that each of us has to each other (and to Mr. Wran and Mr. Humphreys) by virtue of one of us having one property and the other having *any* other.

Every metaphysician worth his or her salt surely feels that there is something "hokey" about such "properties" and "relations," and yet classical logic has no way to rule them out of court. The issue here is not necessarily the ontological one of whether such properties *really exist*, although the issue can be put in this ontological tone of voice if one is so inclined.

Adopting a somewhat neutral vocabulary, but one that clearly looks forward to the use of relevance logics, we shall label the distinction we seek as the distinction between "relevant" and "irrelevant" predications, although occasionally polemic may lead us to speak of the former as "real" or "natural" predications.

We have attempted to write this section so as not to assume that the reader has mastered relevance logic. Many years ago Robert Fogelin informally summarized the formal properties of relevance logic as "no funny business' (§8.21). We shall rely throughout this section on about this level of understanding of relevance logic. The reader who would prefer not just to take remarks on faith, but to "do the calculations," may consult elsewhere in this book. The most mathematical use of relevance logic occurs in §74.8 and §74.11, but even there the reader without the "relevant" technical background should be able to get the philosophical point.

The main gist of this section is that Fogelin's observation regarding relevance logic can be turned around, and relevant implication can be used so as to make sense of what "no funny business" means with respect to predication.

We are well aware that many readers may find this a case of explaining the obscure in terms of the more obscure. Certainly many critics of relevance logic, and even many friends, have wanted to find its home in the notoriously tangled brier patch of epistemological or pragmatic purposes. It may well be that the relation of relevant implication is not part of the objective ontological furniture of the universe, but rather is in some fundamental sense subjective and mind-dependent. Relevance may indeed only be a rough-and-ready way of dividing up the items in the universe according to human concerns (we almost said "*shifting* human concerns," but many of our concerns may well be "hard-wired" into us by evolution). Be that as it may, the same might be said of "relevant predication." It may be only ourselves (and not the universe) saying when a property (or relation) is "natural." It is thus at least interesting to explore the relation between the relevant concepts of implication and predication.

And it could even be that these concepts do reside in the objective universe, and that it is the job of science not just to tell us what items there are in the universe and what facts hold of them, but also to tell us what relevant implications there are among those facts and what are the relevant properties that go into making up those facts. The world is more than "all that is the case," at least given a narrow atomistic, extensionalist reading of those words.

Incidentally, the reader should be told that the "relevance logic" being used is the (first-order) system **R** of relevant implication, not the system **E** of entailment. The system **E** combines both relevance and necessity, which is fine for certain purposes—perhaps, for example, for analyzing essential predication—but is too strong for analyzing merely relevant predication of the more humdrum sort that distinguishes the intimate predication of wisdom to Socrates from the promiscuous predication of Socrates's being wise to someone else, say Alcibiades.

§74.2. Properties (monadic). Consider the following pair of statements:

(1) Socrates is such that he is wise.
(2) Alcibiades is such that Socrates is wise.

Read quickly, they sound quite similar, and yet when we read them with meaning, we are tempted to mark (2) with the linguist's "*" as "deviant." At least we are so tempted if our intuitions have not been "trained" by logic. If they have been so trained, we are tempted to treat (2) as a kind of degenerate case of the logical structure exhibited by (1).

Any reader who has been exposed to the good-natured or bitter polemics of the rest of this book knows what few good things the Relevance Logicians' Manifesto has to say about the way that classical logic has trained our intuitions. The reader will get the main point of this section if he or she understands that we intend that there be a strict analogy between (1) and (2) above, and their correspondents below:

(1′) If anyone is Socrates then he is wise.
(2′) If anyone is Alcibiades then Socrates is wise.

But (1′) is true, as the following valid argument (with presumably true premiss) shows:

(1ˆ) Socrates is wise. Therefore, if ($x=$Socrates) then x is wise.

However the corresponding argument for (2′) is a clear case of irrelevance as understood in relevance logic:

(2ˆ) Socrates is wise. Therefore, if ($x=$Alcibiades) then Socrates is wise.

Thus (1ˆ) is an instance of

(1♯) $Fa \vdash (x=a) \to Fx$ (Indiscernibility), and (1♯) is, presumptively at least, a relevantly valid argument. Given *Fa*, one can get to *Fx* in a natural way (indiscernibility of identicals) *using the assumption* $x = a$.

On the other hand, the validity of (2ˆ) would seem to depend on the dreaded

(2♯) $A \vdash B \rightarrow A$ (Positive Paradox),

and there is no way in the world with "no funny business" that one can get to *A* (even given the premiss *A*), *using B* (see §3).

§74.3. Lambda conversion. Let us back up a bit and see how open sentences are manipulated into predicates in a standard extension of classical first-order logic. We have in mind the device of "lambda-abstraction" due to Church 1941. By means of this device, any formula *Fx* can be made into a predicate:

(3) λxFx "the property of being (an *x* such that *x* is) *F*".

We have applied the lambda operator to an atomic open sentence, but the device is supposed to be applicable to any formula whatsoever:

(3′) λxA, "the property ascribed to *x* is saying that *A*".

Normally *A* in (3′) contains free *x*; however, (3′) is supposed to be applicable even to the case where the formula *A* is a sentence (no free variables):

(3″) λxp, e.g., λx(Socrates is wise).

Of course, the introduction of notation by itself is idle unless rules to govern it are also introduced. Church 1941 introduced the principle of "lambda-conversion," which, in its simplest case is illustrated below:

(4) $(\lambda xFx)a \rightleftarrows Fa$.

Of course (4) must be generalized so that it holds for any formula *A* and for any term *t*, and then it looks like this:

(4′) $(\lambda xAx)t \rightleftarrows At$

Here (and throughout) we use a quite standard suggestive notation to indicate substitution, where "*Ax*" is just another way to name a formula (used to draw attention to the possible free occurrences of the variable *x*), and *At* is the result of substituting *t* for all those free occurrences of *x* (if any), with the proviso of course that *t* be free for *x* in the sense that no such substitution will bind some free variable of *t* (when this proviso is met we talk of *proper* substitution; see §38 for some details). Since it is allowed that there be no free occurrence of *x*, we obtain:

(4″) $(\lambda p)a \rightleftarrows p$.

Now it might be suggested that we can fix things by allowing the formation of a lambda-expression λxA *only* when A actually has at least one free occurrence of the variable x (a similar restriction was placed by Church on his preferred "λ-I calculus," which, it turns out, has very close relations to relevant implication—see §74.9 below for cross-references and citations). But, given the presence of certain logical connectives, such a "restriction" becomes in effect an empty gesture, because of equivalences such as:

(5) $A \rightleftarrows A\&(Fx\vee{\sim}Fx)$,

and

(6) $A \rightleftarrows A\&(A\vee Fx)$.

Equivalences like (5) and (6) allow one to "dummy in" occurrences of variables. The equivalence (5) fails in relevance logic (depending, as it does, on the property that a tautology $Fx\vee{\sim}Fx$ is implied by any sentence A). It might be thought, then, that this is the solution that relevance logic provides to the problem of vacuous predication. Unfortunately, things are not that simple, since (6) is provable even in relevance logic. Perhaps it should be mentioned that it is a common misunderstanding of relevance logic that it rules out the logical principle of "addition" $A\rightarrow.A\vee B$. But in fact—as discussed in §29.6—it does not (at least for the ordinary extensional disjunction $\vee$), and so it licenses $A\rightarrow.A\vee Fx$, the key move in obtaining (6).

So we have to be more subtle.

§74.4. Factor.

We shall make a short technical digression, discussing an important formal feature of relevance logic which will be crucial to developing a theory of relevant predication. The following formula is not a theorem:

(Factor) $A\rightarrow B \rightarrow (A\&C\rightarrow.B\&C)$.

This may come as a kind of shock to those who have opened this book at this section, but Routley and Routley 1972 have linked it with the phenomenon they dub "suppression." If the antecedent of Factor had as an additional conjunct $C\rightarrow C$, then everything would be OK, the resulting formula being a simple instance of

(& int/elim) $(A\rightarrow B)\&(C\rightarrow D) \rightarrow (A\&C\rightarrow.B\&D)$.

But, since it does not, things are not OK. Classical logic allows the "suppression" of logical truths like $C\rightarrow C$, since they are implied by any sentence whatsoever (including Factor's antecedent $A\rightarrow B$). But relevance logic does not.

Indeed, if Factor were added as a theorem to any standard relevance logic, it would collapse to classical logic. We sketch a proof here that, if Factor were added as an axiom scheme to the system **R**, then Positive Paradox would be a theorem (it is well known that adding Positive Paradox to **R** leads to classical logic). The proof is unfortunately "technical" in that it uses the fact, established in effect in §45.1, that the sentential constant $\boldsymbol{t}$ can be added conservatively to **R**, with the axiom scheme (see §R2).

($\boldsymbol{t}$) $\quad A \leftrightarrows (\boldsymbol{t}\rightarrow A)$.

The sketch below can be made into an official proof (without reference to $\boldsymbol{t}$) by consulting §28.2.2 (or, alternatively, in the particular context of the proof below, the reader can simply substitute $(A\rightarrow A)\&(B\rightarrow B)$ for $\boldsymbol{t}$, and fiddle).

1	$A\rightarrow B \rightarrow. (A\&C)\rightarrow C$	from Factor, by weakening the consequent
2	$A\rightarrow B \rightarrow. A\rightarrow A$	from 1, substituting A for C, and applying idempotence of &
3	$\boldsymbol{t}\rightarrow B \rightarrow. \boldsymbol{t}\rightarrow \boldsymbol{t}$	2, substituting $\boldsymbol{t}$ for A
4	$B \rightarrow. \boldsymbol{t}\rightarrow B$	half of $\boldsymbol{t}$ axiom scheme
5	$\boldsymbol{t}\rightarrow \boldsymbol{t} \rightarrow \boldsymbol{t}$	other half
6	$B\rightarrow \boldsymbol{t}$	4, 3, 5, transitivity
7	$\boldsymbol{t} \rightarrow. A\rightarrow A$	from $A\rightarrow .\boldsymbol{t}\rightarrow A$ by permutation
8	$A \rightarrow. B\rightarrow A$	6, 7, transitivity, permutation

The version of Factor above will be called &-Factor, to distinguish it from

($\vee$-Factor) $\quad A\rightarrow B \rightarrow. (A\vee C)\rightarrow(B\vee C)$,

which is equally disastrous from a relevance-logic point of view (the reader is invited to "dualize" the proof given above).

The whole trick of the device on which we elaborate in the next subsection is that the formula

(♯) $\quad x=a \rightarrow [A\&(A\vee Fa)\rightarrow .A\&(A\vee Fx)]$

is not a theorem, and would not be, even were

(♯♯) $\quad x=a \rightarrow. Fa\rightarrow Fx$

to be a theorem. Ordinarily (♯) would follow from (♯♯) by way of first applying $\vee$-Factor and then applying &-Factor (using transitivity, of course). Actually, to make direct contact with the material in the sequel, we should really talk of the permuted form $Fa\rightarrow .(x=a)\rightarrow Fx$ of (♯♯) (and similarly of ♯), but since permutation is valid in **R**, we do not have to be so careful.

§74.5. Indiscernibility of identicals. As (♯) suggests, there is something rather exotic about the treatment of identity in relevance logic. Ordinarily we would expect to have as a theorem something like:

(7) $\quad Aa \rightarrow. (x=a)\rightarrow Ax \qquad$ (Indiscernibility).

Here we use the substitution convention introduced in §74.3 (which is asymmetric between variables and constants, the *variable* x being always replaced by the *constant* a). The reader may have a little difficulty seeing this as "Indiscernibility," since the usual textbook statement tends to go along the following lines (but typically with the identity permuted to the front—we just do not want to introduce that further irrelevant difference):

(7u) $Aaa \rightarrow. (x=a)\rightarrow Aax$,

where Aaa is any formula perhaps involving multiple occurrences of a, and Aax is the result of perhaps replacing one or more (free) occurrences of a by x (but where x does not become accidentally bound in the process). (7u) is awkward for a number of reasons, at least one of which is that a different notion of substitution in *some* occurrences must be introduced in addition to the notion of substitution in *all* occurrences so badly needed for quantification theory. So it is interesting that (7) and (7u) turn out to give the same results. The fiddling is left to the interested reader.

The problem with (7) for relevance logic is that it has as an instance (when x is not free in A)

(7′) $A \rightarrow. (x=a)\rightarrow A$,

which of course is at least nervously close to the dread relevance destroyer:

$A \rightarrow. B\rightarrow A.$

One might think that one could avoid this problem by giving a more rigorous understanding of (7), removing the "perhaps" in the explanation of the notation Ax, and requiring of Ax that it actually contain a free occurrence of x. Unfortunately, this move would ultimately be to no avail for the very same reason that ruling out vacuous lambda terms was to no avail. The same theorem (6) would always allow the dummying in of x.

So, in working out the theory of identity in relevance logic, one must be careful not to take Indiscernibility in its full form (7), at least not for all formulas A. It is worth noting that the problem with full Indiscernibility of Identicals is not the familiar problem associated with so-called "intensional logic." It is surely not that the "context" A in (7′) is "opaque" and that strengthening the identity to an identity of "intensions" (or "hyperintensions" or "hyperhyperintensions" or whatever) of the terms x and a would somehow fix up (7′). The point, put quickly, is that A is no context of x and a at all.

The suggestion is that full Indiscernibility is to be postulated for a formula only when one wants to postulate that the formula is of the kind that determines relevant properties (this is of course different from saying that a particular actually has a relevant property—we will discuss this and other distinctions in §74.8).

But it is not the business of logic, but rather that of metaphysics (or perhaps of whatever field it is whose subject matter is being formalized, e.g., physics) to determine what formulas "really" determine properties, just as it is the business of logic to tell us not what formulas are true, but only what formulas follow from each other. Roughly speaking, logic should tell us only that *if* certain formulas are postulated "really" to determine properties, *then it follows* that certain other formulas "really" determine properties (of course this is only rough, because there are certain special formulas that can be shown "really" to determine properties by way of logic alone, just as there are certain special formulas that can be determined to be true by way of logic alone—see for example Facts 9 and 10 in §74.8 below).

It would be in accord with the intuitions behind logical atomism to think that at least every atomic formula Fx should be one of these, and hence it would be in accord with logical atomism to postulate full Indiscernibility for at least the atomic formulas (see §74.9 below). According to Meyer 197+, Urquhart in fact made the suggestion that the correct axioms for identity in the context of first-order **R** involve Indiscernibility for just the atomic formulas, letting induction on formulas take us where it may with respect to Indiscernibility for compound formulas. This is certainly closely related to the ideas of the present section (despite the disavowal of any special ontological importance of atomic formulas), though there are also some other differences (in particular, the definition of relevant predication using identity, and the Substitution axiom for identity below). But as such the idea seems overly restrictive, although we might like to think that a well-formalized science would have things sorted out so that all atomic formulas did determine relevant properties.

Does this mean that the traditional principle of reasoning known as "substitution of identicals" thus fails for relevance logic? No, since we can always have, in place of Indiscernibility (7), the weaker

(8) $Aa\&(x=a) \rightarrow Ax$ (Substitution axiom).

In relevance logic, $A \rightarrow .B \rightarrow C$ and $(A\&B) \rightarrow C$ are not in general equivalent, although the former always implies the latter. The degenerate instance

(8′) $p\&(x=a) \rightarrow p$

is perfectly harmless (unlike (7′)), being merely of the form $(A\&B) \rightarrow A$. Of course, one may not always want even (8) in intensional contexts, for the usual reasons about "opacity" as mentioned above; but we here assume (8) as a default axiom. It follows that we are thinking of identity as indeed nonmodal—our context is **R**, not **E**—but nevertheless as tighter than the extensional identity often described in conventional modal logic.

The further principles of identity one would want surely include

(Reflexivity) $x=x$,
(Symmetry) $x=y \rightarrow y=x$,

and some form of transitivity.

We have a choice concerning the precise form of transitivity, for we might take it to be either of the following:

(Conjoined Transitivity) $(x=y\&y=z) \rightarrow x=z$,
(Nested Transitivity) $x=y \rightarrow (y=z \rightarrow x=z)$.

The latter implies the former, and we adopt it as our official axiom. If we had to argue for this we would point out that both of the antecedents $x=y$ and $y=z$ are appropriately used in an informal derivation of $x=z$, and we would also borrow on an analogy between identity and the relevant biconditional (in conversations, Meyer has placed much stress on this). But we are not so sure of the absolute validity of the choice that we do not bother to keep track in the sequel of when what we do depends on our having the stronger axiom. Incidentally, notice that Nested Transitivity is really just Indiscernibility of Identicals for formulas that are equations, and Conjoined Transitivity is Substitution.

§74.6. Relevant predication. We turn finally now to a discussion of a formal treatment of "intimate" predication within the framework of relevance logic. Let us recall Juliet's observation that "A rose by any other name would smell as sweet." In rough symbols, letting r be a parameter denoting an arbitrary rose, and letting S be a predicate expressing a particular degree of smelling sweet:

(9) $Sr \rightarrow \forall x(x=r \rightarrow Sx)$ (Shakespeare's law)

We may credit Shakespeare (or Juliet) with the non-Lockean view that sweet smell is a "relevant property" of a rose, and take (9) as a way of stating this. In fact (9) is in effect just a special case of (7) (Indiscernibility), but with the variable x universally quantified and confined to the consequent (all legal moves in relevance logic; see Fact 1 of §74.8).

Indeed, by simple moves in relevance logic, one can reverse the implication of (9) so as to obtain the equivalence:

(10) $Sr \leftrightarrows \forall x(x=r \rightarrow Sx)$ (Relevant predication for rose)

Thus, assuming the right-hand side, one obtains by Universal Instantiation $(r=r)\rightarrow Sr$, and, since $r=r$ is true by Reflexivity, one can get by *modus ponens*, the left-hand side Sr.

This all motivates the definition:

(11) $(\rho x Ax)a =_{\text{df}} \forall x(x=a \rightarrow Ax)$ (Relevant Predication).

This is read in "middle English" as "a relevantly has the property of being (an x) such that A." We would not like to place a lot of stress on the following further motivations for (11), but they are at least worth noting. (i) The definition of relevant predication is in line with the common medieval treatment of affirmative "categorical propositions" with singular subject terms as universal affirmatives (All "Socrates" are wise), at least given the modern analysis of universal affirmatives using the conditional. (ii) To say that "*a* is such that so-and-so" seems to have a ring of universal quantification in it. One is not just saying that *a* is so-and-so, but saying further that *a* is of a *kind* that is so-and-so.

It should now be clear what is supposed to be wrong with sentences like (2). The meaning of "Alcibiades is (relevantly) such that Socrates is wise" would, in symbols, be given by:

(12) $(\rho x p)a =_{\mathrm{df}} \forall x(x=a \rightarrow p)$ ("Irrelevant predication")

And it is clear that the right-hand side of (12) corresponds to a failed relevant implication (even when *p* is true, as it surely is on this example). An *x*'s being identical to Alcibiades has nothing to do with Socrates' being wise.

§74.7. Relations (polyadic). It turns out that the mechanisms of the last subsections can be straightforwardly applied to formulas containing more than one free variable, and so we can develop a theory of "relevant relations." For simplicity we shall discuss the binary case of a formula *Axy*, perhaps having free occurrences of the variables *x* and *y*, extending our substitution conventions in straightforward ways. The main difference between this and the monadic case lies in the many different ways that one can state

Indiscernibility

(13a)	First Position	$x=a \rightarrow (Aab \rightarrow Axb)$
(13b)	Second Position	$y=b \rightarrow (Aab \rightarrow Aay)$
(13→)	Nested	$x=a \rightarrow (y=b \rightarrow (Aab \rightarrow Axy))$
(13&)	Conjoined	$(x=a \mathbin{\&} y=b) \rightarrow (Aab \rightarrow Axy)$.

The reader can imagine how the number would increase with three variables, four variables, etc. The first two, (13a) and (13b), are of course just monadic versions of the sort discussed in §74.5 (thus, e.g., in (13a) the parameter *b* is just "an innocent bystander").

There are certain logical relationships among the statements of indiscernibility above. Thus (13→) implies (13&), (13a), and (13b), but no other implications hold.

The discussion of §74.6 leads one to conclude that "Indiscernibility in a Position" of the types (13a) and (13b) amounts to saying that the formulas determine

Relevant relational properties

(14a) $(\rho x Axb)a \rightleftarrows \forall x(x=a \rightarrow Axb)$

(14b) $(\rho y Aay)b \rightleftarrows \forall y(y=b \rightarrow Aay)$

The fact that *Axy* determines a relational property (in some appropriately strong sense) in one of its positions does not necessarily mean that it does so in the other. As a putative illustration from the history of philosophy, Aquinas said that it is a property of the world that God created it, but not a property of God that he created the world. A more contemporary example might be that although it is a property of us that we are thinking of Little Rock, there is room to doubt that it is a property of Little Rock that we are thinking of it.

It is the strongest Nested Indiscernibility (13→) that says that Axy determines a "relevant relation" between x and y:

Relevant Relation
(14→) $(\rho xyAxy)ab =_{\mathrm{df}} \forall x \forall y(x=a \rightarrow (y=b \rightarrow Axy))$.

Incidentally, (14→) is easily seen to be equivalent (using quantifier confinement) to a formula involving only monadic relevant predication:

(14m) $(\rho x[\rho y(Axy)]b)a$,

i.e., by (11), $\forall x(x=a \rightarrow \forall y(y=b \rightarrow Axy))$.

What of (13&), which is intermediate in strength? It obviously ought to do something intermediate between determining a relevant property in one of its positions and determining a relevant relation. It can be thought of as determining a

Relevant Property of a Pair
(14&) $[\rho(xy)Axy](ab) =_{\mathrm{df}} \forall x \forall y((x=a \,\&\, y=b) \rightarrow Axy)$

An example of a formula that determines properties of pairs (but not relations) is $Fx\&Gy$ (see Fact 8 in §74.8). Undoubtedly the point that Mr. Wran was trying to make to the Royal Commission was that he and Mr. Humphreys have no real relations between them, but at best (worst?) only properties of pairs.

Some motivation can be given to this talk of properties of pairs, by considering the famous

Law of the Ordered Pair
(LOP) $\langle a, b\rangle = \langle c, d\rangle \rightleftarrows (a=b \,\&\, c=d)$.

It is already suggestive that conjunction features prominently in both (LOP) and (14&), but the connection can be made quite explicit, at least if we are generous to bastard notation. Thus let us suppose for a moment that we have the usual angle-bracket notation for ordered pairs $\langle a, b\rangle$. Now formula (11) tells us quite generally when a formula determines a relevant property of an object; so let us just let that object be the ordered pair $\langle a, b\rangle$. Let us suppose then that Ax determines a relevant property of $\langle a, b\rangle$, in symbols

$(\rho x Ax)\langle a, b\rangle$.

By the definition (11) of relevant predication, this is $\forall x(x=\langle a, b\rangle \rightarrow Ax)$, or $\forall\langle x, y\rangle(\langle x, y\rangle=\langle a, b\rangle \rightarrow A\langle x, y\rangle)$. By (LOP), this last can be expressed by $\forall\langle x, y\rangle((x=a \,\&\, y=b) \rightarrow A\langle x, y\rangle)$. For the reader with relatively poor eyesight or memory for notational conventions, the above formula could easily be confused with the right-hand side of (14&). It differs essentially only by the use of the notation "$\langle x, y\rangle$" in an illegitimate position after the universal quantifier. But that can easily be fixed, and indeed (14&) is just the remedy.

Because of the logical relationships among the various kinds of indiscernibility expressed by the formulas (13x) above, we see that there are "metaphysical" relationships among the various kinds of properties and relations that they determine. Thus, e.g., if Axy determines a "relevant relation" (which simply means that (13→) holds), then clearly at the same time Axy determines a property of pairs and also determines relational properties in each of its positions.

Are there any converse relationships? As it turns out, if an open sentence Axy determines a relational property with respect to each of its positions x and y, then it determines a relevant relation as well (see Fact 7 of §74.8). But before going on much more in this way, we need to be more precise with our talk of formulas "determining relevant properties and relations."

§74.8. Formal consequences of the definitions. Before stating some of the formal consequences of the definitions, we draw a few useful distinctions. We want different ways of talking of "formulas' determining relevant properties." Of course, the definition (11) of $(\rho x Ax)a$ tells us when the formula A *actually* determines a relevant property of a (with respect to x). But when does the formula A *potentially* determine a relevant property of a (again with respect to x)? A natural thing to say is that it does so when it satisfies (7) Indiscernibility (with respect to a and x), because then *if* Aa is true *then* $(\rho x Ax)a$, i.e., A actually determines a relevant property of a (see Fact 1 below). (It should be noted, to forestall possible confusion, that the terminology of actual versus potential relevant predication introduced above has the linguistically awkward consequence that "actuality" need not imply "potentiality" in this case.)

There is yet one more distinction to draw. What happens when Indiscernibility holds, not just with respect to a, but for every individual? We have been a bit coy with respect to our use of variables like x, y as opposed to parameters like a, b, but let us now set down a firm policy: the former are to be given the generality interpretation, and understood as implicitly universally quantified, whereas the latter are to be understood as naming specific individuals. This allows us to state a stronger form of Indiscernibility for a formula (in the following, Ay results by proper substitution of y for x in Ax):

$$(7\forall) \qquad Ay \rightarrow (x=y \rightarrow Ax) \qquad \text{(Uniform Indiscernibility).}$$

In this case we want to say that Ax *potentially and* uniformly determines a relevant property (with respect to x). Since this is an awkward phrase, we shall also allow ourselves to say that Ax is a formula of a *kind that determines relevant* properties (with respect to x).

Clearly the distinctions above can be extended to talk of formulas' determining properties of ordered pairs, binary relations, etc.

We now state some formal facts about relevant predication, together with sketches of their proofs. We use dependency numerals in a staightforward way to keep track of when hypotheses have been used, in a way that mirrors the more formal account in the natural deduction system **FR** (see §R3). For the sake of not complicating things, we do not always bother to keep track of whether main hypotheses of theorems have been used, but do so only when we must know that a temporary hypothesis is used in deriving a conclusion, for the sake of proving a relevant implication.

FACT 1. If Ax is a formula of the kind that determines relevant properties then if Aa then $(\rho x Ax)a$.

PROOF

1	$Aa \rightarrow (x=a \rightarrow Ax)$	Assume Ax is of the kind that determines relevant properties
2	$Aa \rightarrow \forall x(x=a \rightarrow Ax)$	1 (taken as implicitly universally quantified), confinement
3	$Aa \rightarrow (\rho x Ax)a$	2, Def. of relevant predication (11)

FACT 2. If $(\rho x Ax)a$, then Aa (in English, if a property holds relevantly of an individual, then it also just plain holds of the individual).

PROOF.

1	$\forall x(x=a \rightarrow Ax) \rightarrow (a=a \rightarrow Aa)$	universal instantiation
2	$a=a$	*reflexivity*
3	$\forall x(x=a \rightarrow Ax) \rightarrow Aa$	1, 2 permutation, *modus ponens*
4	$(\rho x Ax)a \rightarrow Aa$	3, Def. of rel. pred. (11)

FACT 3. If Ax and Bx are formulas of a kind to determine relevant properties, then so are $\sim Ax$, $Ax \& Bx$, $Ax \vee Bx$, and $Ax \rightarrow Bx$. And if C has no free occurrences of x, then (still assuming that Ax is of a kind to determine relevant properties), $C \rightarrow Ax$ and $Ax \rightarrow C$ are also of a kind that determines relevant properties.

PROOF. Given that the usual De Morgan definition of disjunction in terms of negation and conjunction holds in relevance logic, we need explicitly consider only negation and conjunction in the proof below. Also, we omit the proofs for the implicational formulas on the grounds that they are perhaps of more specialized interest.

1	$Ay \rightarrow (x=y \rightarrow Ax)$	Assume Ax of the kind that determines relevant properties
2	$By \rightarrow (x=y \rightarrow Bx)$	Similarly for Bx
3	$x=y \rightarrow (Ay \rightarrow Ax)$	1, permutation
4	$x=y \rightarrow ({\sim}Ax \rightarrow {\sim}Ay)$	3, contraposition in consequent
5	$y=x \rightarrow ({\sim}Ay \rightarrow {\sim}Ax)$	4 (implicitly quantified), universal instantiation
6	$x=y \rightarrow ({\sim}Ay \rightarrow {\sim}Ax)$	5, symmetry of =
7	$x=y \rightarrow (By \rightarrow Bx)$	2, permutation
8	$x=y \rightarrow. (Ay\rightarrow Ax)\&(By\rightarrow Bx)$	3, 7, conjunction introduction
9	$x=y \rightarrow (Ay\&By \rightarrow. Ax\&Bx)$	8, conjunction introduction/ elimination

(The anonymous referee of the paper on which this section is based observes that Fact 3, concerning potential *and uniform* determination, does not seem to extend, as regards negation, to potential nonuniform determination. Thus A can potentially determine a relevant property of a (a a constant), without ${\sim}A$ potentially determining a relevant property of a. The referee suggests that this is a defect of the definition of potential determination, requiring as it does only "one-way" indiscernibility: $(x=a)\rightarrow.Aa\rightarrow Ax$. The referee suggests that the definition should be given in terms of the usual "two-way" indiscernibility: $(x=a)\rightarrow(Aa\rightleftarrows Ax)$. We believe that the referee is correct in his technical observations (although we have no model like those in §74.11 to show this). But we are not convinced of the referee's moral. These are arcane matters, of course, and we do not want to suggest that the referee might not have identified an important notion for certain purposes. But we think that the notion of potential determination as here defined using one-way indiscernibility also has a certain naturalness to it, linking as it does so directly to actual determination.)

FACT 4. Given that Ax relevantly implies Bx, if $(\rho xAx)a$ then $(\rho xBx)a$ (in English, relevant properties are closed under relevant implication). As a particular (somewhat surprising) example, since $A\rightarrow.A\vee B$ is an **R** theorem, if $(\rho xAx)a$ then $[\rho x(Ax\vee B)]a$. (This example will be discussed in §74.9 below.)

PROOF. We can show

$$A\to B \to [(\rho x Ax)a \to (\rho x Bx)a]$$

by assuming the first two antecedents $A\to B$ and $(\rho x Ax)a$ of the nested implication, and then deriving the consequent $(\rho x Bx)a$, *using* both antecedents. The following sequence of moves demonstrates this:

1	$A\to B$	hypothesis
2	$\forall x(x=a \to Ax)$	hypothesis
3	$\forall x(x=a \to Bx)$	from 1, 2 by universal instantiation, transitivity, and universal generalization

FACT 5. If Ax and By are formulas of a kind to determine relevant properties with respect to x and y respectively (and Ax has no free occurrences of y and, similarly, By has no free occurrences of x), then $Ax\&By$ and $Ax\vee By$ do not necessarily determine relevant relations, but $Ax\to By$ does.

PROOF. For the negative facts consult §74.11. For the positive fact we need to show

$$Au\to Bv \to [x=u \to (y=v \to.\ Ax\to By)].$$

This may be shown (again, see the system **FR** of §R3) by assuming as hypotheses for conditional proof each of its several antecedents, marking each with a distinct "dependency numeral" (dep. num.) to keep track of where it has been used, and deriving the consequent By using *all* the antecedents, as follows (note that we are allowed to switch the variables around for convenience in the assumptions 1 and 2, since they are implicitly universally quantified):

1	$Ax \to (u=x \to Au)$	Assume Ax of a kind that determines relevant properties
2	$Bv \to (y=v \to By)$	Similarly for Bx
3	$Au\to Bv$	hypothesis for conditional proof, dep. num. 3
4	$x=u$	hyp. for cond. proof, dep. num. 4
5	$y=v$	hyp. for cond. proof, dep. num. 5
6	Ax	hyp. for cond. proof, dep. num. 6
7	$u=x \to Au$	1, 6 *modus ponens*, dep. num. 6
8	Au	4 (symmetry of $=$), 7 *modus ponens*, dep. num. 4, 6
9	Bv	3, 8 *modus ponens*, dep. num. 3, 4, 6
10	$y=v \to By$	2, 9 *modus ponens*, dep. num. 2, 3, 4, 6
11	By	5, 10 *modus ponens*, dep. num. 2, 3, 4, 5, 6

FACT 6. If a formula Axy potentially {actually} determines a relevant relation (between a and b), then it potentially {actually} determines a property of a pair $\langle a, b\rangle$ and also potentially {actually} determines relevant relational properties (of a and b, respectively) in each of its positions x and y (this holds no matter how many places, although we explicitly treat only the binary case).

PROOF. We first treat the case where the formula potentially determines a relevant relation. That (13→) implies (13&) is an instance of the **R** theorem $[A\rightarrow.(B\rightarrow C)]\rightarrow[(A\&B)\rightarrow C]$. That (13→) implies say (13b) can be obtained by instantiating x to be a and applying *modus ponens*, using $a=a$ (reflexivity) as the minor premiss. The proof for (13a) is essentially the same, involving instantiation of y to b (but also permutation). Now for the case where the formula actually determines a relevant relation, we must show that (14→) implies each of (14&), (14a), and (14b), and this can be shown by moves similar to the above.

FACT 7. If Axy is a formula of a kind to determine relevant properties with respect to each of x and y, then Axy is of the kind to determine a relevant relation; i.e., a criterion for whether a formula defines a relevant relation is whether it determines a relevant relational property in each of its places (this holds no matter how many places, although we have explicitly stated only the binary case).

(The anonymous referee mentioned above, under Fact 3, also asks whether Fact 7 extends to relevant properties of and relations between particular objects (presumably, both potential and actual properties and relations), conjectures that it does not, but observes that the models of the kind in §74.11 do not answer the question. We believe that he is right in his conjecture, but have no models. The referee correctly observes that "it is only in the absence of this extension of Fact 7 that the notion of a relevant relational predication has independent significance.")

PROOF. We can show (by conditional proof) the required

$$Auv \rightarrow [x=u \rightarrow (y=v \rightarrow Axy)]$$

as follows:

1	$Auy \rightarrow (x=u \rightarrow Axy)$	
2	$Axv \rightarrow (y=v \rightarrow Axy)$	
3	Auv	hyp., dep. num. 3
4	$x=u$	hyp., dep. num. 4
5	$y=v$	hyp., dep. num. 5
6	Axv	3, 4 using 1 (instantiating y to v); dep. num. 3, 4
7	Axy	5, 6 using 2; dep. num. 3, 4, 5

FACT 8. If Ax and By potentially {actually} determine relevant properties (of a and b, respectively), then $Ax\&Ay$ potentially {actually} determines a property of the pair $\langle a, b\rangle$, but does not necessarily determine a relevant relation.

PROOF. For the negative fact, see §74.11. For the positive fact, we first suppose that Ax and By potentially determine relevant properties, and we must show that $Ax\&By$ does; i.e., we want

$$(Aa\&Bb) \rightarrow ((x=a \;\&\; y=b) \rightarrow (Ax\&By)).$$

This may be shown by conditional proof from the derivation below (and the positive fact about actual relevant properties follows similarly).

1	$Aa \rightarrow (x=a \rightarrow Ax)$	Assume that Ax potentially determines relevant properties
2	$Bb \rightarrow (y=b \rightarrow By)$	Similarly for By
3	$Aa\&Bb$	hyp., dep. num. 3
4	$x=a \;\&\; y=b$	hyp., dep. num. 4
5	Ax	3, 4, &-elim., 1, *modus ponens*; dep. num. 3, 4
6	By	3, 4, &-elim., 2, *modus ponens*; dep. num. 3, 4
7	$Ax\&By$	5, 6, &-intro.; dep. num. 3, 4

The next three facts concern identity, and might be viewed as answers to traditional metaphysical questions about that strange relation. Note that Fact 11 depends on taking the strong nested form of the transitivity axiom for identity.

FACT 9. (a) Being-identical-to-z is a relevant property of z. (b) Being-such-that-z-is-identical-to-it is also a relevant property of z (and is in fact equivalent to the postulate that identity satisfies symmetry). (c) Having the relevant property of being-identical-to-z is itself a relevant property of z (and this is equivalent to the postulate that identity satisfies nested transitivity). (d) Identity is a relevant relation between z and z (and, given reflexivity, this is equivalent to postulating that identity satisfies both symmetry and nested transitivity).

It is amusing to observe that the various parts of Fact 9 (and also Fact 10) when looked at "from ten yards" all seem to be saying (albeit in various convoluted ways) that z has the property of self-identity. Incidentally, we owe to the anonymous referee the suggestion of including (b) and (c), and the further observation that "nested transitivity is also equivalent to the principle that, for any property, it is always potentially a relevant property . . . to have it as a relevant property—i.e.,

$$(\rho x A x)z \to [\rho y([\rho x A x]y)]z,$$

which can be seen as a principle of iteration for the assertion that predication is relevant."

PROOF.

1. $[\rho x(x=z)]z$ amounts by definition to just $\forall x(x=z \to x=z)$, which is an obvious theorem of **R**.

2. $[\rho x(z=x)]z$ amounts by definition to $\forall x(x=z \to z=x)$, which is just symmetry.

3. $[\rho x(\rho y(y=z)x)]z$ amounts by definition to

$$\forall x[x=z \to \forall y(y=x \to y=z)],$$

which is just a permuted form of nested transitivity (with confinement of the quantifier $\forall y$).

4. $[\rho xy(x=y)]zz$ by definition is $\forall x[x=z \to \forall y(y=z \to x=y)]$, which is a well-known "textbook" postulate combining (in the presence of reflexivity) symmetry and (nested) transitivity (again modulo confinement of a quantifier).

FACT 10. Self-identity is a relevant property of a.

PROOF. $[\rho x(x=x)]a$ just amounts by definition to $\forall x(x=a \to x=x)$, and the following provides a proof of the latter:

1	$x=a$	hyp., dep. num. 1
2	$a=x$	1, symmetry; dep. num. 1
3	$x=x$	1, 2, transitivity

FACT 11. Identity is a relevant relation; more precisely, the formula $x=y$ is of a kind that determines relevant relations.

PROOF. We need to show $a=b \to (x=a \to (y=b \to x=y))$.

1	$a=b$	hyp., dep. num. 1
2	$x=a$	hyp., dep. num. 2
3	$y=b$	hyp., dep. num. 3
4	$x=b$	1, 2 *nested* transitivity; dep. num. 1, 2
5	$x=y$	3, 4 symmetry, *nested* transitivity; dep. num. 1, 2, 3

FACT 12. All arithmetical relations in the system $\mathbf{R}^\sharp$ of relevant arithmetic of Meyer 1976e—see §72—are relevant.

PROOF. Here we must be very sketchy. The main fact (Meyer; see §72) is that the following is a theorem of $\mathbf{R}^{\sharp}$:

$$(15) \qquad x=y \to z=z.$$

This is surely somewhat surprising, and smacks of irrelevance, but it is actually easily proved by first establishing the following special case:

$$(16) \qquad x=y \to 0=0,$$

and then adding z to each side of the consequent. We can establish (16) by first deriving

$$(17) \qquad x=y \to x=x$$

(from the symmetry and transitivity of =, see Fact 10) and then "subtracting" x from both sides of the consequent (the uses of addition and subtraction can be justified by the induction postulate in arithmetic).

We can now begin a structural induction on terms, to show that in general

$$(18) \qquad x=y \to t(x)=t(y).$$

The only real problem arises in the degenerate cases, where perhaps x does not occur in $t(x)$ or no instance of x is replaced by y. But, by (15), we have, for an arbitrary term t,

$$(15t) \qquad x=y \to t=t,$$

which is just "the irrelevancy" needed.

Once we have (18), it is a piece of cake to establish, by structural induction on formulas,

$$(19) \qquad x=y \to .\ Ax \to Ay.$$

The base case for this structural induction is

$$(20) \qquad x=y \to .\ s(x)=t(x) \to s(y)=t(y),$$

which follows from (18) and

$$(18') \qquad x=y \to .\ s(x)=s(y),$$

using symmetry and *nested* transitivity for =.

We are now close to home, since (19), when permuted, becomes

$$(21) \qquad Ax \to (x=y \to Ay),$$

which is just to say that any formula A of $\mathbf{R}^{\sharp}$ is of a kind to determine a relevant property (with respect to each of its variables). Using Fact 7, we then know that each formula is also of a kind to determine relevant relations in its variables.

What are we to make of the fact that relevant arithmetic in the sense of $\mathbf{R}^\#$ cannot distinguish between relevant and irrelevant properties and relations? We think nothing negative. Reflecting on the proofs, particularly of (16) (where it all begins to happen), nothing seems amiss. Fact 12 can just be seen as expressing the strong intuition that, in the domain of numbers, each number is tightly connected to every other number. They are all generated "in a straight line" from 0, and one can get by way of this inductive process from one number to another by addition or subtraction (depending on which is larger). G. Hellman has pointed out to us that a structuralist account of numbers reinforces this view, since, put quickly, a number is nothing but its position in an infinite sequence.

§74.9. Background. For background the reader should consult in this book especially §70 and §71; earlier items include Curry and Feys 1958, Helman 1977 and 1977a, Belnap 197+ and 197+a, Urquhart 1989, Meyer 197+, and Freeman 1975. We use a little discussion of these investigations as a springboard from which to launch an observation or two, but we offer hardly any history. For some disentangling of the strands of the history of the idea of relevant predication, and for the award of due credit, see §9 of Dunn 1987a on which this section is based.

Our main concern is going to be with discussions in the early seventies. Several workers worried about how easily one could validate

(22) $x=y \rightarrow . A \rightarrow A$,

and the same observation was made by others (Urquhart and Meyer) in the context of second-order relevant logic, given (unrestricted) comprehension and the usual definition of identity as sharing all the same properties. The above "thesis" would seem to be somewhat of a paradigm of an irrelevant predication (see §74.3 above). Various proposals were made, most privately, about how to avoid this consequence.

A frequent feature of these proposals was that formulas should be allowed into the comprehension scheme

(23) $\exists F \forall x(Fx \rightleftarrows Ax)$ (Strict Comprehension)

just when Ax is "strict" in x, where this strictness was to be given an entirely syntactical characterization in terms of how the free occurrences of x are distributed in Ax. One proposal was that atomic formulas should always be counted as determining (relevant) properties and that compound formulas should be counted as determining properties only when they meet certain restrictions about "dependence" on their free variables. The actual detailed restrictions varied, the action centering around conjunction (and disjunction).

The discussion of these matters in the early seventies was not always perfectly clear (certainly ours was not), especially as to what was depending on what, and what was intended by the dependence relation (for example, whether it was intended distributively or collectively, or perhaps yet otherwise), but we beg leave to continue waffling or fudging for a few paragraphs in order to make enough conceptual headway to enable us to draw a contrast.

Given some relation of dependence, all parties in this period seemed to agree that $\sim A$ depends on precisely the same set of variables on which A depends (if any), that the dependence conditions for disjunction are precisely the same as those for conjunction (this is natural, given the above decision about negation and De Morgan's laws), and that $A \rightarrow B$ depends on $U \cup V$ when A depends on U and B depends on V (but otherwise on no set of variables).

There are at least three simple proposals regarding conjunction that floated during the period in question. If A depends on a set of variables U and B depends on a set of variables V, then $A \& B$ should depend (1) on $U \cup V$, or (2) on $U \cap V$, or (3) on U when $U = V$ (but otherwise on no set of variables). (1) leads quickly to irrelevance unless compensating restrictions are made, (2) leads to a few strange things—for example, $(Fx \& A) \rightarrow Fx$ is "strict" in x on this account, where A is any sentence (or formula not containing a free occurrence of x); and (3) is just about right. (We remind the reader that we are recounting early proposals; see the analysis of dependence in terms of "used evenly" in §71.3 for the most refined and profitable way to clarify the idea that (3) is aiming for.)

Notice that proposal (2) in effect just takes intersections when it comes to conjunctions. This is reminiscent of certain proposals about how to handle conjunction introduction in relevance logic, wherein it is said that it is always permissible to perform conjunction introduction, with the resultant conjunction depending on the intersection of the sets of hypotheses on which the two premisses depend separately; see Pottinger 1972, or the brief discussion of subscript deletion, which is equivalent, in §27.2, and also §71.3. But this is not the way conjunction introduction is handled in the orthodox system **R** in, say, §23.1 and §26.2. Rather, in **R** one can perform conjunction introduction not always, but only when the premisses depend on precisely the same set of hypotheses, in which case the conjunction depends again on that same set of hypotheses. This is analogous to the proposal (3), and certainly makes it seem natural in the context of **R**.

Fudging some distinctions, let us label by "strictness" the relation of dependence that a formula has to its variables, cashed out as either (2) or (3) (perhaps in some variant, most especially the refined proposal in terms of "used evenly" of §71.3 mentioned above). We shall call the proposal that all and only strict formulas determine properties "The Strict Proposal."

There are a number of similarities, but also contrasts between The Strict Proposal and the ideas of the present section. To our mind the least significant of the contrasts is that The Strict Proposal tended to find its expression in the context of second-order logic, whereas the ideas here have been expressed in the context of first-order logic. As we said in §74.1, one can wax ontological about these matters, and make relevant predication a question of what properties *really* exist, and this is the sort of thing that the Comprehension Axiom encourages. The fundamental question is not, however, ontological; it is logical, and it is instead something like, What are *really* properties of a thing? People of a nominalistic tendency may prefer the account here in that it does not postulate the existence of properties, but we do not mind second-order logic ourselves, and so see this as only of minimal advantage. One could easily extend the ideas of this section to second-order logic, interpreting them as advocating that the Comprehension Axiom should hold only for relevant properties:

$$(24) \qquad \exists F\forall x(Fx \rightleftarrows (\rho yAy)x) \qquad \text{(Relevant Comprehension)}$$

More interesting might be to have two styles of second-order variables, one for relevant properties and one for ordinary "classical" properties, and a separate comprehension axiom for each.

The most significant contrast is that The Strict Proposal is "metalinguistic" and external to the language of relevance logic, whereas the ideas of this section are expressed in the object language, and internal to the language of relevance logic. There is no attempt to figure out some syntactical test for real dependence on variables that is somehow consonant with the ideas of relevance logic. Rather, relevance logic itself is used to determine such real dependence. Thus we can think of "strictness" defined as an object-language notion:

$$(25) \qquad \mathrm{Str}(A, x) =_{\mathrm{df}} \forall x(Ax \to \forall y(y=x \to Ay)) \qquad (y \text{ not free in } A).$$

The careful reader will observe that these are just other words (and a slightly different but equivalent formula) for the idea introduced in §74.8 of a formula A's being "of a kind that determines relevant properties." For the sake of drama, we tentatively state a

CONJECTURE. Suppose we take as axioms all formulas of the form Str(A, x), where A is an atomic sentence and x is a variable occurring in A. Then, for all formulas A (compound as well), Str(A, x) is a theorem iff A is (provably relevantly equivalent to a formula B which is) strict in x in the sense of proposal (3) (or more likely in the refined form due to Helman 1977 and described in §71.3).

Note that this conjecture must be complicated by the addition of the material in parentheses, since without this complexity the conjecture would fail for trivial reasons. Thus $Fx \rightleftarrows .Fx \& (Fx \vee A)$ is a theorem of **R**, and although its left-hand side would clearly pass the syntactical tests for strictness of proposal (3) and its variants, just as clearly its right-hand side would fail. And yet since Str(Fx, x) is a theorem, then (by Replacement of relevant equivalents) we also have as a theorem Str($Fx\&(Fx\vee A)$). It is phenomena such as these that make the proving of the conjecture nontrivial, since a simple structural induction on the form of A messes up on conjunction (and disjunction).

In fact even the above conjecture as amended modulo relevant equivalence is false, when strictness is understood according to proposal (3); for it is easy to see that Str($Fxy\&Gx$, x) is a theorem (given the axioms Str(Fxy, x) and Str(Gx, x)), and yet, according to (3), there is no set of variables on which $Fxy\&Gx$ depends. However, when the conjecture is read giving "strictness" the sense of §71.3, this problem disappears. Indeed, an easy induction verifies half of the biconditional, namely, that if A is strict in x in the sense of §71.3, then Str(A, x) (but the other half is still problematic, for the reasons indicated in the paragraph above).

No matter how the conjecture turns out about what are the kinds of formulas that determine relevant properties (strict formulas), there appears (at first blush anyway) to be a clear formal difference between the ideas from the early seventies about "strictness" and the ideas of this section as to what are the *actual* relevant properties. Thus we know from Fact 4 above that if $(\rho xAx)a$, then also $[\rho x(Ax\vee B)]a$. So pick any actual relevant property of a (say that of being identical to a, from Fact 9 above). Then its disjunction with any arbitrary formula B is also a relevant property of a, even when B is not strict in x. But clearly this is not a property on The Strict Proposal, which would demand that A also be strict in x.

What is one to make of this doctrinal difference between the two otherwise sympathetic accounts of predication? It may be too early to say, but, if one sees them as competing, then we are presently of the mood to favor the account given here (if for no other reasons than the petty one of vested interest). Rhetoric aside, it seems that

(26) $[\rho xFx]a\vdash[\rho x(Fx\vee A)]a$

is just as much a natural part of **R** as its sentential analogue

(27) $B\vdash B\vee A.$

Note, however, that one could not deduce $[\rho x(Fx\vee A)]a$ starting from the assumption that A, which may have something to do with intuitions about the *supposed* failure of the rule of Addition in relevance logic.

It is, then, a mistake to see the two accounts as competing, even though historically they probably developed that way. The right way to look at (actual) relevant predication is not given by way of The Strict Proposal as such. The Strict Proposal tells us which properties are strict, and then it is natural (though mistaken) to identify relevant predication with having a property that is strict. Even though this last move seems mistaken, this does not by itself destroy the interest in looking at strict properties. Indeed, we even saw that a natural definition of strictness (25) emerges in the present framework.

A formal analogy occurs with the work of Bressan 1972 on a modal logic involving quantifiers whose intensional values are individual concepts. The Relevant Predication formula (11) from §74.6, if read in Bressan's system, could come to "*a* has *F* under every description," which is all well and good and useful. There is also in the Bressan framework the notion of a property's being "extensional," which amounts to our object-language definition of strictness (25). We observe that even though, whenever *a* has *F* under every description, it follows trivially that *a* has *F* or *G* under every description, still it does not follow that if *F* is extensional then *F* or *G* is extensional.

So, once again to use a favorite saying that we learned from Our Senior Author, "Let a few flowers bloom."

We note that Kremer 1989, to which we refer the reader, refines and settles a number of questions of the kind posed in the Conjecture above; and further, because of our aim of highlighting the use of the richness of relevance logic to help out here and there, we choose to close this portion of our report with a small result regarding the specification of relevant *functions* "internally." Given first-order **R** with pure identity (we mean Reflexivity, Symmetry, and Nested Transitivity, but not (8) Substitution), augmented perhaps with some nonlogical axioms, one can conservatively add a function letter f subject to the axioms

$$f(x)=y \rightleftarrows Axy$$
$$x=x' \rightarrow f(x)=f(x')$$

if and only if the following formulas are theorems (of the possibly augmented theory):

(a) $\forall x\exists yAxy$ (Existence)
(b) $Axy \rightarrow (Axy' \rightarrow y=y')$ (Uniqueness)
(c) $x=x' \rightarrow (Axy \rightarrow Ax'y)$ (Dependence).

It is perhaps interesting to note that this requires only that Axy determine a relevant property in the "first position." Also we would like to "readvertise" the following:

PROJECT. Develop a theory of computable relevant functions. Constraint: it should be entertaining.

§74.10. Philosophical applications. One of us (JMD) plans a follow up on the investigations reported in the present section, to examine at least potential philosophical applications. A quick way to give the flavor is to say that the apparatus of relevant predication should be able to stop, once and for all, the "funny business" that fuels a whole philosophical industry of the "Chisholming" kind. By this label we by no means mean to denigrate the real philosophical contributions made by people working in this tradition. Indeed they are to be admired for their craftsmanship using the primitive tool of classical logic. But there should be no need to develop clauses upon clauses, just to stop what were clearly inappropriate counterexamples (relevant implication is also to be recommended in this regard).

A quick list of the topics that might be discussed under applications includes intrinsic properties and Cambridge change, internal relations, essential properties, intentional relations, predication involving fictional and other nonexistent objects, questions about whether existence and truth are properties (and whether exemplification is a relation), Russell's paradox of predication, Goodman's grue-bleen paradox, Goodman's problems about aboutness, and Barwise and Perry's "slingshot."

§74.11. Technical appendix. Many of the remarks above depend on the fact that certain mentioned formulas are not theorems of first-order **R** with identity. Let us understand by this system the result of adding Reflexivity, Symmetry, (Nested) Transitivity, and Substitution (from §74.5) to the axioms of first-order **R**.

It is easy to produce a model for these axioms that brings out some of the features of relevant predication, using as a value algebra a *Sugihara chain* (S, $\leq$, $-$), where (S, $\leq$) is any linearly ordered set, and $-$ is an *involution* on S, i.e., for $x, y \in S$, $x \leq y$ only if $-y \leq -x$, and $--x=x$. Such chains were employed in §§26.9, 29.3, and 29.4. Typical examples are the integers, rationals, or reals (with or without 0), and we shall have special use for finite versions S_n consisting of the nonzero integers $-n$ through $+n$. The *designated elements* (thought of as "true") consist of all those elements x such that $-x \leq x$. In the sequel we shall restrict our attention to these finite versions, although it is easy to adapt our modelings to any Sugihara chain with a least element.

It is possible to get some intuitive feel for a Sugihara matrix by thinking of the designated elements as "degrees of truth." The reader should be cautioned, though, that Sugihara matrices provide a characteristic semantics not for the relevance logic **R**, but instead for a stronger—and simpler—logic, **RM** (see §29.3).

We can now use a Sugihara chain in the valuation of formulas as follows: let D be any nonempty set thought of as the domain, and (following Smullyan 1968) let actual elements of this domain enter into formulas (thought of as finite sequences) in place of variables. A D-*sentence* is, then, such a formula with no free variables (to simplify presentation we shall suppose for the moment that we have no individual constants or function letters). A valuation will then be any mapping φ of the D-sentences into S_n satisfying the following clauses:

$$\begin{aligned}
\varphi(c=d) &= +i \text{ for some } i \leq n \text{ if } c \text{ equals } d, \text{ and} \\
&= -n \text{ otherwise;} \\
\varphi({\sim}A) &= -\varphi(A); \\
\varphi(A\&B) &= \min(\varphi(A), \varphi(B)); \\
\varphi(A \vee B) &= \max(\varphi(A), \varphi(B)); \\
\varphi(A \rightarrow B) &= \max(-\varphi(A), \varphi(B)) \text{ if } \varphi(A) \leq \varphi(B); \\
&= \min(-\varphi(A), \varphi(B)) \text{ if } \varphi(A) > \varphi(B); \\
\varphi(\forall x A) &= \min\{A(d/x)\text{: } d \in \mathrm{D}\}.
\end{aligned}$$

This interpretation can easily be extended to formulas containing free individual variables or constants (and even function letters). Thus let φ now also assign to each individual variable or constant a member of D, and let it assign to each n-placed function letter an n-place operation on D. Then of course $\varphi(ft_1, \ldots, ft_n) = (\varphi f)(\varphi t_1, \ldots, \varphi t_n)$.

Then the idea is that φ should assign to a formula A precisely the same value that it assigns to the corresponding D-formula A' that results from replacing terms with the actual elements of the domain that φ assigns them.

It is well known that such an interpretation satisfies the axioms of first-order **R** (see §53 and references therein), and it is easy to see that this interpretation satisfies the identity axioms above. Thus, that Reflexivity and Symmetry always take on designated values follows almost immediately from the valuation clauses, using the fact that our equality ("equality in the metalanguage") has the corresponding properties.

That Substitution always takes on a designated value is also easy to see. There are two cases: either c equals d, in which case Ac and Ad are really the same D-sentences and the question of validity is the same as that of $(A\&B)\rightarrow A$. Or else c does not equal d, in which case $c=d$ evaluates to the least element $-n$ (which "implies" everything), and hence so does the conjunction $Ac\&(c=d)$.

Nested Transitivity is only slightly more difficult. Thus let us consider the D-sentence

$$c=d \rightarrow (d=e \rightarrow c=e).$$

There are two main cases, depending on whether c and d are the same element or not. If not, then $c=d$ evaluates to the least element $-n$, which "implies" everything, and so we are immediately OK. If c and d are the same element, then we are really looking at the D-sentence

$$c=c \rightarrow (c=e \rightarrow c=e).$$

Again we have two subcases, depending on whether c and e are the same element or not. If not, $c=e$ evaluates to $-n$, and so the consequent evaluates to the greatest element $+n$ ("implied" by everything), and so we are OK. If c and e are the same element, then we are looking at

$$c=c \rightarrow (c=c \rightarrow c=c),$$

or, in effect,

$$+i \rightarrow. +i \rightarrow +i,$$

which evaluates as $+i \rightarrow +i$, i.e., $+i$.

But, using S_2, it is easy to see that the formula from §74.4,

$$(\#\#) \qquad x=a \rightarrow. Fa \rightarrow Fx$$

is not a theorem: let φ assign to x and a the very same object $d \in D$, let $\varphi(d=d) = +2$, and let $\varphi(Fd) = +1$.

In a similar manner one can establish the other claims in the body of the text concerning unprovability. We shall run through one more example. This is the claim that, even when Fx and Gy determine relevant properties of a and b respectively, $Fx\&Gy$ need not determine a relevant relation between a and b. We need then a model in which the following two formulas are assigned designated values:

$$\text{(a)} \qquad \forall x(x=a \rightarrow Fx)$$
$$\text{(b)} \qquad \forall y(y=b \rightarrow Gy),$$

but in which the next formula is not:

$$\text{(ab)} \qquad \forall x \forall y(x=a \rightarrow (y=b \rightarrow. Fx\&Gy)).$$

It turns out that the formula (ab) can be easily falsified in the Sugihara chain S_2 in a domain with two elements c and d. Thus let φ assign $c=c$ the value $+2$, and $d=d$ the value $+1$, and let $c=d$ and $d=c$ be assigned the value -2. Let $\varphi(Fc) = +2$, and let $\varphi(Gd) = +1$. It is easy to see that the formula (a) then takes on the designated value $+2$, and that the formula (b) takes on the designated value $+1$. However, (ab) takes on the value -2. Thus look at the following D-formula instance:

$$\text{(cd)} \qquad c=c \rightarrow (d=d \rightarrow. Fc\&Gd).$$

Then $Fc\&Gd$ takes on the minimum value $+1$, and so we are in effect looking at $+2 \rightarrow. +1 \rightarrow +1$, which can be seen to compute to $+2 \rightarrow +1$, i.e., -2. But (ab) takes on the minimal value of its instances, and so takes on the undesignated value -2.

§75. Relevant implication and conditional assertion (by Daniel Cohen). Belnap 1973 offers an account of the use of conditional locutions to make "conditional assertions." The notion of a conditional assertion derives from a suggestion credited to Rhinelander and reported in Quine 1950, distinguishing the (categorical) assertion of a conditional from the conditional assertion of its consequent. The assertion of a proposition, B, on the condition that A, is symbolized A/B. It asserts just what B asserts alone, when the condition of assertion is met. When that condition is not met, there is no assertion; the statement is "nonassertive." Proof-theoretic accounts of conditional assertions can be found in Dunn 1975, van Fraassen 1975a, Manor 1971, Cohen 1983, and Cohen 1986. A more detailed semanticopragmatic account of conditional assertions is given in Belnap 1973, which is presupposed here.

Belnap 1973 shows how the slash of conditional assertion can be used to effect restricted quantification. This is a consequence of the peculiar status of nonassertive statements in his calculus of statements: they neither contribute to nor detract from the truth-functional compounds in which they occur. They "drop out," as it were, of zero degree formulas.

The ability to effect restricted quantification is desirable for the relevance logics—and indeed for any formal system whose conditional connective is strong. The objections that might be raised against such systems, based on the now-traditional treatment of Aristotle's A-propositions as quantified conditionals, can all be forestalled by choosing the appropriate conditional for a given generalization.

Consider, for example, the conditional connective $\rightarrow$ of the relevance logic **R**. Suppose that this is the conditional used in formalizing universal generalizations. Then, "All crows are black" might be rendered as

$$\forall x(Cx \rightarrow Bx).$$

Perhaps this asserts too much. It says that, in every case, something's being a crow relevantly implies its being black. Although this may be unobjectionable in this case, not all universal generalizations are meant to assert something that strong. Strong conditional connectives make for strong generalizations. However, accidental generalizations are often our concern.

In contrast, the "material implication" connective $\supset$, which can be expressed in **R** using disjunction and negation, is too weak. In particular, it cannot support the inference from an instance of the subject class to the corresponding instance of the predicate class. That is,

$$(\forall x(Cx \supset Bx) \& Ca) \rightarrow Ba$$

is not **R**-valid. What is needed is a conditional connective not too strong to express accidental generalizations but not too weak to ground implications. The slash of conditional assertion is such a connective.

This section lays the foundation for the eventual application of the connection between conditional assertion and restricted quantification to formal systems with stronger conditional connectives. The system to be presented is a result of combining the relevance logic **R** and the logic of conditional assertions, **CA**. The result, called **RCA**, is a "multiconditional logic." It is axiomatized and proved sound and complete.

§75.1. Assertivity functions.

Axiomatizing the logic of conditional assertions involves some problems all its own. As noted in Dunn 1975, the presence of nonassertive formulas—which can be ignored as conjuncts or disjuncts of larger formulas—means that there are no generally *valid*, i.e., always-true, formula schemata. When A is nonassertive, so is $A \vee {\sim} A$. Even worse, in that case, $(A \vee {\sim} A) \vee B$ takes the value of B, which may be false. Dunn's solution is to abandon the use of axiom schemata in favor of actual axioms. This exploits the categorical nature of atomic formulas: nonassertiveness arises only from the failure of a conditional assertion; so atomic sentences are always assertive. Of course, some of the *instances* of the always-true axioms will still be untrue or even false; so unrestricted substitution cannot be unrestrictedly available. A similar problem occurs when an implication connective is at hand: the tautological entailment

$$A \& B \rightarrow . B \vee C$$

loses its relevant connection entirely when B is nonassertive.

A three-part solution to these difficulties is adopted here. First, the axiomatization to be presented is an axiomatization of the *never-false* formulas of the language, not the always-true ones. This is in line with the pragmatic characterization of failed conditional assertions in Belnap 1973. (However, Dunn notes that success in axiomatizing either the always-trues or the never-falses guarantees success in the other endeavor.) This alone does not solve the problems because, as we have seen, some classically valid schemata have false instances in the new language. It does point to the rest of the solution, though. The substitution of nonassertive formulas creates the problems, but nonassertive instances are not themselves the problem. Nonassertive instances are not to count against the validity of a formula schema. Nonassertiveness arises from the falsity of the antecedent, the condition of assertion, of a conditional assertion. Therefore, if the assertiveness of a formula is explicitly made the condition of assertion for substituting that formula into an always-true one, then the resulting instance will either be true, if the substituted formula does assert, or nonassertive, as a failed conditional assertion. That is, if

$$\Phi(\ldots p \ldots)$$

is an always-true formula (not schema), then

$(A \text{ is assertive})/\Phi(\ldots A \ldots)$

will never be false.

All that is needed to implement this is an object-language representation of "A is assertive." To that end, an "assertivity function" is introduced. Briefly, the assertivity function is a metalinguistic function mapping each sentence in the object language onto another. The value of the function represents the "conditions of assertiveness" for the first sentence. Letting α represent that function, the necessary and sufficient conditions for the adequacy of α are as follows:

(i) αA is true iff A is assertive (true or false), and
(ii) αA is false iff A is nonassertive.

(For a fuller discussion of the nature of and possibilities for determining the conditions of assertiveness, see Cohen 1983, pp. 64–73.)

These criteria are met by the following inductive definition, which also exploits the categoricity of atomic formulas:

0. If A is a propositional constant, αA is $A \vee {\sim} A$.
1. If A is of the form ${\sim} B$, αA is αB.
2. If A is of the form $B \vee C$, αA is $\alpha B \vee \alpha C$.
3. If A is of the form $B \& C$, αA is $\alpha B \vee \alpha C$.
4. If A is of the form $B \rightarrow C$, αA is $\alpha B \& \alpha C$.
5. If A is of the form B/C, αA is $B \& \alpha C$.

An assertive {disjunction} conjunction must have at least one assertive {disjunct} conjunct. Relevant implication is understood as a relation between propositions (or *assertive* statements) only. The nonfalsity of the antecedent and the assertiveness of the main clause are both needed for the assertiveness of a conditional assertion.

The assertivity function can now be used to accompany genuine axioms. However, that function can also be used in the axiomatization itself, obviating the need for any kind of substitution rule. That is, the third and final step is to abandon axioms—$\Phi(\ldots p \ldots)$—and conditioned substitution instances—$\alpha A/\Phi(\ldots A \ldots)$—in favor of simply taking the latter as part of the (conditioned) axiomatization.

§75.2. Axiomatization. The axioms of **RCA** fall into three categories: the axioms of **R**, the axioms of **CA**, and the axioms governing the merger. A formula A is an axiom of **CA** iff there are formulas B, C, and D such that

A is identical to one of the following (the slash is associated to the right):

CA1	$B \vee {\sim} B$
CA2	${\sim}\alpha B/B$
CA3	$(B \& {\sim} B)/{\sim}\alpha B$
CA4	${\sim}\alpha\alpha B/C$
CA5	${\sim}(B/C)/(B/{\sim}C)$ and $(B/{\sim}C)/{\sim}(B/C)$
CA6	$B/C/(B \& C)$
CA7	$(B \& C)/B$ and $(B \& C)/C$
CA8	$(B/C)/(D/C)/((B \vee D)/C)$
CA9	$B/C/B$
CA10	$(B/C/D)/(B/C)/(B/D)$.

The additional axioms are given by:

RCA1	${\sim}\alpha C/((B \vee C) \to B)$ and ${\sim}\alpha B/((B \vee C) \to C)$
RCA2	${\sim}\alpha C/(B \to (B \& C))$ and ${\sim}\alpha B/(C \to (B \& C))$
RCA3	$C \to (B/C)$
RCA4	$\alpha C/C/(C \to B)/B$.

The sole rule of inference for **RCA** is detachment for the slash:

DET from B and B/C to infer C.

CA2 can be read as expressing the system's tolerance for nonassertive theorems. In fact, every instance of CA2 is nonassertive. If the antecedent is true, then the consequent is nonassertive and so is the whole. If, on the other hand, the consequent is assertive, then the antecedent is false and the entire conditional assertion is nonassertive on that account. The fourth axiom governing the merger—$\alpha C/C/(C \to B)/B$—expresses the kind of *modus ponens* (MP) available for relevant implications. A nonassertive component makes a statement of implication itself nonassertive; so a nonassertive antecedent is no basis for inferring the consequent from a nonfalse implication.

Several points about this axiomatization are worth noting. The first is that the axioms of **CA**, together with DET, constitute a sound and complete subsystem relative to the semantics suggested in Belnap 1973. Proofs are to be found in Cohen 1983.

Second, the fact that *modus ponens* is not generally available in an unrestricted form does mean that the theorem schemata of **R** are not all provable in **RCA**. The form cited above—$A \& B \to . B \vee C$—is such an example. In fairness, it should be noted that all the axiom schemata of **R** are unfalsifiable in the semantics for **RCA** to be presented (a testimony, perhaps, to the insight of the architects of **R**). Further, since nonassertiveness arises only from the failure of a conditional assertion, the slash-free formulas of **R** are always assertive, and indeed **RCA** is a conservative extension of **R** (and of **CA**).

Third, the axiomatization of **RCA** is quite unusual. The assertivity function is not one-one: $p \vee \sim p$ is the value of αp as well as of $\alpha(\sim p)$, $\alpha(\sim\sim p)$, etc. Also, the assertivity function is not a homomorphism with respect to substitution: sub(A for p in αB) is not the same as α(sub(A for p in B)). Thus, the axiomatization provides neither a finite set of axioms nor even a finite set of axiom schemata. Both $\sim(p \vee \sim p)/p$ and $\sim(p \& (q \vee \sim q))/(p/q)$ are instances of CA2, although there is no single valid schematic form under which they both fall. Nevertheless, the axiom set is decidable. There are only a finite number of these quasi-schemata, and it can be decided whether or not a given formula is an instance of one. Each quasi-schema is either a genuine schema (among those presented without the assertivity function), a case in which the assertivity function is applied only to some other subformula (and the assertivity function is functional; it is easily decided whether some subformula is the value of α at that other), or else a case of CA4 (and this is decidable, because αB is bivalent, always assertive, and so $\alpha\alpha B$ is always a case of an excluded middle with assorted other disjuncts).

§75.3. Semantics. An **RCA** *model structure*, like an **R** model structure as defined in §48.5, is an ordered quadruple (K, 0, R, *), where K is a set, 0 is an element of K, R is a three-placed relation on those elements (R$\subseteq$K$\times$K$\times$K), and * is an operation on the members of K. The relation R, as for **R**-models, must meet the following conditions:

R1. R0aa.
R2. Raaa.
R3. If Rabc and Rcde then $\exists$f(Radf and Rfbe) (the "Pasch property").
R4. If R0ax and Rxbc then Rabc.
R5. If Rabc, then Rac*b*.
R6. a** = a.

RCA model structures are distinguished in that, in addition, 0* must be equivalent with 0. This can be formalized by adding a seventh condition:

R7. R000*.

This last is tantamount to requiring that the 0-points of **RCA** be consistent in that no sentence and its negation are both assigned the value T at 0. That this last condition can be imposed without damaging the structure is established in Meyer and Routley 1973.

An **RCA**-*model* is an **RCA** model structure together with a valuation. A *valuation* V is an assignment that maps the cross product of the set of atomic sentences and the set K into the set $\{T, F\}$ that satisfies the "Hereditary property":

H: If R0ab and V(p, a) = T, then V(p, b) = T.

It is extended into a mapping from the cross product of the set of *all* formulas and K into the set $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{F}\}$ by the following rules:

$$\begin{array}{lll}
(\mathrm{V}\&) & \mathrm{V}(A\&B, \mathrm{w}) = \boldsymbol{T} & \text{if } \mathrm{V}(A, \mathrm{w}) = \mathrm{V}(B, \mathrm{w}) = \boldsymbol{T}, \text{ or} \\
 & & \quad \mathrm{V}(A, \mathrm{w}) = \boldsymbol{N} \text{ and } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{T}, \text{ or} \\
 & & \quad \mathrm{V}(A, \mathrm{w}) = \boldsymbol{T} \text{ and } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{N} & \text{if } \mathrm{V}(A, \mathrm{w}) = \mathrm{V}(B, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{F} & \text{otherwise.} \\
(\mathrm{V}\vee) & \mathrm{V}(A\vee B, \mathrm{w}) = \boldsymbol{T} & \text{if } \mathrm{V}(A, \mathrm{w}) = \boldsymbol{T} \text{ or } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{T}; \\
 & \qquad\qquad = \boldsymbol{N} & \text{if } \mathrm{V}(A, \mathrm{w}) = \mathrm{V}(B, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{F} & \text{otherwise.} \\
(\mathrm{V}\sim) & \mathrm{V}({\sim}A, \mathrm{w}) = \boldsymbol{T} & \text{if } \mathrm{V}(A, \mathrm{w}^*) = \boldsymbol{F}; \\
 & \qquad\qquad = \boldsymbol{N} & \text{if } \mathrm{V}(A, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{F} & \text{otherwise.} \\
(\mathrm{V}\rightarrow) & \mathrm{V}(A\rightarrow B, \mathrm{w}) = \boldsymbol{T} & \text{if } \mathrm{V}(A, \mathrm{w}) \neq \boldsymbol{N} \text{ and} \\
 & & \quad \mathrm{V}(B, \mathrm{w}) \neq \boldsymbol{N} \text{ and } \forall\mathrm{x}\forall\mathrm{y} \text{ (if Rwxy and} \\
 & & \quad \mathrm{V}(A, \mathrm{x}) = \boldsymbol{T} \text{ then } \mathrm{V}(B, \mathrm{y}) = \boldsymbol{T}); \\
 & \qquad\qquad = \boldsymbol{N} & \text{if } \mathrm{V}(A, \mathrm{w}) = \boldsymbol{N} \text{ or } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{F} & \text{otherwise.} \\
(\mathrm{V}/) & \mathrm{V}(A/B, \mathrm{w}) = \boldsymbol{T} & \text{if } \mathrm{V}(A, 0) \neq \boldsymbol{F} \text{ and } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{T}; \\
 & \qquad\qquad = \boldsymbol{N} & \text{if } \mathrm{V}(A, 0) = \boldsymbol{F} \text{ or } \mathrm{V}(B, \mathrm{w}) = \boldsymbol{N}; \\
 & \qquad\qquad = \boldsymbol{F} & \text{otherwise.}
\end{array}$$

As in **CA**, nonfalsity—as opposed to truth—is the required state for valid formulas. As in **R**, it is only the 0-points of models that matter. A formula is **RCA**-*valid*, then, iff it is nonfalse at the 0-point of every **RCA**-model.

§75.4. Soundness.

THEOREM. **RCA** is sound; i.e., every theorem of **RCA** is valid and cannot be falsified at the 0-point of any **RCA**-model.

The proof, which is only sketched here, is a straight induction from the validity of the axioms and the validity-preserving nature of the only rule of inference. Verifying the axioms involves the use of inductions, because of the presence of the inductively defined assertivity function. These are most easily carried out in the following three lemmas. Proofs are omitted:

LEMMA 1. For all sentences A, $\mathrm{V}(\alpha A, 0) \neq \boldsymbol{N}$.

LEMMA 2. For all sentences A, $\mathrm{V}(\alpha A, 0) = \boldsymbol{T}$ iff $\mathrm{V}(A, 0) \neq \boldsymbol{N}$.

LEMMA 3. For all sentences A, if $\mathrm{V}(A, \mathrm{w}) = \boldsymbol{N}$ for some $\mathrm{w} \in \mathrm{K}$, then $\mathrm{V}(A, y) = \boldsymbol{N}$ for all $\mathrm{y} \in \mathrm{K}$.

The following three useful facts are also noted without proof:

FACT 1. For all sentences A, $V(A, 0) = \boldsymbol{F}$ iff $V({\sim}A, 0) = \boldsymbol{T}$.

FACT 2. For all sentences A and B, if $V(A/B, w) = \boldsymbol{F}$, then $V(B, w) = \boldsymbol{F}$.

FACT 3. For all sentences A and B, if $V(A/B, w) = \boldsymbol{F}$, then $V(A, 0) \neq \boldsymbol{F}$.

The verifications of two axioms will be offered here as examples.

CA2: ${\sim}\alpha B/B$. Suppose ${\sim}\alpha B/B$ is false at the 0-point of some **RCA**-model. Then ${\sim}\alpha B$ must be nonfalse at the 0-point, by Fact 3, while B is false at 0, by Fact 2. By Lemma 2, though, if B is false at 0, αB must be true at 0. Thus, by Fact 1, ${\sim}\alpha B$ has to be false at 0. This contradicts the conclusion that ${\sim}\alpha B$ is nonfalse at 0; so ${\sim}\alpha B/B$ cannot be falsified at the 0-point of any **RCA**-model.

RCA1: ${\sim}\alpha A/((A \vee B) \rightarrow B)$. Suppose this formula is false at the 0-point of some **RCA**-model. Then $V({\sim}\alpha A, 0) \neq \boldsymbol{F}$ by Fact 3—and $V({\sim}\alpha A, 0) = \boldsymbol{T}$ by Lemma 1, so $V(\alpha A, 0) = \boldsymbol{F}$ by Fact 1–while $(A \vee B) \rightarrow B$ is false at 0. By the truth conditions for the arrow, for some w, x, R0wx and $V(A \vee B, w) = \boldsymbol{T}$ while $V(B, x) = \boldsymbol{F}$. If $A \vee B$ is true at w, then either $V(A, w) = \boldsymbol{T}$ or $V(B, w) = \boldsymbol{T}$. But A is nonassertive at 0 since αA is false at 0, so A is nonassertive at w also, by Lemma 3. So, B must be true at w. If B is true at w then it is true at x also, since R0wx and **RCA**-models respect the "Hereditary property." This contradicts the conclusion that $V(B, x) = \boldsymbol{F}$; so ${\sim}\alpha A/((\mathrm{A} \vee \mathrm{B}) \rightarrow \mathrm{B})$ cannot be falsified at the 0-point of any **RCA**-model.

§75.5. Completeness. This section is given over to the proof of the following

THEOREM. **RCA** is complete; i.e., all valid formulas are theorems, and all nontheorems of **RCA** can be falsified at the 0-point of some **RCA**-model.

PROOF. The proof of completeness is a modification of the proof of the completeness of **R** in Routley and Meyer 1973. For any given nontheorem, a canonical model can be constructed at whose 0-point the unprovable formula is false.

Let D be a formula that cannot be proved in **RCA**.

The proof begins with the construction of a regular, prime, **RCA**-theory, excluding D, where, by definition, a set G of formulas is an **RCA**-*theory* iff whenever formulas A and A/B are in G then B is in G; an **RCA**-theory is *prime* iff whenever $A \vee B \in \mathrm{G}$ then $A \in \mathrm{G}$ or $B \in \mathrm{G}$; and an **RCA**-theory is *regular* iff it is a superset of the theorems of **RCA**.

Let

$$G_0 = \{A: \vdash A\}.$$

Next enumerate all the formulas: $C_1, C_2, \ldots$. Then define G_{n+1} inductively as follows:

$$\begin{aligned} G_{n+1} &= G_n \cup \{C_{n+1}\} \text{ if } G_n \cup \{C_{n+1}\} \nvdash D; \\ &= G_n \text{ otherwise.} \end{aligned}$$

Finally, let

$$G = \cup\{G_n: 0 \leq n\}.$$

It is noted without proof that $G \nvdash D$ and that G is a regular and prime **RCA**-theory.

This set G is now partitioned into two sets that jointly exhaust G and are mutually exclusive. This move is similar to that made in Dunn 1975. Let

$$\begin{aligned} T &= \{A: A \in G \text{ and } {\sim}A \notin G\} \text{ and} \\ N &= \{A: A \in G \text{ and } {\sim}A \in G\}. \end{aligned}$$

These sets will be used to define the canonical valuation. The formulas in the set N will be counted as nonassertive. Those in set T and their negations will be assertive in this model. Accordingly, a formula A will be said to be *assertive* iff $\alpha A \in T$.

There are three important facts about T. First, T is itself an **RCA**-theory. Second, T is α-closed, where, by definition, a set S is *α-closed* iff, whenever $A \in S$, $\alpha A \in S$. Third, T is α-regular, where, by definition, a set S is *α-regular* iff whenever $\vdash A$ and $\alpha A \in S$, $A \in S$. The key fact concerning N is this: for all formulas A, $\alpha A \notin N$. The proof rests on the fact that $\alpha A/{\sim}\alpha A/{\sim}\alpha\alpha A$ and ${\sim}\alpha\alpha A/D$ are theorems.

The set T will be the 0-point of the canonical model to be constructed, as well as the cornerstone of that construction. With 0 in hand, the notion of a 0-theory can be defined: a set of formulas w is a *0-theory* iff (i) whenever $B \rightarrow C \in 0$ and $B \in w$, $C \in w$, and (ii) whenever $A \in w$ and $B \in w$, $A \& B \in w$. A 0-theory w is *assertive* (or 0-assertive) iff, in addition, for all formulas A, if $A \in w$ then $\alpha A \in 0$; i.e., iff every formula in w is assertive.

Let K be the set of all prime, assertive 0-theories. It is trivial to establish that 0 (=T), in addition to being α-closed and α-regular, is itself a prime, assertive 0-theory and, therefore, that $0 \in K$.

The next step is to define the *-operation. Let

$$w^* = \{A: {\sim}A \notin w \cup N\}.$$

It must be shown that * is an operation on the elements of K, i.e., that, for all $w \in K$, $w^* \in K$. An immediate consequence of the definitions of * and 0 is that $0 = 0^*$. The proof that $*$ is an operation on the elements of K follows:

LEMMA. If w is prime, assertive 0-theory, then so is w*.

PROOF. Suppose, for the proof of this lemma, that w is a prime, assertive 0-theory.

(i) Suppose $B \rightarrow C \in 0$ and $B \in w^*$, but $C \notin w^*$. Then $\sim B \notin w \cup N$, so $\sim B \notin w$. Since $C \notin w^*$, $\sim C \in w \cup N$. But, since $B \rightarrow C \in 0$ (making $B \rightarrow C$ and, hence, B, C, and $\sim C$ all assertive), $\sim C \notin N$. So $\sim C \in w$. It is a theorem of **RCA** that $B \rightarrow C \rightarrow . \sim C \rightarrow \sim B$, and this formula is assertive, since B and C are assertive. By α-regularity, this formula is in 0. Since 0 is a 0-theory and $B \rightarrow C$ is also in 0, $\sim C \rightarrow \sim B \in 0$. Then, since w is a 0-theory and $\sim C \in w$, $\sim B \in w$. This contradicts the assumption that $\sim B \notin w$. So, if $B \rightarrow C \in 0$ and $B \in w^*$ then $C \in w^*$.

(ii) Suppose B and C are in w*, but $B \& C$ is not. Then neither $\sim B$ nor $\sim C$ is in $w \cup N$, so neither is in N, and, hence, both B and C are assertive. Since $B \& C \notin w^*$, $\sim(B \& C) \in w \cup N$. But $\sim(B \& C) \notin N$, since both B and C are assertive, making $\sim(B \& C)$ assertive also. So $\sim(B \& C) \in w$. It is a theorem of **RCA** that $\sim(B \& C) \rightarrow . \sim B \vee \sim C$, and this formula is assertive, since B and C are assertive. By α-regularity, this formula is in 0. Then, since w is a 0-theory and $\sim(B \& C) \in w$, $\sim B \vee \sim C \in w$. Moreover, w is a *prime* 0-theory; so $\sim B \in w$ or $\sim C \in w$, each contradicting the assumption that neither $\sim B$ nor $\sim C$ is in w. So, if B and C are in w* then $B \& C$ is too. Therefore w* is a 0-theory.

(iii) Suppose $A \vee B \in w^*$, but $A \notin w^*$ and $B \notin w^*$. Then $\sim A$ and $\sim B$ are members of $w \cup N$, but $\sim(A \vee B) \notin w \cup N$. Thus $\sim(A \vee B) \notin N$ and $\alpha \sim(A \vee B) \in 0$. That is, $\alpha A \vee \alpha B \in 0$. 0 is prime, though, so $\alpha A \in 0$ or $\alpha B \in 0$. Suppose it is αB that is in 0. Then B, $\sim B \notin N$; so $\sim B \in w$. Now, $\sim A$ cannot be in w since, in that case, $\sim A$, $\sim B \in w$, $\sim A \& \sim B \in w$, and $\sim(A \vee B) \in w$, which would contradict the premiss. So, $\sim A$ must be in N, $\alpha \sim A \notin 0$, and $\sim \alpha \sim A \in 0$ (since $\alpha \sim A \notin N$, by the key fact about N mentioned above). The formula $\sim \alpha \sim A / (\sim B \rightarrow (\sim A \& \sim B))$ is an axiom; so $\sim B \rightarrow (\sim A \& \sim B)$ is in G and is, indeed, in 0 (since B was supposed assertive: $\alpha B \in 0$). So $\sim A \& \sim B \in w$ and $\sim(A \vee B) \in w$, which contradicts the premiss that $\sim(A \vee B) \notin w$. Similar reasoning applies if it is αA that is in 0. So w* must also be prime.

(iv) If $A \in w^*$ then $\sim A \notin w \cup N$. So $\sim A \notin N$. Then $\alpha \sim A \in 0$. But $\alpha \sim A$ is the same formula as αA; so $\alpha A \in 0$. Therefore, w* is an assertive, prime 0-theory. This completes the proof of the lemma.

The final element needed for an **RCA** model structure is the three-place assessibility relation R on members of K. To that end, a two-place operation

+ is defined on the elements of K:

$$a+b = \{A: \exists B(B \rightarrow A \in a \text{ and } B \in b)\}.$$

The three-place relation R is then defined as follows:

Rabc iff $a+b \subseteq c$.

The set $a+b$ is more than simply another set of formulas. The significant fact to note is that, whenever a and b are assertive 0-theories, $a+b$ is also an assertive 0-theory. Additionally, each of the following properties holds for +:

1 $a+(b+c) \subseteq (a+b)+c$
2 $a+b = b+a$
3 $(a+b)+b \subseteq a+b$
4 If $a \subseteq b$, then $a+c \subseteq b+c$
5 $a \subseteq 0+a$
6 $0+0 \subseteq 0$.

Establishing these is routine, so is omitted here.

What we have at this point is an ordered quadruple (0, K, R, *). This is not yet established as an **RCA** model structure, though, since the R-properties (see §75.3) have not been demonstrated. Certain R-properties can be established easily, such as R2—Raaa—which follows immediately from the definitions of R and +. Similarly, R4—if R0ax and Rxbc, then Rabc—can be demonstrated directly from the definitions and 4 and 5 above. R7 follows immediately from the fact that $0 = 0^*$. Others are less easily established. In particular, the "Pasch" property poses a problem. This one is proved here.

The Pasch property is this:

If Rabc and Rcde then ∃f(Radf and Rfbe).

If we let $a+b = c$, $a+d = g$, and $g+b = e$, then $(a+b)+d = c+d$ and $c+d \subseteq (a+d)+b = g+b = e$. So we have

Rabc and Rcde.

What is required by Pasch is that there be some point f in K such that

Radf and Rfbe.

The point g, which was set equal to $a+d$, is very nearly such a point. Indeed,

Radg and Rgbe

follow immediately. The problem is that $a+d$ might not be an element of K. If a and d are both prime, assertive 0-theories, and so in K, then, while $a+d$ is an assertive 0-theory, it might not be prime. A prime, assertive

0-theory is required by Pasch. Finding a suitable prime, assertive 0-theory requires a fair bit of machinery. We begin with some definitions.

1. $A \rightarrow_Z B$ iff $A \rightarrow B \in Z$.

2. A set X is *Z-assertive* iff, for every sentence A, if $A \in X$ then $\alpha A \in Z$. (This is a generalization of the notion of assertiveness for sets. For this reason, the earlier case might have been more perspicuously, but less simply, labeled "0-assertive.")

3. Two sets X and Y are *Z-consistent* iff it is never the case that $A \rightarrow_Z B$, where the formula A is a conjunction of $A_i \in X$ and B is a disjunction of $B_j \in Y$.

LEMMA 0. If 0 is an α-closed and α-regular **RCA**-theory and X and Y are 0-assertive and X is 0-consistent with Y, then there is an X′ such that $X \subseteq X'$, $X' \cap Y = \varnothing$, and X′ is a prime, assertive 0-theory.

PROOF. Enumerate all the assertive formulas, $C_1, C_2, \ldots$, that is, all the formulas C_i such that $\alpha C_i \in 0$. Define inductively a chain of pairs of sets as follows:

$$\begin{aligned}\langle X_0, Y_0\rangle &= \langle X, Y\rangle;\\ \langle X_{n+1}, Y_{n+1}\rangle &= \langle X_n \cup \{C_{n+1}\}, Y_n\rangle \text{ if } X_n \cup \{C_{n+1}\} \text{ is 0-consistent}\\ &\quad\ \text{with } Y_n;\\ &= \langle X_n, Y_n \cup \{C_{n+1}\}\rangle \text{ otherwise.}\end{aligned}$$

SUBLEMMA 1. For all n, $\langle X_n, Y_n\rangle$ is 0-consistent.

The proof is by induction. $\langle X_0, Y_0\rangle$ is 0-consistent by the hypothesis of Lemma 0. Suppose, for reductio, that $\langle X_n \cup \{C_{n+1}\}, Y_n\rangle$ and $\langle X_n, Y_n \cup \{C_{n+1}\}\rangle$ are both 0-inconsistent. Then

$$A \& C_{n+1} \rightarrow B \in 0 \quad \text{and} \quad A' \rightarrow B' \vee C_{n+1} \in 0,$$

where A and A' are conjunctions of sentences in X_n, and B and B' are disjunctions of sentences in Y_n. 0 is an α-regular **RCA**-theory; so

$$((A \& A') \& C_{n+1}) \rightarrow (B \vee B') \in 0 \text{ and}$$
$$(A \& A') \rightarrow ((B \vee B') \vee C_{n+1}) \in 0.$$

Since 0 is α-closed, the derived rule of **R** that yields

$$(A \& A') \rightarrow (B \vee B') \in 0$$

is available. This contradicts the inductive hypothesis that $\langle X_n, Y_n\rangle$ is 0-consistent.

Let $X' = \bigcup\{X_n : 0 \leq n\}$ and $Y' = \bigcup\{Y_n : 0 \leq n\}$.

Sublemma 2. $\langle X', Y'\rangle$ is 0-consistent.

If not, then $A\rightarrow B \in 0$, where A is some conjunction from X′ and B is some disjunction from Y′. Since the X_ns and Y_ns are in a $\leq$ chain, for some n, $\langle X_n, Y_n\rangle$ must be 0-consistent; but this contradicts Sublemma 1.

Note that X′ and Y′ exhaust the 0-assertive formulas; i.e., $X'\cup Y' = \{A\colon \alpha A \in 0\}$.

Sublemma 3. X′ is a prime, assertive 0-theory.

Since only the 0-assertive formulas were enumerated in the construction and since X′ and Y′ exhaust these formulas, showing that X′ is an assertive 0-theory is straightforward and will not be rehearsed here. To prove that X′ is also prime, assume that $A\vee B \in X'$ while $A\notin X'$ and $B\notin X'$. Since X′ is 0-assertive, $\alpha(A\vee B) \in 0$, which is to say that $\alpha A\vee \alpha B \in 0$. But 0 *is* prime, so $\alpha A \in 0$ or $\alpha B \in 0$. If $\alpha A \in 0$, then A must be in Y′, since X′ and Y′ exhaust the assertive formulas and A was supposed not to be in X′. If B is also 0-assertive—and so also in Y′—then $\langle X', Y'\rangle$ is 0-inconsistent: $A\vee B \in X'$, A, $B \in Y'$, and $A\vee B\rightarrow .A\vee B \in 0$. This contradicts Sublemma 2. If, instead, B is not 0-assertive, then $\alpha B \in 0$. Then ${\sim}\alpha B \in 0$, since 0 is prime and $\alpha B \notin N$. But ${\sim}\alpha B/((A\vee B)\rightarrow A) \in 0$, since this is 0-assertive (${\sim}\alpha B$ and αA are already both in 0), it is an axiom, and 0 is α-regular. 0 is also an **RCA**-theory; so $(A\vee B)\rightarrow A \in 0$. But this makes $\langle X', Y'\rangle$ 0-inconsistent. Thus, if A is assertive, then, whether or not B is assertive, whenever $A\vee B \in X'$, $A \in X'$ or $B \in X'$. The same line of reasoning applies if B is the one that is assertive. Since at least one of A and B must be assertive, X′ is prime. This completes the proof of Sublemma 3 and with it the proof of Lemma 0.

Lemma 1. If c is an assertive 0-theory—where 0 is an α-regular **RCA**-theory—not containing D, then there is a prime, assertive 0-theory, c′, such that $c \subseteq c'$ and $D\notin c'$.

Proof. Suppose that $D\notin c$, where c is an assertive 0-theory, but that c is not 0-consistent with $\{D\}$. Then $B\rightarrow D \in 0$, where B is a conjunction from c. Because c is a 0-theory, that conjunction must be in c and so must D. This contradicts the assumption that $D\notin c$; so c is 0-consistent with $\{D\}$. Lemma 0 can now be invoked to prove Lemma 1.

Squeeze lemma. If Rabc′, where a, b, and c′ are assertive 0-theories and c′ is prime, then there is a b′ such that $b \subseteq b'$, Rab′c′, and b′ is a prime, assertive 0-theory.

Proof. Let w = $\{A\colon \exists B(A\rightarrow B \in \mathrm{a}$ and $B \in -\mathrm{c}')\}$, where $-\mathrm{c}'$ is the assertive complement of c′; i.e., $\{A\colon \alpha A \in 0$ and $A \notin \mathrm{c}'\}$.

Suppose b is not 0-consistent with w. Then $C_1 \& \ldots \& C_n \rightarrow . D_1 \vee \ldots \vee D_m \in 0$, for some conjunction of C_is in b and some disjunction of D_js in w. Since b is a 0-theory, that disjunction of D_js must be in b. By the definition of w, for each D_j there is a B_j such that $D_j \rightarrow B_j \in \mathrm{a}$ and $B_j \in -\mathrm{c}'$. Since each $D_j \rightarrow B_j \in \mathrm{a}$, $D_1 \vee \ldots \vee D_m \rightarrow . B_1 \vee \ldots \vee B_m \in \mathrm{a}+\mathrm{b}$, because the disjunction of D_js is in b, an assertive 0-theory, and that disjunction is 0-implied by a conjunction in b. So, $B_1 \vee \ldots \vee B_m$ is in c′ since $\mathrm{a}+\mathrm{b} \subseteq \mathrm{c}'$, which is how the assumption that Rabc′ is unpacked. The theory c′ is a prime theory; so $B_i \in \mathrm{c}'$, for some i. But each B_i is an element of $-\mathrm{c}'$, by the definition of w. Thus, to avoid contradiction, b must be 0-consistent with w.

Thus, by Lemma 0, there is a b′ such that $\mathrm{b} \subseteq \mathrm{b}'$, b′ is a prime, assertive 0-theory, and $\mathrm{b}' \cap \mathrm{w} = \varnothing$.

It remains to establish that $\mathrm{a}+\mathrm{b}' \subseteq \mathrm{c}'$ so that Rab′c′. Suppose $A \in \mathrm{a}+\mathrm{b}'$. Then $\exists B(B \rightarrow A \in \mathrm{a}$ and $B \in \mathrm{b}')$, by the definition of the +-operation. If A were not in c′, then B would be in w, by the definition of w. Then B would be in $\mathrm{b}' \cap \mathrm{w}$, which is known to be empty. So, for all formulas A, if $A \in \mathrm{a}+\mathrm{b}'$, it must be in c′ as well. Thus, $\mathrm{a}+\mathrm{b}' \subseteq \mathrm{c}'$, $\mathrm{b} \subseteq \mathrm{b}'$, and b′ is prime, assertive, and a 0-theory. This completes the proof of the Squeeze lemma and, with it, the proof of the Pasch property.

What we have so far is a set 0, a set K of prime, assertive 0-theories, an operation, * on members of K, and a ternary relation R among the elements in K. The conditions that are placed on this ternary relation can now all be shown to be fulfilled. In short, we have an **RCA** model structure. To complete the model, we make the following "canonical" assignment to atomic sentences:

V(p, w) = ***T*** iff $p \in$ w, and
V(p, w) = ***F*** otherwise.

This provides the basis for the Canonical Valuation lemma and, ultimately, for the proof that the Hereditary property holds.

Canonical Valuation lemma. If the canonical assignment is extended to a valuation for all sentences in the manner prescribed, then, for all sentences A and for all points w in K,

(i) V(A, w) = ***T*** iff $A \in$ w,
(ii) V(A, w) = ***N*** iff $A \in$ N, and
(iii) V(A, w) = ***F*** iff $A \notin \mathrm{w} \cup \mathrm{N}$.

Proof. The proof is by induction on the complexity of formulas. For conjunctions, the crucial axiom is $\sim\alpha A/(B \rightarrow (A \& B))$; for disjunctions it is

$\sim\alpha A/((A\vee B)\to B)$. With these noted, the various cases are all unproblematic with the exception of one: if $\mathrm{V}(B\to C, \mathrm{w}) = \boldsymbol{T}$, then $B\to C\in \mathrm{w}$. The proof of this one case will be the only one presented here.

Suppose, to prove the contrapositive, $B\to C\notin \mathrm{w}$, for an arbitrary point w in K. Either $B\to C\in \mathrm{N}$ or $B\to C\notin \mathrm{N}$. If $B\to C\in \mathrm{N}$, then $\alpha(B\to C)$, which is $\alpha B\&\alpha C$, is not in 0. Since the absence of an α-sentence from 0 implies the presence of its negation (this was proved in passing in the proof of Sublemma 3 above), $\sim\alpha B\vee\sim\alpha C\in 0$. 0 is prime; so either B or C is nonassertive. Hence, $\mathrm{V}(B\to C, \mathrm{w})\neq \boldsymbol{T}$, by V$\to$.

Suppose, then, that $B\to C\notin \mathrm{w}$ and $B\to C\notin \mathrm{N}$. The first step in falsifying $B\to C$ at w is to build sets b and c. Let $\mathrm{b} = \{A\colon B\to A\in 0\}$ and $\mathrm{c} = \mathrm{w}+\mathrm{b}$. Note that Rwbc. As defined, b is an assertive 0-theory. Proof of this claim: (i) If $A\in\mathrm{b}$ and $A\to E\in 0$ then $B\to A\in 0$. Additionally, A, B, and E are all 0-assertive. So $B\to A\to.A\to E\to.B\to E$ is 0-assertive also. This is an axiom, so it is in α-regular 0. 0 is a 0-theory; so $A\to E\to.B\to E$ and, then, $B\to E$ are in 0. Since $B\to E\in 0$, $E\in\mathrm{b}$, by the definition of b. (ii) If A and E are in b, then $B\to A$ and $B\to E$ are in 0, again by the definition of b. $B\to A\to.B\to E\to.B\to.A\&E$ is provable and assertive and thereby in 0, as must be $B\to.A\&E$. This suffices to show that $A\&E\in\mathrm{b}$. The set b is, therefore, a 0-theory. (iii) If $A\in\mathrm{b}$, then $B\to A\in 0$ and $B\to A$ must be assertive. So A must be assertive, too. The set b must be an assertive 0-theory.

Since w is also an assertive 0-theory—$\mathrm{w}\in\mathrm{K}$—$\mathrm{w}+\mathrm{b}$, which is c, is also an assertive 0-theory.

Suppose $C\in\mathrm{c}$. Then $C\in\mathrm{w}+\mathrm{b}$, and there is a formula A such that $A\to C\in\mathrm{w}$ and $A\in\mathrm{b}$. If $A\in\mathrm{b}$, then $B\to A\in 0$. But the formula $B\to A\to.A\to C\to.B\to C$ is provable and assertive and so in 0. Then $A\to C\to.B\to C\in 0$ and, finally, $B\to C\in\mathrm{w}$. This contradicts the main premiss; so $C\notin\mathrm{c}$. By Lemma 1, there is a c′ such that $\mathrm{c}\subseteq\mathrm{c}'$, $C\notin\mathrm{c}'$, and c′ is a prime and assertive 0-theory. Further, $\mathrm{w}+\mathrm{b}\subseteq\mathrm{c}\subseteq\mathrm{c}'$; so Rwbc′.

Since B is assertive, $B\to B$ is also, and since this formula is provable it is in 0, by α-regularity. Thus, $B\in\mathrm{b}$. By the Squeeze lemma, there is a b′ such that $\mathrm{b}\subseteq\mathrm{b}'$, Rwb′c′, and b′ is a prime, assertive 0-theory. Also, since $B\in\mathrm{b}$ and $\mathrm{b}\subseteq\mathrm{b}'$, $B\in\mathrm{b}'$. Thus, there is a b′ and c′ such that Rwb′c′, $B\in\mathrm{b}'$, $C\notin\mathrm{c}'$, and b′ and c′ are prime, assertive 0-theories. So, finally, there are sets b′ and c′ which are elements of K, which stand in the relation Rwb′c′, and at which $\mathrm{V}(B, \mathrm{b}') = \boldsymbol{T}$ while $\mathrm{V}(C, \mathrm{c}') = \boldsymbol{F}$, by the inductive hypothesis of the proof of the Canonical Valuation lemma. Therefore, $\mathrm{V}(B\to C, \mathrm{w})\neq\boldsymbol{T}$.

The proof of the Hereditary property is straightforward: if R0ab and $\mathrm{V}(A, \mathrm{a}) = \boldsymbol{T}$, then $A\in\mathrm{a}$, by the Canonical Valuation lemma. Since $\mathrm{a}\subseteq 0+\mathrm{a}$ and $0+\mathrm{a}\subseteq\mathrm{b}$ (by R0ab), $A\in\mathrm{b}$ and, by that same lemma, $\mathrm{V}(A, \mathrm{b}) = \boldsymbol{T}$. Thus, the quadruple $(0, \mathrm{K}, \mathrm{R}, {}^*)$, as defined, together with the Canonical Valuation, is an **RCA**-model.

Therefore, for an unprovable formula D, an **RCA**-model can be constructed whose 0-point excludes D, making D false under the Canonical Valuation for that model. This establishes the nonvalidity of the given nontheorem. This completes the completeness proof for **RCA**.

§75.6. Quantification. The logic of conditional assertions, **CA**, is enriched with the proof-theoretic and model-theoretic apparatus for individual quantification as follows: a predicate letter followed by the appropriate number of terms is treated as atomic; so $\alpha(Fa_1 \ldots a_n)$ is $Fa_1 \ldots a_n \vee {\sim} Fa_1 \ldots a_n$. The assertivity function is extended to apply to quantified formulas in a manner parallel to its treatment of conjunctions and disjunctions. Thus, $\alpha\forall xA$ and $\alpha\exists xA$ are both $\exists x\alpha A$.

The system $\mathbf{CA}^{\forall x}$ results from combining CA1–CA10 with

QCA1 $\forall xBx \,/\, Ba$,
QCA2 $\forall x(B \,/\, Cx) \,/\, B \,/\, \forall xCx$, and
QCA3 $\forall x(Cx \,/\, B) \,/\, \exists xCx \,/\, B$,

where, for QCA2 and QCA3, it is provided that the variable is not free in B. In addition to DET, $\mathbf{CA}^{\forall x}$ has generalization, GEN, as a rule of inference, with the usual rider that one cannot generalize on a term that occurs free in the conclusion.

A universal generalization is nonassertive iff all its instances are; it is true iff no instances are false and some are true, and it is false iff some instances are false. Similar clauses govern the other quantifier. These clauses can be relativized to possible worlds without hitches. $\mathbf{CA}^{\forall x}$ is sound and complete relative to this semantics. Some care must be taken in proving these clauses: it is particularly worth remembering that a nonfalse instance of a universal generalization is an untrustworthy "witness" to that formula's own nonfalsity. Full proofs are found in Cohen 1983. Alternative axioms and proofs are found in Dunn 1975.

Although quantification and conditional assertion are easily combined, and conditional assertion and relevant implication have been welded into a single system, formal capture of the relation between individual quantification and relevant implication has remained somewhat elusive. Specifically, as shown in §52, the systems $\mathbf{R}^{\forall x}$ and $\mathbf{E}^{\forall x}$ are not complete with respect to the constant-domain semantics that one would expect. The same holds true when conditional assertion is added, that is, for the system $\mathbf{RCA}^{\forall x}$ whose axioms are those of **RCA**, $\mathbf{CA}^{\forall x}$, and $\mathbf{R}^{\forall x}$. On the other hand, the presence of a second conditional does increase one's deductive options, and the system *is* sound with respect to the natural constant-domain semantics. Moreover, the system does meet several important desiderata:

1. $(\forall x(Gx/Mx)\&Gs)\rightarrow Ms$ is valid. Universally quantified conditional assertions are strong enough to support implications from instances of the subject class to the corresponding instances of the predicate class.

2. $\forall x(Gx/Mx)\rightarrow\forall x(Gx\rightarrow Mx)$ is invalid. Generalizations expressed by quantified conditional assertions are only accidental. They are, to use the language, nonnecessitives, not too strong but not too weak.

3. $\forall x(Cx/Bx)\rightarrow\forall x({\sim}Cx\vee Bx)$ is valid. That all crows are black implies that everything is either not a crow or black. The converse fails, done in by a possible lack of crows (at some non-0 point x such that R0xx, when there are crows in the world 0).

4. $\forall x(Gx/Hx)\,/\,(\forall x(Hx/Mx)\rightarrow\forall x(Gx/Mx))$ is valid. This is the formulation of the syllogism Barbara in Belnap 1973. It is the major premiss that does the implying, given the presence of the minor.

Finally, it should be noted that the ability to effect restricted quantification may be of special value in relevantly formalizing arithmetic. In particular, the slash of conditional assertion can be put to good use in formulating the principles of mathematical induction. Weak induction is well translated by the following schema:

$$(\Phi(0)\&\forall x(Nx/(\Phi(x)\rightarrow\Phi(x'))))\rightarrow\forall x(Nx/\Phi(x)).$$

Replacing the first occurrence of the arrow by a slash results in an invalid principle, even though the use of the slash is suggested by the quantification. Strong mathematical induction can also be accommodated:

$$\{\Phi(0)\&\forall n(Nn/\forall x(x<n/(\Phi(x)\rightarrow\Phi(n))))\}\rightarrow\forall n(Nn/\Phi(n)).$$

The conclusion of the strong induction neither is based on the "vacuous truth" that all of 0's predecessors have the property in question nor requires that it be relevantly provable that 0 has the property from the assumption that its (nonexistent) predecessors do. What is required is simply that 0 have the property, and that cannot be avoided by any logical prestidigitation.

CHAPTER XII

APPLICATIONS AND DISCUSSION

§80. Entailment and the disjunctive syllogism

§80.1. Tautological entailment. We turn now and for the rest of this section to entailment as a relation between sentences. Chapter III motivated and then proposed as adequate to the relational idea of entailment the concept of *tautological entailment.*

§80.1.1. Review. This venerable concept may conveniently be described as follows. To test whether $A \rightarrow B$ is a tautological entailment, first put A in a disjunctive normal form A' and B in a conjunctive normal form B', using only De Morgan's, Double Negation, and Distribution principles. Then, for each disjunct A^* of A' and each conjunct B^* of B', ask whether some conjunct of A^* is exactly the same as some disjunct of B^*, and count $A \rightarrow B$ a tautological entailment just in case the answer is an invariable "yes."

The reader can easily check that the following few examples are correctly sorted:

Tautological entailments

(1) $p \& {\sim} p \rightarrow p$
(2) $p \rightarrow p \vee q$
(3) ${\sim} p \& (p \vee q) \rightarrow q \vee (p \& {\sim} p)$

Not tautological entailments

(4) $p \& {\sim} p \rightarrow q$
(5) $p \rightarrow q \vee {\sim} q$
(6) $p \& ({\sim} p \vee q) \rightarrow q$
(7) ${\sim} p \& (p \vee q) \rightarrow q$
(8) $(p \& {\sim} p) \vee q \rightarrow q$
(9) $p \rightarrow p \& (q \vee {\sim} q)$

§80.1.2. The disjunctive syllogism. Among those on the "bad side" it has been (7) perhaps which has received most attention. It represents, of course, the disjunctive syllogism (the d.s.), the argument pattern: from $A \vee B$ and ${\sim} A$ to infer B. The rejection of the d.s., we must say, has been taken as noxious by many logicians, both relevant and irrelevant, who are critical of our enterprise. On the one hand this reaction does not surprise us; the d.s. was, after all, one of the Stoics' "Five Indemonstrables." Still, on the other hand, we do take some small solace in the fact that it was the fifth of them—recalling the tradition of notoriety, starting with Euclid's *Elements*, regarding fifth postulates. Also, we suppose that it is better to deny an Indemonstrable than a Demonstrable.

Still, we can imagine a reader who does not find these considerations conclusive. For the sake of this reader, who doubtless shares with the critics at least surprise at the relevant failure of the d.s., we are going to devote the lion's share of our considerations to the conflict between relevance on one side and the d.s. on the other.

We pause to portend confusion. Since it happens that, by double negation, (7)—the d.s.—and (6)—*modus ponens* for material "implication"—come to the same thing, we shall on occasion take the liberty of consciously confusing the two, using "the d.s." as the name for both.

§80.1.3. Relevance logic and relevantism. In the course of these considerations we should like to address ourselves to a number of questions. In order to do so in a convenient way, we presume to impose on the reader some terminology. There are in the first place those who believe that it is worth paying attention to the concept of a relevant connection between statements and worth using formal techniques in an effort to get clear on the idea of relevance; such a person we label a *relevance logician in the wide sense*. Within this group there are those who adopt the position that the concept of tautological entailment as just described represents a stable and accurate and interesting analysis of relevant implication; these are relevance logicians in the narrow sense, but, for purposes of this section and with all due respect to our wide-sense colleagues, we are going to drop "narrow" and thus reserve *relevance logician* for members of this group. (Contrary to the relevance logician is the *irrelevant* logician, the fellow who thinks there is nothing whatsoever, beyond freshman confusion, to the topic of relevance in logic; but we shall have very little to say about this not-so-rara avis.)

There is a further distinction to be made for which we lay the groundwork by reference to intuitionism. We all know that there are those who believe that intuitionists have got hold of an interesting idea and are prepared to try to throw formal light on this idea without themselves being intuitionists, e.g., Kripke 1965a. In contrast, there are those who (e.g., Brouwer 1913) take intuitionistic principles as the very standard of reasoning and by so much reject classical two-valued logic. A *relevantist* we define to be like Brouwer—he or she rejects classical logic and instead adopts tautological entailmenthood as the proper standard of correct inference. (By so much we can be seen to be giving a narrow sense to "relevantist" in this section.) The contrary of the relevantist is the *classicalist*, who subscribes to two-valued logic as *the* organon of inference. Evidently one can be a relevance logician without being a relevantist; and indeed, in contrast to the case of intuitionism, it is not clear that there are any full-blooded relevantists, though there are certainly lots of relevance logicians.

This terminology permits us to describe what we are up to as follows: we propose to *defend* the claim of the relevance logician and to *investigate* the claim of the relevantist. That is, we shall be defending the view of the relevance logician that tautological entailment is indeed a stable and interesting concept of relevant logical implication; but we are not in this section going to defend relevantism. Instead, we shall be investigating the nature, coherence, and ramifications of the relevantist claims that tautological entailmenthood represents the correct norm of inference and that the irrelevant inferences of the classicalist, with all their presumed subtlety, must in the end be labeled as little better than exhibitions of brute cunning (§25.1).

§80.1.4. Our plan. In more detail, we plan to proceed as follows. In §80.2 we begin to speak of relevantism, and in particular we discuss the mysterious concept known as "Boolean negation" and what it means or doesn't mean to a relevantist. In the next subsection, §80.3, we shall probe more deeply into the language and logic used by the relevance logicians, especially as regards the admissibility of the d.s., asking to what extent relevance logicians could, if they wished, count themselves as relevantists. Finally, in §80.4, we essay a discussion of what it would really be like to be a relevantist, centering our discussion on the two related topics of the admissibility of and the use of the disjunctive syllogism.

It is just as well to note that we do not happen to touch on a number of issues involved in the debate concerning the Lewis argument for the validity of the disjunctive syllogism and associated issues (§16.1). Some further discussions of those matters are located in or can be found through the following: Curley 1972, Barker 1975, Stephenson 1975, Meyer 1978, Burgess 1981 and 1983 and 1984, Mortensen 1983 and 1986, and Read 1981, 1983, and 1983a.

§80.2. Boolean negation. There are some things one sometimes wishes had never been invented or discovered: e.g., nuclear energy, irrational numbers, plastic, cigarettes, mouthwash, and Boolean negation. The reader may possibly not yet have heard of this last-named threat, and it is our present purpose to inform and caution him regarding it.

§80.2.1. Background. First we give some background. Meyer and Routley 1973 discovered that it was possible to add to relevance logics a new negation $\neg$, called *Boolean negation*, which contrasts with the usual negation $\sim$ present in the relevance logics, called *De Morgan negation*. Boolean negation, unlike De Morgan negation, has such surprising properties as that $A\&\neg A$ entails B and that $\neg A\&(A\vee B)$ entails B. Meyer 1974a uses Boolean negation to give elegant axiomatizations of the relevance logics, similar in style to the well-known Gödel-Lemmon axiomatizations of the Lewis modal logics, in which one starts with a base containing all classical tautologies (these being expressed with classical negation interpreted as Boolean, not De Morgan negation).

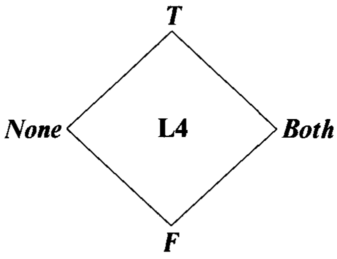
The interesting thing about the Meyer 1974a axiomatizations, besides their style, is that they seem to demonstrate that in the full systems of relevance logic (implication as a connective) De Morgan and Boolean negations can live side by side in peaceful coexistence. Indeed, Meyer 1974a shows that his axiomatizations are "conservative extensions" of the usual ones in that no new theorems arise in the standard vocabulary of relevance logic (which, of course, excludes Boolean negation).

We should stress, though, that such new derivable rules do arise in the Meyer 1974a axiomatizations; so these axiomatizations are not "conservative extensions" in an extended sense appropriate to an understanding of "logic" that takes its sorting of inferences or rules to be more basic than its sorting of sentences. In particular one can derive the terrible *ex falsum quod libet* in rule form with De Morgan negation: $A\&{\sim}A \vdash B$. This makes problematic any straightforward use of axiomatizations of relevance logic involving Boolean negation for the purpose of developing interesting (potentially) inconsistent theories, at least if one wants those theories to be closed under derivability. Indeed, any theory that contained the logical theorems ("regular" theories—see §28.3.1) would be so closed (the other theories are merely closed under entailment—there might be some merit in looking at *these* in the context of the Meyer 1974a axiomatizations). Closely connected with the above is the fact, recently noted by Meyer 197+i, that if propositional quantifiers are allowed then Boolean negation forces the extension to be nonconservative even in the unextended sense: $\exists p[p\&(p\&A\&{\sim}A\rightarrow q)]$ witnesses the trouble, with p chosen, of course, as $\neg(A\&{\sim}A)$.

The idea of Boolean negation first arose in the context of the relational semantics for relevance logic described in §48, but in the setting in which we have placed ourselves in this section (entailment as a relation), it is easier to discuss it in the context of a four-valued semantics described in §81 below.

The problem in that section was to devise a good logic for computers (mechanical question-answering systems) to use when there is real risk that the data base from which answers to questions are to be inferred may be inconsistent, and it was suggested that, in such a situation, sentences be considered to have not two but four values, ***T***, ***F***, ***None***, ***Both***, representing the four cases in which the reasoner: has been told about a certain sentence that it is true but hasn't been told that it is false; has been told that it is false, but not that it is true; has not been told anything; and has been told both that the sentence is true and that the sentence is false.

The four values form the "logical lattice" **L4** when they are ordered as follows:



An upward-directed path from a to b is to be thought of as a's *implying* b. Operations are defined on these four values so that a&b = the greatest lower bound of a, b; a$\vee$b = the least upper bound of a, b; and $\sim$***T*** = ***F***, $\sim$***F*** = ***T***, $\sim$***Both*** = ***Both***, $\sim$***None*** = ***None***.

These operations give rise to a four-valued logic that is easily seen to be the same as tautological entailment. We want to focus attention first on the definition of $\sim$. Presumably we do not have to argue for the definitions of $\sim$***T*** and $\sim$***F***; and, given what we *mean* by the four values, we feel that the definitions of $\sim$***Both*** and $\sim$***None*** are almost as obvious. Thus, if the computer (or anyone else for that matter) has been told that A is both true and false, then there is certainly a point to saying that it has been told that $\sim A$ is also both true and false (true because A is said to be false, false because A is said to be true). And there is a similar point to saying that if the computer has been told nothing about the truth value of A, then it has been told nothing about the truth value of $\sim A$ either.

Following Meyer 1974a, let us call this operation $\sim$ *De Morgan negation.* Now the problem of this section is that there is another operation on the four values with some claim to be regarded as negation, again following Meyer 1974a; this is *Boolean negation* $\neg$, defined so that it behaves the same as the De Morgan negation $\sim$ on ***T*** and ***F***, but so that $\neg$***Both*** = ***None*** and $\neg$***None*** = ***Both***. One can imagine motivating $\neg$***Both*** = ***None*** by saying something like this: if a sentence A is marked as both true and false, then $\neg A$ cannot be marked as true, since for this to be the case it would have to be that A is not marked as true (but it is). And, similarly, $\neg A$ cannot be marked as false, since then A would have to be not marked as false (but it is). So $\neg A$ must be marked as ***None***. And the reader can go through similar moves for himself in order to motivate $\neg$***None*** = ***Both***.

In a nutshell, the difference between $\sim$ and $\neg$ *would seem to be* that $\sim$ is a kind of "internal" negation, whereas $\neg$ is a kind of "external" negation; $\sim A$ might be read as "A is false," whereas $\neg A$ should be read as "It is not the case that A is true."

At least two questions force themselves upon us. (1) Which is the *real* negation? (2) Even supposing that Boolean negation is not the real negation, why should we not have it in our logic anyhow? (1) of course drags along with it a subsidiary question: Just what kind of question is it anyway? Is it a question of metaphysics, linguistics, or logic? We shall not presume to put these questions to rest here, but we do wish to address ourselves to some ancillary questions they raise.

§80.2.2. A dilemma. The orthodox relevantist, or relevance logician playing the role of the relevantist, has an account of the interaction of negation with entailment that goes something like this. In logic we find the ordinary truth-functional connectives, in particular conjunction, disjunction, and negation, and what classical logic tells us about them is partly true and partly false. The true part is what it tells us about which sentences involving only the truth functions (and quantifiers) are logically true. The false part is what it tells us about which such sentences entail one another. In particular, in §16 and §25 and elsewhere, we complain bitterly about *ex falsum quodlibet* (A and not-A entails B) and the d.s. The relevantist presumes to be talking about the same truth-functional connectives (and quantifiers) as the classical logician, but urges that a different, tighter relationship be taken as entailment.

Now, once the existence of Boolean negation is noticed by the classicalist, he can well reply that the relevantists are doing more than merely insisting that he should use "entailment" in a nonclassical sense—they are also insisting that he use "negation" in a nonclassical sense. Classical negation is Boolean negation, and *ex falsum quod libet* and the d.s. hold for Boolean negation, as the reader can easily check for himself, using the logical lattice above. And the classicalist could go on to point out that, even if some relevantist should succeed by some clever argument in showing that De Morgan negation was after all the "real" negation, perhaps even the same after all as "classical negation," Boolean "negation" would still remain. And, on the face of it, that $A\&\neg A$ entails B would seem to be as objectionable on relevantist intuitions about relevance as that $A\&{\sim}A$ entails B.

Now it seems to us that the relevantist has a reply to the classicalist, and we shall attempt to sketch it here. But we want to admit at the outset that Boolean negation is a complicated topic, and we may just be confused.

The gist of our reply for the relevantist is that he does not have to, indeed should not, recognize the legitimacy of Boolean negation.

Let us look closely at the distinction between De Morgan and Boolean negation, using a construct of Dunn 1976. There, in effect, ***Both*** was interpreted as the set $\{\boldsymbol{t}, \boldsymbol{f}\}$, ***None*** as the empty set $\varnothing$, ***T*** as $\{\boldsymbol{t}\}$, and ***F*** as $\{\boldsymbol{f}\}$. For any sentence A, let $|A|$ be the value of A. Then De Morgan and Boolean negations may be neatly compared and contrasted in their truth (and falsity) conditions as follows:

De Morgan: $(\boldsymbol{t}{\sim})$ $\boldsymbol{t} \in |{\sim}A| \Leftrightarrow \boldsymbol{f} \in |A|$
$(\boldsymbol{f}{\sim})$ $\boldsymbol{f} \in |{\sim}A| \Leftrightarrow \boldsymbol{t} \in |A|$
Boolean: $(\boldsymbol{t}\neg)$ $\boldsymbol{t} \in |\neg A| \Leftrightarrow \text{not } (\boldsymbol{t} \in |A|)$
$(\boldsymbol{f}\neg)$ $\boldsymbol{f} \in |\neg A| \Leftrightarrow \text{not } (\boldsymbol{f} \in |A|)$

These clauses certainly seem to support the kind of distinction between De Morgan and Boolean negations suggested earlier in this section, where De Morgan negation is "internal," the Boolean "external." They do anyway, until one stops to ask what kind of metalinguistic "not" it is that occurs in the Boolean clauses above.

This is a profound question, and upon it we construct a dilemma along the following lines. If the "not" is a De Morgan negation (as surely it would be for the relevantist) then, given plausible semantical principles, the "internal/external" distinction collapses and $\neg A$ co-entails $\sim A$, and so we have only one kind of negation after all. On the other hand, if the "not" is Boolean, then the relevantist can simply and consistently claim not to understand it. Thus, recognizing only one kind of "not" (the De Morgan one) is at least a stable position. But we shall go on to argue that not only *can* the relevantist take such a position; he *should*, given motivations of concern for reasoning in situations of possibly inconsistent information.

§80.2.3. Horn 1. That was the bare bones of the dilemma; we now begin to flesh it out. Taking the first horn then, let us suppose that the metalinguistic "not" is De Morgan. Let us symbolize it by $\sim$, letting context determine whether $\sim$ is metalinguistic or objectlinguistic. Surprisingly, we can show, given plausible semantical assumptions,

$$(*) \qquad \boldsymbol{f} \in |A| \Leftrightarrow \sim(\boldsymbol{t} \in |A|).$$

We say "surprisingly," since (*) seems to be a rejection of the four-valued semantics; (*) seems to say that a sentence A is assigned precisely one of the two truth values (never ***Both*** or *None*).

What are the plausible assumptions about the semantics of the metalanguage? They are:

(1) the metalanguage too should be given a four-valued interpretation (where φ is a metalinguistic expression, we shall let $\|\varphi\|$ be the value of φ);

(2) the metalinguistic $\sim$ should be evaluated as in the clause for De Morgan negation above (replacing $|\ \ |$ by $\|\ \ \|$, etc.); and

(3) the metalinguistic sentences $\boldsymbol{t} \in |A|$ and $\boldsymbol{f} \in |A|$ should be evaluated as follows:

$$\|\boldsymbol{t} \in |A|\| = |A|; \qquad \|\boldsymbol{f} \in |A|\| = |{\sim}A|.$$

It is not our point here to defend all these assumptions as the only ones that could have been made about the semantics of the metalanguage; we wish merely to point out that these are all plausible assumptions for a relevantist to make. Assumption (2), of course, merely represents our choice to explore one horn of the dilemma we are constructing. Assumptions (1) and (2) have definite Tarskian-Davidsonian overtones about them, and reflect some sort of decision to treat the object language and the metalanguage in very similar ways. (3) has Ramseyan undertones about a redundancy or disappearance theory of truth (and falsity).

We now go about deriving (*). To make our presentation concise, we adopt yet one more use for the symbol $\sim$. We already have $\sim$ as a connective of the object language and $\sim$ as a connective of the metalanguage. We now want as well $\sim$ as an operation upon the four values as explained at the beginning of this section ($\sim \boldsymbol{T} = \boldsymbol{F}$, $\sim \boldsymbol{F} = \boldsymbol{T}$, $\sim$***Both*** $=$ ***Both***, $\sim$***None*** $=$ ***None***), with these four values now thought of as sets of the usual two truth values $\mathbf{t}$, $\mathbf{f}$, of course. Then

(i)	$\|\mathbf{f} \in \|A\|\| = \|{\sim}A\|$	semantic assumption (3)
(ii)	$\|{\sim}A\| = {\sim}\|A\|$	easy to check
(iii)	${\sim}\|A\| = {\sim}\|\mathbf{t} \in \|A\|\|$	semantic assumption (3), substitution of identicals
(iv)	${\sim}\|\mathbf{t} \in \|A\|\| = \|{\sim}(\mathbf{t} \in \|A\|)\|$	semantic assumption (2), and the checking at (ii)
(v)	$\|\mathbf{f} \in \|A\|\| = \|{\sim}(\mathbf{t} \in \|A\|)\|$	(i)–(iv), transitivity of $=$

Then (*) is an immediate consequence of (v), since (v) says that the left-hand and right-hand sides of the biconditional (*) are evaluated alike.

Now that we have (*), its dual form:

$$(**) \qquad \mathbf{t} \in |A| \Leftrightarrow {\sim}(\mathbf{f} \in |A|)$$

follows by contraposition and double negation (both relevantly valid principles).

Finally, $\mathbf{t} \in |{\sim}A| \Leftrightarrow \mathbf{t} \in |\neg A|$ follows directly from ($\mathbf{t}\sim$) and ($\mathbf{t}\neg$) (interpreting the "not" in ($\mathbf{t}\neg$) as $\sim$, since we are on horn one), using (*). And, similarly, $\mathbf{f} \in |{\sim}A| \Leftrightarrow \mathbf{f} \in |\neg A|$ follows from ($\mathbf{f}\sim$) and ($\mathbf{f}\neg$), using (**). So $|{\sim}A| = |\neg A|$, and ${\sim}A$ and $\neg A$ co-entail each other, and so are fundamentally indistinguishable, as promised.

We find these considerations more than a little perplexing, and we shall return to reflect and moralize upon them a bit later. But first let us turn to exploring the second horn of the dilemma. The point there is simply that if the "not" in the clauses ($\mathbf{t}\neg$), ($\mathbf{f}\neg$) is Boolean, then the relevantist can claim not to understand it, recognizing, as he does, only one negation. But the further question is, of course, should he understand it? We think not, for reasons we shall now develop.

§80.2.4. Horn 2. Let us put ourselves in the existential situation of the computer of §80.2.1 (or other reasoner, perhaps ourselves) having to make inferences in an environment of possibly inconsistent information. It seems to us that such a reasoner has no conceivable use for Boolean negation.

Let us suppose that this reasoner's sentences are being marked in the four-valued way. Then ${\sim}A$ makes perfect sense, both as input and as output (input occurs when the reasoner is told things—by a programmer, informant, nature, whatever—and output occurs when the reasoner is asked things). Thus, e.g., if the reasoner receives ${\sim}A$ marked $\mathbf{t}$ as input, then A is to be marked $\mathbf{f}$; and the reasoner can output ${\sim}A$ marked $\mathbf{t}$ if A is already marked as $\mathbf{f}$. This is just the practical content of the clause ($\mathbf{t}\sim$), and ($\mathbf{f}\sim$) has similar practical content about $\sim A$ being marked as f if and only if A is marked as t.

But what possible practical content can be ascribed to the clauses ($t\neg$) and ($f\neg$)? We can *say* that ($t\neg$) instructs that $\neg A$ should be marked as t if and only if A is not marked as t. But what does this mean, practically speaking, from an input-output point of view?

On the input side, does it mean that the reasoner receiving $\neg A$ marked t as input, should "unmark" A as t (erase any marking of A as t)? Then what is the reasoner supposed to do when it receives $A\&\neg A$ marked t as input? Both mark and unmark A as t (or mark A as t and then erase)? Trying both to mark and to unmark A as t seems to invite psychotic breakdown (and the parenthetical alternative of marking and erasing is not without problems: it would seem that $A\&\neg A$ would differ from $\neg A\&A$; and there are deeper troubles, to which we shall advert after we discuss the rule of $\neg A$ as output).

Let us think about the conditions under which the reasoner can produce $\neg A$ marked t as output. The reasoner would first have to verify that it has not been told A, *either explicitly or implicitly*. This last is most important, and we have not stressed it previously. If the reasoner can ultimately deduce A (output A marked t), then we would not want the reasoner to report out $\neg A$ (output $\neg A$ marked t) purely on the basis of its not yet having got around to the appropriate deduction of A. Indeed, $\neg A$ is a claim that such a deduction does not exist. Thus $\neg$ would be an "ineffective" connective in the technical sense, since it is well known (Church's theorem) that there is in general no mechanical procedure for determining whether such deductions of A exist (at least if quantifiers are present, which we suppose they are in any interesting case).

Reflecting upon what has just been said reveals new problems for $\neg A$ on the input side as well. Taking $\neg A$ as an instruction to "erase A" is not really a viable alternative, at least in the absence of some formal (mechanical?) model of how a reasoner "takes things back." On being told $\neg A$, we do not want the reasoner merely to "keep quiet about A," even though all its information points to A's being true. It is still then *implicitly* told A. What we would want, we guess, is for the reasoner to correct its information so that A is no longer deducible. But how is it to do this? We need a theory of theory correction, and there seems to us to be no such fully developed theory on the market. Without such a theory, $\neg A$ cannot be viewed as an explicit instruction—it is at best a pious hope. (This point relates to one which could be put more technically in the language of §81: Boolean negation is not "ampliative." It is *also* true that it is not "continuous" in the sense of that section and thus would be ruled out by what is there called "Scott's thesis.")

Let us back away from the horrid detail of the problems of treating $\neg A$ as input/output to a question-answering machine, and summarize our feelings in a somewhat metaphorical way. Classical negation (ordinary two-valued negation) is a connective fit only for God. It is an ontological negation which can be used as an epistemic negation only by the omniscient. De Morgan negation is the appropriate epistemic negation for the poor finite reasoner, be it machine or human. Four-valued Boolean negation is a temptation which must be resisted, promising as it does to combine the ontological and epistemic. Thus $\neg A$ is supposed to mean that A *really* is not *marked* $\boldsymbol{t}$.

Boolean negation seems to us to be best understood as an attempt to push down into a four-valued object language the two-valued negation of the classical metalanguage. Relevance logicians have so far invariably used a classical metalanguage—a practice which might be excused by the relevantist as "preaching to the heathen in his own language." But the true relevantist should for himself use a relevant metalanguage, with the only negation being De Morgan negation. And then, as we have seen in exploring the first horn of our dilemma, it would appear that the truth (and falsity) conditions for Boolean negation can no longer be stated so as to distinguish it from De Morgan negation.

§80.2.5. A puzzle. The reader may still feel a sense of puzzlement about all this (we do). How is it that what started as a four-valued semantics set forth in a classical metalanguage ends up as a two-valued semantics when reinterpreted in a relevant metalanguage (see (*) above)?

One point that can be made is that one would not have expected that a four-valued semantics would have been *needed* with a relevant metalanguage. If all the connectives in sight are "relevant," then, on Tarskian-Davidsonian intuitions, there could not be too much wrong with the "homophonic"

$$(\sharp) \qquad ({\sim}A) \text{ is } \boldsymbol{t} \Leftrightarrow {\sim}(A \text{ is } \boldsymbol{t}).$$

On the other hand, one would have definitely expected that some additional apparatus beyond the usual two-valued approach was needed for doing a semantics for relevance logic in a classical metalanguage.

But these considerations do not by themselves dispel the puzzle. Although one *need* not give a four-valued semantics in a relevant metalanguage, *couldn't* one? The answer seems to be "no," given at least the semantical assumptions we adopted in exploring the first horn of our dilemma. But why is it "no"? It seems to us that the answer is something like the following.

There is no way in a relevant metalanguage (without Boolean negation) to say that a sentence A takes on, for example, just the value $\boldsymbol{t}$ (and *not* f as well). The most one can say is that A is at least $\boldsymbol{t}$ ($\boldsymbol{t} \in |A|$). The metalinguistic sentence

$$(\boldsymbol{t} \in |A|) \& {\sim}(f \in |A|)$$

does not do the trick it might be thought to do, since (at least on our plausible semantical assumptions) it asserts that A is at least true and it is *not* the case that A is at least false. But this last conjunct does not mean that A is *really not* at least false; all it means in the end, given our analysis of $\sim$, is that A is at least true, the same as the first conjunct.

Similarly, there is no way to say that A takes on just the value f—all one can say is that A is at least f. The two values of the relevant metalanguage are, as it were, "at least t" and "at least f," whereas the usual two values of a classical metalanguage are, as it were, "just t" and "just f."

The surprising result (*) above should now look much less surprising. The shock of it was that it appeared to say that A took on precisely one of the values t, f—that A was at least false if and only if A was *not* at least true. But now we see that since the "not" in question is De Morgan negation, the right-hand side does not mean *really* that A is *not* at least true. All it means is that it is at least false that A is at least true, which reduces (on our semantical principles) to the left-hand side—A is at least false. Got it?

§80.3. Relevant arguments for the admissibility of the disjunctive syllogism.

Relevance logicians have used formal or formalizable arguments in order to establish various facts about relevance logics; can such facts be established by using only arguments that the relevantist takes as valid? This question, which relevance logicians have discussed among themselves for a number of years, *without, let it be said, making much headway*, has been posed in conversation in most strenuous terms by Kripke in the following guise: one of the principal results concerning relevance logics is that, for the central cases, the d.s. is an admissible rule. For example, whenever $\sim A$ and $A \vee B$ are theorems of the system **E** (which includes the arrow as an object sign), so is B. Kripke asks: Is there a proof of this fact which does not itself *use* the disjunctive syllogism in the course of the proof?

The question is complicated (Meyer 1985 discusses some of these matters); we cannot hope to answer it here; we hope only to explain a bit just why it is so complicated and to clarify it some.

There are several choices which complicate the question, choices we take up in turn.

§80.3.1. Readings.

There is a question whether the various natural-language expressions occurring in the various arguments and conclusions in the work on the admissibility of the d.s. should be given (a) wholly extensional or (b) partly relevant readings—the latter in the sense that the relevant arrow shall be used in translating at least some of the English conditionals that occur (explicitly or implicitly). There appear to us to be four important possibilities.

Option 1. The conclusion of the argument and, furthermore, all the language in the argument itself are given a purely extensional reading. The conclusion, on this reading, is that either $\sim A$ is not an **E**-theorem, or $A \vee B$

is not an **E**-theorem, or B is an **E**-theorem; this we call *extensional admissibility*. And we have added as part of this option that the entire proof be given an extensional gloss. Under this option it seems to us straightforward that *none* of anyone's arguments for the d.s. are relevantly valid. But this doesn't have much to do with the fact that it is the admissibility of the d.s. which is being proved, as the following example shows. Let all natural-language constructions be given their "usual" extensional reading. Now suppose someone argues "If A then B; but A; so B." He is quite evidently employing the d.s. (since the first premiss is a material "implication") and so is not arguing relevantly. He may be (under this construal) a relevance logician, but he is no relevantist.

We take it that the investigations reported in §72 into the relevant arithmetic $\mathbf{R}^\#$ indicate that we have not uncovered some surprising scandal; typically, arguments for purely extensional theorems of $\mathbf{R}^\#$ need to take a detour through relevant connectives. If one had tried to argue for, say, the Kleene theorems using only extensional ideas, one would have been *forced* into using relevantly bad arguments. So much for this option.

Option 2. The conclusion of the argument is given a relevant reading. On this option, the admissibility of the d.s. is to be taken as the statement that the joint **E**-theoremhood of $\sim A$ and $A \vee B$ relevantly implies the **E**-theoremhood of B; this we call *relevant admissibility*. We know of no mathematical proof one way or the other as to whether the d.s. is relevantly admissible in **E**, but several relevance logicians have opinions. Meyer (in correspondence), for example, takes it that the d.s. is *not* relevantly admissible. And, after stating the third alternative, we give a reason why we tend to agree, noting at this point that since, on this option, the conclusion of the argument is relevant, it is certain that any proof would also require relevant connectives.

Option 3. Suppose we combine options 1 and 2 in the following way. As in option 1, the conclusion of the argument is purely extensional: either $\sim A$ is not an **E**-theorem or $A \vee B$ is not an **E**-theorem or B is an **E**-theorem (extensional admissibility). But, as in option 2, the proof is allowed to involve relevant connectives. Thus we have the *typical* case for $\mathbf{R}^\#$-extensional conclusions requiring relevant proofs. Obviously this option is a stand-in for a host of options, since there are so many choices possible in giving readings to various parts of the argument; there may or may not be a relevant reading of some extant proof of the d.s. which renders it entirely acceptable to the relevantist. To this extent, we can only commend the enterprise suggested by this option as interesting. But still, we do have a guess.

Every one of the extant proofs (Meyer and Dunn 1969 (see §25.3), Routley and Meyer 1973 (see §48.8), Meyer 197+ (see §42—this is the "metavaluation" approach to "The Way Down" of §42.3)) involves certain pleasant "theories." One finds (never mind how) a theory T with the **E**-theorems $\sim A$

and $A \vee B$ *in* it and the non-**E**-theorem B (for *reductio*) *not* in it. The theory is sufficiently pleasant so that $\sim A$'s being in it implies that A is *not* and that $A \vee B$'s being in it implies that either A *or* B is in. Now it might appear that the extant proofs proceed by the d.s. to conclude that B is in—a contradiction.

If appearance is reality, then there are consequences for both options 2 and 3. With respect to 2, the classicalist will have to reject the claim that relevant (but not extensional) admissibility has been established by the extant proofs. For, though the d.s. is classically acceptable, even the classicalist knows that it cannot suffice to establish a relevant connection. With respect to option 3, if appearance is reality, then the relevantist will have to reject the claim that (even) extensional admissibility has been relevantly established, for he forbids all use of the d.s.

But is appearance reality? One matter up for grabs is whether the "or" in "either A or B is in" is extensional or intensional ("A's not being in relevantly implies B's being in"). If the "or" were to be intensional, then clearly this part of the argument would serve as no bar to a fully relevant proof of full relevant admissibility; but after substantial (if not wholly conclusive) analysis of the extant arguments, we report to the reader our view that in fact this "or" cannot be taken intensionally. But this does not settle the matter; for even with an extensional "or" the argument can be just slightly restructured to avoid use of the d.s.; indeed it was originally so structured in the extant proofs mentioned above. It is, as we noted, a *reductio*, for which *any* absurdity will do. And it is easy to see that, although relevant moves will not produce "B is in and not in" without the d.s., it is straightforward to get "either B is in and not in, or A is in and not in"—which is enough. But at this point there is a divergence between options 2 and 3, laid bare by asking, Enough for what? To make the point, let us suppose (without doxastic commitment) that all the other parts of the extant arguments are relevantly acceptable. Then this *reductio* argument would in fact suffice to establish *in a relevant way* the extensional form of admissibility, but not the relevant form. Upshot: we think the extant proofs do not give a proof of relevant admissibility; but they might turn out to give a relevant proof of extensional admissibility.

Option 4. The fourth option is to conclude that the expressive power of the languages so far investigated by relevance logicians is not enough to handle their arguments in a relevant way, but that instead additional features must be added.

One possibility for translating the extant arguments lies in Boolean negation (§80.2). By a *classical-relevantist* let us mean one who takes as an organon the logics with both Boolean and De Morgan negation (see §80.2 above). Now in contexts in which neither arrow nor De Morgan negation appears, i.e., contexts involving only Boolean negation and positive extensional connectives, including quantifiers, it is trivial that there is no difference between the classicalist and the classical-relevantist. It follows that, if we take a proof of the admissibility of the d.s. and translate everything, conclusion and argument alike, extensionally, but with (only) Boolean negation, the argument is bound to be classical-relevantly valid since it is classically valid. So, in the presence of Boolean negation and the foregoing policy, the whole question of the provability of the admissibility of the d.s. in a relevant way is trivialized and cannot be sensibly asked. We must rule out Boolean negation to make the question interesting, even for the classicalist. (The relevantist, as we suggested in §80.2, doesn't recognize Boolean "negation" anyhow.)

A last possibility is to introduce some logical apparatus that is *both* relevant *and* of use. We don't have any firm candidates for this role; still, with all hesitation, we mention the possible relevant usefulness of some forms of restricted quantification, which should be generalizations of conjunction and disjunction, as in §75. (Neither $\forall x(\sim Ax \vee Bx)$ nor $\forall x(Ax \rightarrow Bx)$ is such a generalization, and similarly for existential quantification). This seems to us an important line to pursue; but we cannot here take the space to follow it up, beyond indicating what the d.s. itself might come to on this reading: consider the space of all formulas B such that for some A, $\sim A$ and $A \vee B$ are **E**-theorems; within this restricted space, everything is an **E**-theorem.

§80.3.2. "Equivalent" forms. Let the d.s. be stated materially, in accordance with option 3 of §80.3.1. Still not all is settled, for what it means will depend on what is meant by "**E**-theoremhood." In a way, perhaps, this is noncontroversial, but it is worth mentioning because there may be differing accounts of **E**-theoremhood which can be shown to be provably materially "equivalent." Now suppose we have a relevantly valid proof of the admissibility of the d.s. under one of these accounts of **E**-theoremhood. In the absence of the d.s. itself, it is clear that we cannot use that proof as the front end of a proof of the admissibility of the d.s. for a materially "equivalent" account of **E**-theoremhood.

Perhaps the most important example of this phenomenon is due to our having both a syntax and a semantics for **E** and to having a proof that (in the usual terminology) **E**-theoremhood on the syntactic side is "equivalent" to **E**-validity on the semantic side (see §48). It could therefore be that, at some time, someone will provide a relevant proof of the d.s. for **E**-theoremhood, but not a proof of the d.s. for **E**-validity, even given the proven "equivalence" of **E**-validity and **E**-theoremhood.

In what follows we decide to concentrate on the presently available syntactic version of **E**-theoremhood. And we note that, in the usual Gödel way, we can represent **E**-theoremhood in the vocabulary of $\mathbf{P}^{\#}$ and hence in $\mathbf{R}^{\#}$ of §72. So we can find, in the purely extensional language of arithmetic, a sentence that can be read as a formulation of the d.s.—let's call this arithmetic sentence $DS^{\#}$. (Note: the relevantist might already object to some of these classical moves; we ignore this possibility, but not because it's not a real one.)

We are thereby led to the following surprisingly definite question: Is $DS^{\#}$ provable in $\mathbf{R}^{\#}$? We just don't have much information about this. $DS^{\#}$ *may* be like some of Kleene's arithmetic theorems, which are provable in $\mathbf{R}^{\#}$ even though Kleene's own arguments for them are relevantly invalid (§72.2). But the analogy can't really be close in any straightforward sense, because the extant proofs not only involve steps raising the question of relevance; they also all involve *second-order* considerations—quantification over *sets* of sentences (theories). By so much the arguments cannot be merely "relevantized" to become available in $\mathbf{R}^{\#}$ (which is *first order*). We return to this point. The second alternative is that $DS^{\#}$ may be unprovable in $\mathbf{R}^{\#}$. In the latter case, $DS^{\#}$ would be unprovable even in $\mathbf{P}^{\#}$ because, as Meyer has pointed out in correspondence, $DS^{\#}$ is a "secondary unequation," and so, by results recorded in Meyer's unpublished work, *cannot* be in $\mathbf{P}^{\#}$ without being in $\mathbf{R}^{\#}$. We here record the sober guess that $DS^{\#}$ is not provable in $\mathbf{P}^{\#}$, hence not in $\mathbf{R}^{\#}$, since (as we said above) the extant proofs of the admissibility of the d.s. involve second-order considerations. But, who knows, the picture might change again for second-order $\mathbf{R}^{\#}$: it could be that the extant arguments can be carried out in second-order $\mathbf{R}^{\#}$, or it could even be that $DS^{\#}$ is provable in second-order $\mathbf{P}^{\#}$ without being so in $\mathbf{R}^{\#}$. The results about secondary unequations mentioned above do not, as far as we know, extend in any immediate way to second-order $\mathbf{R}^{\#}$.

§80.3.3. Extensional admissibility is useless for a relevantist. The foregoing makes little mention of the difference between a classicalist and a relevantist, but our next point relies on this distinction. Consider $\mathbf{R}^{\#\#}$ and $\mathbf{P}^{\#\#}$ instead of $\mathbf{R}^{\#}$ and $\mathbf{P}^{\#}$ (§72.4). Of course, any classicalist will believe that $DS^{\#}$ is provable in $\mathbf{R}^{\#}$, since he will believe that it is true and, hence, in $\mathbf{P}^{\#}$, and that infinite induction gets $\mathbf{P}^{\#\#}$ inside $\mathbf{R}^{\#\#}$. So he could come to believe that $DS^{\#}$ is relevantly provable with infinite induction (in $\mathbf{R}^{\#\#}$), but not with only finite induction (in $\mathbf{R}^{\#}$). On the other hand, the true relevantist could not use this argument if the lemma on which it depends is that $\mathbf{P}^{\#\#}$ is only materially "contained" in $\mathbf{R}^{\#\#}$; for the conclusion that $DS^{\#}$ is inside $\mathbf{R}^{\#\#}$ would then involve the d.s. (Furthermore, the relevantist would want to have a look at the proof that $\mathbf{P}^{\#\#}$ is, even materially, inside $\mathbf{R}^{\#\#}$; but that is just another level of the same kind of complexity.)

Let us put to one side, now, the question of how one has arrived at the admissibility of the d.s., and just suppose, now, that one believes that $\sim A$ and $A \vee B$ are never provable in E while B is not. An interesting phenomenon arises: if one is a classicalist, being given a provable $\sim A$ and a provable $A \vee B$, one will not hesitate to infer the provability of B. But of course the relevantist cannot do this, for so to infer would be precisely to employ the d.s.! Which brings us to our next (and last) topic.

§80.4. The phenomenology of relevantism. What we are after in this section might be called "the phenomenology of relevantism," at least in the vicinity of the d.s. We want at this point not to argue for relevantism, but only to indicate some features of the relevantist situation as best we can.

To be a little concrete, we ask you to imagine a relevantist who accepts a proof of the "extensional admissibility" of the d.s. as described at the end of the last section, and who has also got proofs in **E** of some $\sim A$ and $A \vee B$. He knows, as a true relevantist, that he cannot infer the **E**-provability of B. So what does he say to himself?

§80.4.1. I'm all right, Jack.

One might think as follows. The point of relevantism is to take seriously the threat of contradiction. But there is in this vicinity (that of fairly low-level mathematics) no real such threat. So here it is OK to use the d.s. and conclude the theoremhood of B. That *sounds* OK; but is it? After all, we suppose that "here there is no threat of contradiction" is to be construed as an added premiss. But a little thought shows that *no* such added premiss should permit the relevantist to use the d.s., for a very simple reason: as we said, avoidance of the d.s. was bound up with the threat of contradiction, and one thing that is clear is that *adding* premisses cannot possibly *reduce* that threat. If in fact the body of information from which one is inferring *is* contradictory, then it surely doesn't help to add as an extra premiss that it *is not*. That way lies madness.

We take the opportunity to point out a disanalogy between the relevantist situation and the intuitionist situation. The intuitionist overhearing with dismay the meanderings of some classicalist can always say: 'Poor fellow! He actually thinks he is reasoning. Still, there is some sense that can be made of his musings. What he seems to be doing is assuming (without warrant) a bunch of excluded middles. So I can charitably interpret him as constructing an enthymematic argument which can be made (intuitionistically) correct by adding the appropriate excluded middles as premisses." The relevantist, as we have seen, cannot make an analogous charitable interpretation.

§80.4.2. The relevantist/deductivist parallel.

One is reminded of induction: if induction is risky to begin with, it does no good to add as an extra premiss that *here* after all it is all right to use it. We all know about the circles we wind up in if we start out in that direction. So in this, and the next couple of subsections, we want to consider a parallel between the relevantist and *the deductivist*. The deductivist, as the reader will recall, is that hero of elementary logic texts who says that the only correct arguments are deductively valid arguments. We suppose the deductivist must have his moments when he wonders just what it is that seduces those other poor fools (and himself as well on occasion) to use inductive principles—is it just blind irrationalism, or can some justification be given? Both the deductivist and the relevantist have set themselves severe standards of personal conduct with regard to their reasoning, and both should feel the need to explain, or at least apologize for, any lapses from these standards.

§80.4.3. The leap of faith. One tack that has been taken with respect to induction is just to count it as involving some *judgment* (not to be construed as an extra—useless—premiss) that in the particular case induction is appropriate, together with a *leap* to the conclusion, a leap *known to be risky*, a leap based perhaps partly on *faith* as well as judgment.

Perhaps the relevantist could or should take a precisely parallel tack, given his proof of the extensional admissibility of the d.s. and given the E-theoremhood of $\sim A$ and $A \vee B$. He cannot and must not count the inference to the E-theoremhood of B as "good logic," but perhaps he could *judge* that nevertheless it would be appropriate to *leap* to the E-theoremhood of B anyhow, even though (because of the threat of contradiction) he knows that the leap is *risky* and hence based partly on *faith.*

§80.4.4. The toe in the water. There is another option, which, like the foregoing, pictures the relevantist as at least tempted to jump to the E-theoremhood of B from the E-theoremhood of $\sim A$ and $A \vee B$, but with a great deal of hesitancy. This option involves just a little technicality. Observe first that although the inference (6) of §80.1, from $\sim p$ and $(p \vee q)$ to q, is relevantly abhorrent, there is nothing wrong with the inference (3), also of §80.1, from $\sim p$ and $(p \vee q)$ to $q \vee (p \& \sim p)$—i.e., to "q, unless there is something awfully wrong in our information about p."

This suggests the technical maneuver of introducing a propositional constant f, to be interpreted (in this context; see also §27.1.2) as the generalized disjunction of *all* contradictions (given propositional quantifiers, f could be defined as $\exists p(p \& \sim p)$). Then it is easy to see that

$$(3') \qquad \sim p \& (p \vee q) \rightarrow q \vee f$$

is relevantly acceptable—from $\sim p$ and $p \vee q$ to infer: q unless we've got ourselves a contradiction.

Given f, there is then available to the relevantist a sort of copycat procedure: whenever the classicalist infers q, the relevantist infers $q \vee f$ instead. (It turns out, as is also easy to see, that additional uses of the d.s. do not require somehow more and more fs—one f will do the trick for the whole argument.) The relevantist thus can give a charitable interpretation of the classicalist's "reasoning" which is dual to the intuitionist's interpretation. The relevantist can say of the classicalist: "Poor incautious fellow! He concludes that q outright, when what he should conclude is merely that q unless his information is all screwed up."

Such a relevantist might go further and come to see *himself* as employing a special speech act, that of "disjoining f" *sotto voce*; so that every time the relevantist was heard to assert "B is an E-theorem" (say), he would be understood as having added "or else there's real trouble." One could certainly do that without collapsing the meanings of q and $q \vee f$, just as one can presume that a claim to warranted belief is part of the speech act of assertion without being part of the meaning of what is asserted. (The OED might let us call our new speech act of diffident assertion, "diffidation.")

One is reminded here too of induction, because there are people of basically deductivist persuasion who urge that one should never conclude by inductive argument something like "All crows are black"—all one is really entitled to conclude is something like "*probably* all crows are black" (or maybe, "it is probable, *relative to my data*, that all crows are black").

But it seems to us in the end that diffident assertion is a pretty shabby ploy for the true relevantist to use, since, on his own account, the inference from $q \vee f$ to q is invalid (it is precisely equivalent to the d.s., as it turns out)—and so the practice of such a speech act would conceal what are for him real differences. Furthermore, in the actual case, the relevantist generally has more information than a barren disjoined f; he knows, if he has done his homework, *which* contradiction is at issue. Of course this point will not interest the classicalist, who cannot tell the difference between one contradiction and another. But the relevantist can, and so for him using f, whether suppressed or not, is to lose information.

We conclude that this option has demerits without compensating advantages, except, perhaps, for the profoundly dubious values of dissembling. It is not so easy to tell apart the classicalist and the *sotto voce* relevantist we have described. For this reason, someone might want to claim to be *really* a relevantist after all, even though he has no qualms about using the d.s. Such a person's defense of the apparent inconsistency in his policy might be that he is not after all using the d.s., but using only its relevantly valid cousin with disjoined f, coupled with the practice of systematically suppressing such disjunctions.

We think that such a position is probably coherent, but we certainly do not admire it. It smacks of waffling, wavering, backsliding, and similar characteristics not to be encouraged.

§80.4.5. The true relevantist. The final option that we describe is the one to which we have become more and more attracted (as a description of the true relevantist) in the course of developing this section: the true relevantist should not even be tempted to use the d.s. After all, the temptation presumably comes from continuing to take $p \vee q$ as *some* kind of logical link between p and q—perhaps a weak one, but still more than nothing. Perhaps the true relevantist should just stop before he starts, declaring $p \vee q$ to be no link at all.

Apply this to the case at hand. The relevantist so described *might* be interested in "extensional admissibility." But not at all because of hoping to be able to use it as some kind of major premiss once he has got an E-provable $\sim A$ and an E-provable $A \vee B$. We grant that it is hard to see why he would then be interested—that is the point of the "might."

And we are then led to our last thought. We *do* think that "admissibility" has some kind of an "if-then" in it. So if what we have heretofore called "extensional admissibility" does not, then what is needed is a new theorem—not just a relevant proof of an old one. Perhaps it would be a new theorem using restricted quantification as described in §80.3.1(4). But, however that comes out, this true relevantist has to look around the logical landscape and say not just that he has not seen a *relevant* proof of the admissibility of the d.s., but that he has not seen even a bad proof of it. Of course such a claim would outrage a classicalist; but Our Hero should not let that bother him.

In any event, we applaud the steadfast courage of the true relevantist as described here. In contrast to the old-fashioned logical empiricists and the new-fashioned nominalists and such, the true relevantist is truly tough-minded, with nary a soft spot in his head. His brow wrinkles, his jaw juts, and he will never, ever use the d.s.

A sober closing: see Lance 1988 for a detailed argument that deep philosophical commitments based on understanding language from a truly social perspective support the use of some form of relevance logic as the *only* viable standard of all reasoning in any social context.

§81. A useful four-valued logic: How a computer should think.

The work of the previous section can be understood from a number of points of view. On one of these we can see it as working out a four-valued logic, the values being the various subsets of $\{T, F\}$. We propose that this four-valued logic should sometimes be *used.*

§81.1. The computer.

A lot of work has been done recently on applying many-valued logics to the design of computer circuitry and thus giving them application (see Wolf's bibliography in Dunn and Epstein 1977); so what, you may ask, is special about offering a four-valued logic as "useful"? In fact we think we are indeed involved in an odd sort of enterprise; for in the present context we want to use "logic" in a narrow sense, the old sense: "logic" in the sense of an *organon,* a tool, a canon of inference. And it is our impression that hardly any of what individual practitioners of many-valued logic have done is directly concerned with developing logics to use as practical tools for inference. Hence the peculiarity of our task, which is to suggest that a certain four-valued logic ought to be used in certain circumstances as an actual guide to reasoning.

Our suggestion for the utility of a four-valued logic is a local one. It is not the Big Claim that we all ought always to use this logic (unlike the rest of this book, this section does not comment on that claim), but the Small Claim that there are circumstances in which someone—not you—ought to abandon the familiar two-valued logic and use another instead. It will be important to delineate these circumstances with some care.

The situation we have in mind may be described as follows. In the first place, the reasoner who is to use this logic is an artificial information processor, that is, a (programmed) *computer*. This already has an important consequence. People sometimes give as an argument for staying with classical two-valued logic that it is tried and true, which is to say that it is imbued with the warmth of familiarity. This is a good (though not conclusive) argument for anyone who is interested, as we occasionally are, in practicality; it is akin to Quine's principle of "minimal mutilation," though we specifically want the emotional tone surrounding familiarity to be kept firmly in mind. But, given that in the situation we envisage the reasoner is a computer, this argument has no application. The notion of "familiarity to the computer" makes no sense, and surely the computer does not care what logic is familiar to *us*. Nor is it any trouble for a programmer to program an unfamiliar logic into the computer. So much for emotional liberation from two-valued logic.

In the second place, the computer is to be some kind of sophisticated question-answering system, where by "sophisticated" we mean that it does not confine itself, in answering questions, to just the data it has explicitly in its memory banks, but can also answer questions on the basis of *deductions* that it makes from its explicit information. Such sophisticated devices barely exist today, but they are in the forefront of everyone's hopes. In any event, the point is clear: unless there is some need for reasoning, there is hardly a need for logic.

Thirdly, the computer is to be envisioned as obtaining the data on which it is to base its inferences from a variety of sources, all of which may be supposed to be on the whole trustworthy, but none of which can be assumed to be that paragon of paragons, a universal truth-teller. There are at least two possible pictures here. One puts the computer in the context of a lot of fallible humans telling it what is so and what is not, or, with rough equivalency, a single human feeding it information over a stretch of time. The other picture paints the computer as part of a network of artificial intelligences with whom it exchanges information. In any event, the essential feature is that there is no single, monolithic, infallible source of the computer's data, but that inputs come from several independent sources. In such circumstances *the* crucial feature of the situation emerges: *inconsistency threatens*. Elizabeth tells the computer that the Pirates won the Series in 1971; Sam tells it otherwise. What is the computer to do? If it is a classical two-valued logician, it must give up altogether talking about anything to anybody, or, equivalently, it must say everything to everybody. We all know all about the fecundity of contradictions in two-valued logic: contradictions are never isolated, infecting as they do the whole system. Of course the computer could refuse to entertain inconsistent information. But in the first place that is unfair both to Elizabeth and to Sam, each of whose credentials are, by hypothesis, nearly impeccable. And in the second place, as we know all too well, contradictions may not lie on the surface. There may be in the system an *undetected* contradiction, or, what is just as bad, a contradiction that is not detected until long after the input that generated it has been blended in with the general information of the computer and has lost its separate identity. But still we want the computer to use its head to reason to just conclusions yielding sensible answers to our questions.

Of course we want the computer to report any contradictions that it finds, and in that sense we by no means want the computer to ignore contradictions. It is just that where there is a possibility of inconsistency, we want to set things up so that the computer can continue reasoning in a sensible manner even if there is such an inconsistency, discovered or not. And, even if the computer has discovered and reported an inconsistency in its baseball information, such as that the Pirates both won and did not win the Series in 1971, we would not want that to affect how it answered questions about airline schedules. But if the computer is a two-valued logician, the baseball contradiction will lead it to report that there is no way to get from Bloomington to Chicago. And also, of course, that there are exactly 3,000 flights per day. In an incisive phrase, S. C. Shapiro calls this "polluting the data." What we are proposing is to Keep Our Data Clean. (Shapiro and Wand 1976 and also Shapiro separately have independently argued for the utility of relevance logics for question-answering systems, and have suggested implementation; see §83 for a detailed account.)

So we have a *practical* motive for dealing with situations in which the computer may be told both that a thing is true and that it is false (at the same time, in the same place, in the same respect, etc., etc., etc.).

There is a fourth aspect of the situation, concerning the significance of which we remain uncertain, but which nevertheless needs mentioning for a just appreciation of developments below: our computer is *not* a complete reasoner, who should be able to do something better in the face of contradiction than just report. The complete reasoner should, presumably, have some strategy for *giving up* part of what it believes when it finds its beliefs inconsistent. Since we have never heard of a practical, reasonable, mechanizable strategy for revision of belief in the presence of contradiction, we can hardly be faulted for not providing our computer with such. In the meantime, while others work on this extremely important problem, our computer can only accept and report contradictions without divesting itself of them.

This aspect is bound up with a fifth: in answering its questions, the computer is to reply strictly in terms of what it has been told, *not* in terms of what it could be programmed to believe. For example, if it has been told that the Pirates won and did not win in 1971, it is to so report, even though we could of course program it to recognize the falsity of such a report. The point here is both subtle and obvious: if the computer would not report out contradictions in answer to our questions, we would have no way of knowing that its data base harbored contradictory information. (We could, if we wished, ask it to give a *supplementary* report, e.g., as follows: "I've been told that the Pirates won and did not win; but of course it ain't so"; but would that be useful?)

Approximation lattices. Always in the background and sometimes in the foreground of what we shall be working out is the notion of an *approximation lattice*, due to Scott 1970b, 1972, 1973a; see the *Compendium* Gierz, Hofmann, Keimel, Lawson, Mislove, Scott 1980. Let us say a word about this concept before getting on. You are going to be disappointed at the mathematical definition of an approximation lattice: mathematically it is just a complete lattice. That is, we have a set A on which there is a partial ordering $\sqsubseteq$, and for arbitrary subsets X of A there always exist least upper bounds $\bigsqcup X \in A$ and greatest lower bounds $\bigsqcap X \in A$ (two-element ones written $x \sqcup y$ and $x \sqcap y$). But we don't call a complete lattice an approximation lattice unless it satisfies a further, nonmathematical condition: it is appropriate to read $x \sqsubseteq y$ as "x approximates y." Examples worked out by Scott include the lattice of "approximate and overdetermined real numbers," where we identify an approximate real number with an interval, and where $x \sqsubseteq y$ just in case $y \subseteq x$. The (only) overdetermined real number is the empty set. As a further example Scott offers the lattice of "approximate and overdetermined functions" from A to B, identified as subsets of A × B. Here we want $f \sqsubseteq g$ just in case $f \subseteq g$.

In such lattices the *directed* sets are important: those sets such that every pair of members x and y of the set have an upper bound z also in the set. For such a set can be thought of as approximating by a limiting process to its union $\bigsqcup X$. That is, if X is directed, it makes sense to think of $\bigsqcup X$ as the limit of X. (An ascending sequence $x_1 \sqsubseteq \ldots \sqsubseteq x_i \sqsubseteq \ldots$ is a special case of a directed set.) And now when we pass to the family of functions from one approximation lattice into another (or of course the same) approximation lattice, Scott has demonstrated that what are important are the *continuous* functions: those which preserve nontrivial directed unions (i.e., $f(\bigsqcup X) = \bigsqcup\{fx: x \in X\}$, for nonempty directed X). These are the only functions that respect the lattices qua *approximation* lattices. This idea is so fundamental to developments below that we choose to catch it in a "thesis" to be thought of as analogous to Church's thesis:

SCOTT'S THESIS. In the presence of complete lattices A and B, naturally thought of as approximation lattices, pay attention only to the continuous functions from A into B, resolutely ignoring all other functions as violating the nature of A and B as approximation lattices.

(Though honesty compels us to attribute the thesis to Scott, the same policy bids us note that the formulation is ours and that, as it is stated, Scott may not want it or may think that some other thesis in the neighborhood is more important, for example, that every approximation lattice (intuitive sense) is a continuous lattice (sense of Scott 1972a).)

You will see how we rely on Scott's thesis in what follows.

Program. The rest of this section is divided into three parts. Part 1 (§81.2) considers the case in which the computer accepts only *atomic* information. This is a heavy limitation, but provides a relatively simple context in which to develop some of the key ideas. Part 2 (§81.3) allows the computer to accept also information conveyed by *truth-functionally compounded sentences*; and in this context we offer a new kind of meaning for formulas as certain mappings from epistemic states into epistemic states. In Part 3 (§81.4) the computer is allowed also to accept *implications construed as rules* for improving its data base.

§81.2. Part 1. Atomic inputs. We first consider the computer receiving only "simple" or atomic bits of information on the basis of which to answer (possibly complex) questions.

§81.2.1. Atomic sentences and the approximation lattice A4. Now and throughout this paper you must keep firmly fixed in mind the circumstances in which the computer finds itself, and especially that it must be prepared to receive and reason about inconsistent information. We want to suggest a natural technique for employment in such cases: when an item comes in as asserted, mark it with a "told True" sign, and when an item comes in denied, mark with a "told False" sign, treating these two kinds of tellings as altogether on a par. In a phrase we have used elsewhere, this is a "double-entry bookkeeping" and it is easy to see that it leads to four possibilities. For each item in its basic data file, the computer is going to have it marked in one of the following four ways: (1) just the "told True" sign, indicating that that item has been asserted to the computer without ever having been denied; (2) just the value "told False," which indicates that the item has been denied but never asserted; (3) no "told" values at all, which means the computer is in ignorance, has been told nothing; (4) the interesting case: the item is marked with both "told True" and "told False." (Recall that allowing this case is a *practical necessity* because of human fallibility.)

These four possibilities are precisely the four values of the many-valued logic we are offering as a practical guide to reasoning by the computer. Let us give them names:

T: just told True (warning: not same as ***T*** of §50 and elsewhere)
F: just told False (ditto)
None: told neither True nor False
Both: told both True and False

So these are our four values, and we baptize: **4** = {***T***, ***F***, ***None***, and ***Both***}. Of course four values do not a logic make, but let us nevertheless pause a minute to see what we have so far.

The suggestion requires that a system using this logic code each of the atomic statements representing its data base in some manner indicating which of the four values it has (at the present stage). It follows that the computer cannot represent a class merely by listing certain elements, with the assumption that those not listed are not in the class. For, just as there are four values, so there are four possible states of each element: the computer might have been told none, one, or both of "in the class" and "not in the class." Two procedures suggest themselves. The first is to list each item with one of the values ***T***, ***F***, or ***Both***, for these are the elements about which the computer has been told something; and to let an absence of a listing signify ***None***, i.e., that there is no information about that element. The second procedure would be to list each element with one *or both* of the "told" values, "told True" and "told False," not listing elements lacking both "told" values. This amounts to the §50 *relations* of formulas to (told) truth values. Obviously the procedures are equivalent, and we shall not in our discourse distinguish between them, using one or the other as seems convenient.

The same procedure works for relations, except that it is ordered pairs that get marked. For example, a part of the correct table for Series winners, conceived as a relation between teams and years, might look like this:

⟨Pirates, 1971⟩ ***T***	or	⟨Pirates, 1971⟩ True
⟨Orioles, 1971⟩ ***F***		⟨Orioles, 1971⟩ False

But if Sam slipped up and gave the wrong information after Elizabeth had previously entered the above, the first entry would become

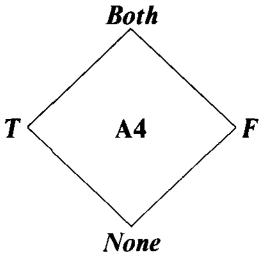
⟨Pirates, 1971⟩ ***Both*** or ⟨Pirates, 1971⟩ True, False

To be specific, we envision (in this Part of the section) the epistemic state of the computer to be maintained in terms of a table giving one of four values to each atomic sentence. We call such a table a *set-up* (following an isomorphic use of Routley and Routley 1972; see §16.2.1); i.e., a set-up is, mathematically, a mapping from atomic sentences into the set **4** = {***T***, ***F***, ***None***, ***Both***}. When an atomic formula is entered into the computer as either affirmed or denied, the computer modifies its current set-up by adding a "told True" or "told False" according as the formula was affirmed or denied; it does not subtract any information it already has, for that is the whole point of what we are up to. In other words, if p is affirmed, it marks p with ***T*** if p was previously marked with ***None***, with ***Both*** if p was previously marked with ***False***; and of course leaves things alone if p was already marked either ***T*** or ***Both***. So much for p as input.

The computer not only accepts input, but answers questions. We consider only the basic question as to p; this it answers in one of four ways: Yes, No, Yes and No, or I don't know, depending on the value of p in its current set-up as ***T***, ***F***, ***Both***, or ***None***. (It would be wrong to suppose that these four answers are either dictated by the four-valued logic or excluded by the two-valued logic; it is just that they are made more useful in the four-valued context. See Belnap 1963 and Belnap and Steel 1975.)

Warning—or, as N. Bourbaki says, *tournant dangereux* (☡): "told True" is *not* equivalent to ***T***. The relationships are rather as follows. In the first place, the computer is told True about some sentence A just in case it has marked A *either* with ***T*** *or* with ***Both***. Secondly, the computer marks A with ***T*** just in case it has been told True of A *and* has not been told False of A. And similarly for the relation between ***F*** and "told False." These relationships are certainly obvious, but also in practice confusing. It might help always to read "told True" as "told *at least* True," and ***T*** as "told *exactly* True."

We now make the observation that consitutes the foundation of what follows: these four values naturally form a lattice under the lattice-ordering "approximates the information in"; indeed they form an *approximation lattice* in the sense we described above:



In this Hasse diagram joins ($\sqcup$) and meets ($\sqcap$) are least upper bounds and greatest lower bounds, respectively, and $\sqsubseteq$ goes uphill. ***None*** is at the bottom because it gives no information at all; and ***Both*** is at the top because it gives too much (inconsistent) information, saying as it does that the statement so marked is held both told True/told False. As we mentioned above, Scott has studied approximation lattices in detail and in a much richer setting than we have before us; yet still this little four-element lattice is important for much of his work. We remarked above that, according to Scott's thesis, the important functions in the neighborhood of an approximation lattice like A4 are the continuous ones. We do not, fortunately, have to deal with continuity for a while, since in the finite case it turns out that for a function f to be continuous is just for it to be *monotonic*, i.e., for it to preserve the lattice ordering: $a \sqsubseteq b$ implies $fa \sqsubseteq fb$.

For example, suppose a function g on **A4** is such that it takes ***T*** into ***F*** and ***F*** into ***T***: $g(\boldsymbol{T}) = \boldsymbol{F}$, $g(\boldsymbol{F}) = \boldsymbol{T}$. Then, given that g is monotonic, since

T $\sqsubseteq$ ***Both*** we must have ***F*** $\sqsubseteq$ g(***Both***) and similarly ***T*** $\sqsubseteq$ g(***Both***). So we must have g(***Both***) = ***Both***. In a similar way, it is easy to calculate that g(***None***) = ***None***—if g is to be monotonic, as all good functions should be.

§81.2.2. Compound sentences and the logical lattice L4. Now this function g is no *mere* example of a monotonic function on the lattice **A4** of approximate and inconsistent truth values. In fact we are in the very presence of *negation*, which some have called the original sin of logic, but which we clearly need in a sufficiently rich language for our computer to use—just to be able to answer simple yes–no questions. To see that g really *is* negation, consider first that the values ***T*** and ***F***, representing as they do the pure case, should act like the ordinary truth values the True and the False; so obviously we want $\sim$***T*** = ***F***, and $\sim$***F*** = ***T***. And then Scott's thesis now imposes on us a ***unique*** solution to the problem of extending negation to the values of our foursome; we *must* have $\sim$***None*** = ***None*** and $\sim$***Both*** = ***Both*** if negation is to be an acceptably monotonic function on the approximation lattice **A4**.

We can summarize the argument in a small table for negation.

	None	***F***	***T***	***Both***
$\sim$	m ***None***	tt ***T***	tt ***F***	m ***Both***

Here "tt" in the upper right-hand corner means that the value was given by truth-table considerations, and "m" indicates that monotonicity was invoked.

Having put negation in our pocket, let's turn to conjunction and the disjunction. We start with truth-table considerations for the ***T***–***F*** portion of the tables, and then invoke monotony (in each argument place) and easy considerations to extend them as indicated.

&	***None***	***F***	***T***	***Both***
None	m ***None***		m ***None***	
F		tt ***F***	tt ***F***	
T	m ***None***	tt ***F***	tt ***T***	m ***Both***
Both			m ***Both***	m ***Both***

∨	***None***	***F***	***T***	***Both***
None	m ***None***	m ***None***		
F	m ***None***	tt ***F***	tt ***T***	m ***Both***
T		tt ***T***	tt ***T***	
Both		m ***Both***		m ***Both***

With just ordinary truth tables and monotonicity, it would appear we have to stop with these partial tables; on this basis neither conjunction nor disjunction—unlike negation—is uniquely determined. Of course we might make some guesses on the basis of intuition, but this part of the argument is founded on a desire not to do that; rather, we are trying to see how far we can go on a purely theoretical basis.

It turns out that, if we ask only that conjunction and disjunction have some minimal relation to each other, then every other box is uniquely determined. There are several approaches possible here, but perhaps as illuminating as any is to insist that the orderings determined by the two in the standard way be the same; which is to say that the following equivalence (see e.g., the end of §28.2.1.) holds:

$a\&b = a$ iff $a \vee b = b$
$a\&b = b$ iff $a \vee b = a$

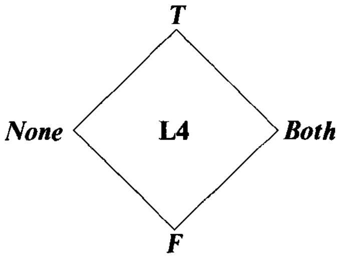
For look at the partial table for conjunction. One can see that ***T*** is an identity element: a&***T*** = a, for all a. So, if conjunction and disjunction fit together as they ought, we must have a∨***T*** = ***T***, for all a, which fills in two boxes of the ∨-table. And similar arguments fill in all except the corners.

For the corners we must invoke monotonicity (*after* the above lattice argument). For example, since ***F*** ⊑ ***Both***, by monotonicity (***F*** & ***None***) ⊑ (***Both*** & ***None***), so ***F*** ⊑ (***Both*** & ***None***). Similarly, ***None*** ⊑ ***F*** leads to (***Both*** & ***None***) ⊑ (***Both*** & ***F***), i.e., (***Both*** & ***None***) ⊑ ***F***. So, by antisymmetry in A4, (***Both*** & ***None***) = ***F***. These additional results are brought together in the following tables, where "f" indicates use of the above fit between & and ∨, and "m" again indicates monotonicity.

&	***None***	***F***	***T***	***Both***
None	***None***	f ***F***	***None***	m ***F***
F	f ***F***	***F***	***F***	f ***F***
T	***None***	***F***	***T***	***Both***
Both	m ***F***	f ***F***	***Both***	***Both***

∨	***None***	***F***	***T***	***Both***
None	***None***	***None***	f ***T***	m ***T***
F	***None***	***F***	***T***	***Both***
T	f ***T***	***T***	***T***	f ***T***
Both	m ***T***	***Both***	f ***T***	***Both***

We don't know whether we should be surprised or not, but in fact these tables (isomorphic to Smiley's matrix of §15.3, which is their historical as opposed to theoretical source) do constitute a lattice, with conjunction as meet and disjunction as join; a lattice which can be pictured as follows:



Let us agree to call this the *logical* lattice **L4** (it is the SL of §34.1), to distinguish it from the *approximation* lattice **A4**. The ordering on **L4** we write as $a \leq b$; we write meets as $a\&b$, and joins as $a \vee b$. We note that in the logical lattice, each of the values ***None*** and ***Both*** is intermediate between ***F*** and ***T***; and this is as it should be, for the worst thing to be told is that something you cling to is false, *simpliciter*. You are better off (it is one of your hopes) either being told nothing about it or being told both that it is true and also that it is false; while of course best of all is to be told it is true, with no muddying of the waters. Nevertheless, surely most of you must be puzzled, if you are thinking about it, concerning the rules for computing the conjunction and disjunction of ***None*** and ***Both***: (***None*** & ***Both***) = ***F***, while (***None*** ∨ ***Both***) = ***T***. We ask you for now only to observe that we were driven to these equations by only three considerations: ordinary truth tables, monotonicity, and fit between & and ∨. But we shall have more to say about this.

We can now use these logical operations on **L4** to induce a semantics for a language involving &, ∨, and ~, in just the usual way. Given an arbitrary set-up s—a mapping, you will recall, of atomic formulas into **4**—we can extend s to a mapping of *all* formulas into **4** in the standard inductive way:

$$s(A\&B) = s(A)\&s(B)$$
$$s(A \vee B) = s(A) \vee s(B)$$
$$s({\sim}A) = {\sim}s(A)$$

And this tells us how the computer should answer questions about complex formulas based on a set-up representing its epistemic state (what it has been told): just as it does for answering questions about atomic formulas, it should answer a question as to *A* by Yes, No, Yes and No, or I don't know, according as the value of *A* in s (i.e., s(A)) is ***T***, ***F***, ***Both***, or ***None***.

The preceding discussion will have struck you as abstractly theoretical; we should next like to take up negation, conjunction, and disjunction from an altogether different and more intuitive point of view. The question to which we are going to address ourselves is this: Given our intuitive understanding of the meaning of the four truth values as indicating markings of sentences with either or both of the True and the False, what is a plausible way to extend these values to compound sentences when we know the values of the ingredient sentences?

In the context of our enterprise, we can sharpen this question and distinguish it from others. We are *not* asking, What is meant by the truth-functional connectives of English—or any language, informal or formal? Rather, we are simply asking, How do *we* want our computer to answer *our* questions about compound sentences, given that we have decided how it is supposed to answer our simpler questions? The question presupposes something nontrivial: namely, that in fact we want a functional relationship between how the computer answers questions about the parts and how it answers questions about the whole when its input is entirely atomic. The reason for this presupposition? We think it tends to simplify our dealings with the computer by increasing our intellectual control over what we ourselves have created, and, furthermore, it is likely to be more easily and efficiently managed by the computer than some other than truth-functional alternatives. But it is not an article of faith.

Let us take up negation first. The inevitable thing to say seems to be that $\sim A$ should be marked "told True" just in case A is marked "told False," and vice versa. In other words, we want the computer to answer "Yes" to "$\sim A$?" just in case it answers "No" to "A?" and vice versa. But then consider the correspondences:

None: marked with neither
F: marked with just told False
T: marked with just told True
Both: marked with both.

It immediately comes out that we should mark $\sim A$ with ***Both*** if A is, with ***None*** if A is, and with ***T*** or ***F*** if A is ***F*** or ***T***. For example, if A is marked ***None***, i.e., with neither told True nor told False, then $\sim A$ should also be marked with neither. If you know nothing about A, then you know nothing about $\sim A$. And the same reasoning works for ***Both***: if you know too much about A, then you also know too much about $\sim A$.

In a similar way, we can give intuitive clauses for evaluation of conjunctions and disjunctions, as follows:

Mark (A&B) with at least told True just in case both A and B have been marked with at least told True.

Mark (A&B) with at least told False just in case at least one of A and B have been marked with at least told False.

This completely determines how to mark conjunctions.

Mark ($A \vee B$) with at least told True just in case at least one of A and B have been marked with at least told True.

Mark ($A \vee B$) with at least told False just in case both A and B have been marked with at least told False.

And this similarly uniquely determines disjunction, given our intuitive correspondence between our four values ***None***, ***F***, ***T***, ***Both*** on the one hand, and markings with neither, one, or both of told True and told False on the other.

There is another way to read this table, which we give only for one case: we are declaring that we want the computer to answer "Yes" to "($A \vee B$)?" just in case it is prepared to answer "Yes" to either "A?" or "B?" And we note that this is plausible *only* when, as at present, all input is atomic, so that all disjunctions are decided. In general, commencing with Part 2, we might often expect that the computer can tell us that it was told "$A \vee B$" when it wasn't told either A or B: "told" does not in general distribute over disjunction. But it does in the special, present case, when the entire epistemic state is to be thought of as carried by a set-up.

This intuitive "double-entry bookkeeping" account of the connectives is exactly that of §50.3 with regard to its structure. What we can now go on to observe is that the intuitive account exactly agrees with the theoretically based account deriving from Scott's approximation lattices. For example, consider one of the odd corners, (***Both*** & ***None***) = ***F***. Well, suppose A has been marked both told True and told False, and B with neither (corresponding to ***Both*** and ***None***, respectively). Then the computer must mark (A&B) at least told False, since one of its components is marked at least told False; and it must *not* mark it at least told True, since not both of its components are so marked (assuming atomic input only). So we must mark it exactly told False. So (***Both*** & ***None***) = ***F***. In other even more informal words, in this circumstance the computer has a reason to suppose (A&B) told False, but none to suppose it told True. So, although the oddity of (***Both*** & ***None***) = ***F*** doesn't go away, it anyhow gets explained.

§81.2.3. Entailment and inference: The four-valued logic. Where are we? Well, we haven't got a logic, i.e., rules for generating and evaluating inferences. (In our case we really want the former; we want some rules for the computer to use in generating what it implicitly knows from what it explicitly knows.) What we do have is four interesting values, with indications as to how these are to be used by friend computer, and three splendid connectives, with complete and well-motivated tables for each. And, as we all know, lots of other connectives can be defined in terms of these; so for our purposes three is enough.

Suppose we have an argument involving these connectives. The question is, when is it a good one? Again we want to give an abstractly theoretical answer, and then an intuitive answer. (And then several more answers, too, if there is time enough. For the question is fascinating.)

The abstract answer relies on the *logical* lattice we took so much time to develop. It is: entailment goes uphill. That is, given any sentences A and B (compounded from variables by negation, conjunction, and disjunction), we will say that A *entails* or *implies* B just in case for each assignment of one of the four values to the variables, the value of A does not exceed (is less-than-or-equal-to) the value of B. In symbols: $s(A) \leq s(B)$ for each set-up s. This is a plausible definition of entailment whenever we have a lattice of values that we can think of as somehow being graded from bottom to top; and, as we suggested when first presenting you with the logical lattice, we can indeed think of ***None*** and ***Both*** as being intermediate between awful ***F*** and wonderful ***T***.

Now for an account which is close to the informal considerations underlying our understanding of the four values as keeping track of markings with told True and told False: say that the inference from A to B is valid, or that A entails B, if the inference never leads us from told True to the absence of told True (preserves Truth), *and also* never leads us from the absence of told False to told False (preserves non-Falsity). Given our system of markings, to ask this is hardly to ask too much.

(We note that in §50.6 we have shown that it suffices to mention truth preservation, since if some inference form fails always to preserve non-Falsity, then it *also* fails to preserve Truth. But, as we suggested in §50.6, the False really is on all fours with the True; so it is profoundly natural to state our account of "valid" or "acceptable" inference in a way that is neutral with respect to the two.)

Finally we have a logic, that is, a canon of inference, for our computer to use in making inferences involving conjunction, negation, and disjunction, as well of course as whatever can be defined in terms thereof. We note that this logic has two key features. In the first and most important place, it is rooted in reality. We gave reasons why it would be good for our computer to think in terms of our four values, and why the logic of the four values should be as it is. In the second place, though we have not thrown around many hen-scratches, it is clear that our account of validity is mathematically rigorous. And obviously the computer can decide by running through a truth-tabular computation whether or not a proposed inference is valid. But there is another side to the logician's job, which is codifying inferences in some axiomatic or semi-axiomatic way that is transparent and accordingly usable. If this sounds mysterious, it is not; we just mean that a logician, given a semantics, ordinarily tries to come up with proof theory for it; a proof theory that is consistent and complete relative to the semantics.

Fortunately the job has already been done. It will come as no surprise to anyone, other than the odd reader who elected to read this section first, that we have given still another characterization of the tautological entailments—the system $\mathbf{E}_{\mathrm{fde}}$—of §15.1. And although we cannot help taking note of this additional evidence for the stability of $\mathbf{E}_{\mathrm{fde}}$, our interests in this section lie in altogether different directions.

§81.2.4. Observations. Some observations now need to be made before pushing further.

First, we note that not derivable from these principles, and not semantically valid, are the paradoxes of "implication" $A\&{\sim}A \rightarrow B$ and $A \rightarrow B\vee{\sim}B$. In context, the failure of these principles is evident. The failure of the first simply means that just because we have been told both that A is True and that A is False, we cannot conclude: everything. Indeed, we may have been told nothing about B, or just that it is False. And the failure of the second is equally evident: from the fact that we have been told that A is True, we cannot conclude that we know something about B. Of course B is *ontologically* either True or False, and such ontological truth values will receive their due; but for $B\vee{\sim}B$ to be marked told True is either for B to be marked told True or for B to be marked told False; and it may have neither mark. Or, for a different way of counterexampling $A \rightarrow B\vee{\sim}B$, A may have just been told True while $(B\vee{\sim}B)$ has *both* values because B does.

These inferences are not wanted in a scheme that is designed not to break down in the presence of "contradictions"; and since contradictions really do threaten in the circumstances we describe, their absence is welcome.

We would be less than open, however, if we failed to point out the absence of what at first sight looks like a more harmless principle: our old friend (γ), $(A\vee B)\&{\sim}A \rightarrow B$. Surely, one would think, our computer should be able to argue that if it is told that one of A and B is True, and it is told that A is False, then it must have been told that B is True. That's true; unless—and of course this is a critical "unless"—there is an inconsistency around. In fact the inference that the canon allows is just exactly

$$(A\vee B)\&{\sim}A \rightarrow (A\&{\sim}A)\vee B.$$

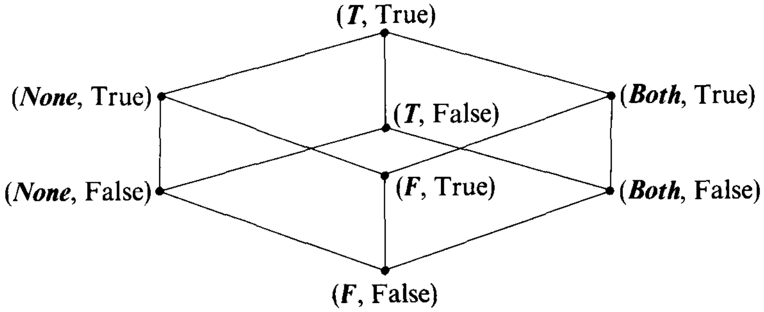
That is, having determined that the antecedent is at least told True, we allow the computer to conclude: either B is at least told True, or something funny is going on; i.e., it's been told that A is both True and False. And this, you will see, is right on target. If the *reason* that $(A\vee B)\&{\sim}A$ is getting thought of as a Truth is because A has been labeled as both told True and told False, then we certainly do *not* want to go around inferring B. The inference is wholly inappropriate in a context where inconsistency is a live possibility.

The second observation is that our four values are proposed only in connection with *inferences*, and are definitely not supposed to be used for determining which formulas in &, $\vee$, and $\sim$ count as so-called "logical truths." In fact, no formula takes *always* the value $\boldsymbol{T}$; so *that* property surely won't do as a semantic account of logical truth. There are, on the other hand, formulas that never take the value ***F***, e.g., $A \vee \sim A$; but this set is not even closed under conjunction and does not contain $(A \vee \sim A)\&(B \vee \sim B)$, which can take ***F*** when A takes ***None*** and B takes ***Both***. So just don't try to base logical truth on these values.

Thirdly, let us consider ontology versus epistemology. One of the difficulties that often arise in relating many-valued logics to real concerns is that one tends to vacillate between reading the various values as epistemic on the one hand, and ontological on the other. Does Łukasiewicz's middle value, $\frac{1}{2}$, mean "doesn't have a proper truth value," or does it mean "truth value unknown"? In informal explanations of what is going on, logicians sometimes move from one of these readings to the other in order to save the interest of the enterprise.

Our four values are unabashedly epistemic. According to our instructions, sentences are to be marked with either a ***T*** or an ***F***, a ***None*** or a ***Both***, according to what the computer has been told; or, with only a slight (but dangerous) metaphor, according to what it believes or knows. Does this somehow make the enterprise wrong-headed? Or not logic? No. Of course these sentences *have* truth values independent of what the computer has been told; but who can gainsay that the computer cannot *use* the actual truth value of the sentences in which it is interested? All it can possibly *use* as a basis for inference is what it knows or believes, i.e., what it has been told.

But we can do better than this. Let us get the ontology into the act by *splitting* our four epistemic values into two, one representing the case in which the sentence is ontologically true, the other the case in which it is false. Obviously we then get eight values instead of four, each of which we may visualize as an ordered pair, the left entry of which is an epistemic value ***T***, ***F***, ***None***, or ***Both***, while the right entry is one of Frege's ontological values the True and the False. Giving the usual classical two-valued tables to the connectives, and also and equivalently, interpreting the implicative connective in the usual way, we are led to the following lattice picture (this is *not* an approximation lattice; it is isomorphic to M_0 of §18.4 and elsewhere):



The &s and ∨s can be computed, respectively, as greatest lower bounds and least upper bounds, while negation-pairs are: two left, two center, two right, and top-bottom (*not* the Boolean way). The values of this new many-valued logic have a mixed status: they are in part epistemological and in part ontological. Should we then move to this logic? It is entertaining to observe that there is no need to do so for inferences; for *exactly* the same inferences are valid with this as with our four-valued canon of inference. Nor should this be surprising, for two reasons. In the first place, as we already observed, the only things we can actually *use* in inference are the epistemic values ***T***, ***F***, ***None***, and ***Both***, representing what we know, believe, or in any event have been told by authority we by and large trust. Secondly, and more prosaically, observe that all the inferences sanctioned by the four-valued canon are already approved in two-valued logic; so adding as a condition that ontological truth is to be preserved is to add a condition that is already satisfied and yields no new constraints. So for practical reasons there is no need to move from four to eight values for judging inferences. In the words of a famous philosopher, "Do not multiply many values beyond necessity."

If, however, for some reason (we do not just now know what) someone wanted an account of logical truth in &, ∨, and ~, then one could invoke as a criterion: being always True (by the right entry of the pair) regardless of what you've been told (according to the left entry). Then, not surprisingly, one finds out that the two-valued tautologies are precisely the logical truths on this account. Not surprisingly, because we invoke values only ontological, throwing away (in the eight-valued case) all the information of the epistemic values.

Let us say explicitly, if it is not obvious, that we think this codification of truth-functional logical truths not important to the computer; for what was *wanted* was a way of *reasoning* from and to truth-functional compounds, not a sorting of these compounds.

Our fourth "observation" is not so much an observation as it is an inconclusive discussion of the role of ***Both*** and ***None***. The problem is that one is inclined to the view that they should be identified, that the computer is in the same state having been told that A is both True and False as it is having been told nothing about A. Both S. Haack and A. Kenny have each, in quite different ways, suggested something like this. If you will be satisfied with a dialectical flourish, we can supply one of the form "Wrong, but understandable." It goes like this. In the first place, it is somehow magnificently obvious that ***Both*** and ***None*** should not be identified, as H. S. Harris noted in conversation, just because we want the computer to distinguish *for us* when it has been told a contradiction from when it has been told nothing. This is surely essential on anyone's view. In the second, our developments can be taken as explaining the feeling that they should be identified, for just look at the logical lattice **L4**: there ***Both*** and ***None*** occupy (distinct but) absolutely symmetrical positions between ***F*** and ***T***, and in this sense are "identified." For instance, we allow the inference from neither to ***F***, and to neither from ***T***, and thus treat them alike.

Still, though this response may be helpful, we are not altogether happy with it. And we much prefer to leave the discussion as at this stage incomplete.

Our penultimate observation concerns the suggestion that the computer keep more information than we have allowed it to keep. Perhaps it should count the number of times it has been told True or told False, or perhaps it should keep track of its sources by always marking, for example, "told True by Sam at 2200:03 on 4 August 1973." We do not see why these ideas should not be explored, but two comments are in order. The first is that it is by no means self-evident how this extra information is to be utilized in answering questions, in inference, and in the input of complex sentences. That is, one should not be misled by the transparency of the idea in the case of atomic sentences. The consequence of this first comment is merely that the exploration lies ahead. The second is the practical remark that there are severe costs in carrying extra information, costs which may or may not be worth incurring. And if there are circumstances in which they are not worth incurring, we are back to the situation we originally described.

Lastly, we want to mention some alternatives without (much) discussing them. Gupta has noted that one could define the value of A in s not directly as we have done, but rather by reference to all the consistent sub-set-ups of s. Definitions: s′ is a *sub-set-up* of s if it approximates it: $s' \sqsubseteq s$. And s′ is *consistent* if it never awards ***Both***. Finally, let s(A) be defined, in Gupta's way, by $s(A) = \bigsqcup \{s'(A): s' \text{ is a consistent sub-set-up of } s\}$, where s′($A$) is as already defined. The idea is clearly dual to van Fraassen's 1969a definition of true-in-a-valuation by reference to all the complete (i.e., all truth-value gaps filled) "supervaluations" of a given valuation. One notes that if s(p) = ***Both*** then the question as to p (on s) will be answered "Yes and No" as before, while the question as to $p \& \sim p$ will be answered just "No," instead of "Yes and No."

A related idea is to follow van Fraassen 1969a directly by looking at all the complete super-set-ups of a given set-up; this would give always "Yes" to $p \vee \sim p$. And carrying this idea to its logical conclusion would combine the two ideas (if possible).

All these things are possible. One would hope, however, that the discussion of the alternatives would circulate around the question, How in fact do we want the computer to answer our questions? Thus they would not be *mere* possibilities.

Quantifiers. Quantifiers introduce a number of subtleties to which we shall merely tip our hat, while recognizing that treating them in detail is quite essential to our enterprise.

There is in the first place the question of whether "the" domain is finite or infinite. Both cases can plausibly arise. In the latter case, there is a question of how the computer is to represent infinite information with its finite resources, but one should not infer from the existence of this problem that the computer can't or shouldn't involve itself with quantification over infinite domains. Surely it should be allowed to answer "Is there a number such that . . . ?" queries (if it can).

In the second place, there is the question of whether the computer has a name for everything in "the" domain, so that we can employ the substitutional interpretation of the quantifiers, or, on the other hand, does not have a name for each entity in "the" domain, so that the domain-and-values interpretation is forced. Again: both cases can plausibly arise, though attending to standard examples like baseball queries or airline flights might have made one think that in the computer situation everything always has a name. But, for example, in some of Isner's work (1975) the computer is told "there is something between a and b" in a context in which it hasn't got a complete list of either the names or the entities against which to interpret this statement. And still it must work out the consequences, and answer the questions it is given. (Of course it is OK for the computer to make up its own name for the "something" between a and b; but that is both an important and an entirely different matter.)

In any event, the semantics given for the connectives extend to universal and existential quantifiers in an obvious way, and we suppose the job done. And the various alternatives mentioned above turn out not to make any difference to the *logic* (with the obvious exception of the finite everything-has-a-name case): the valid "first degree entailments" of §40 do admirably (supplemented, in the finite case, with the principle that a conjunction that runs through the domain implies the appropriate universal statement).

§81.3. Part 2. Compound truth-functional inputs.

We can pause now if we like with regard to the overall title of this section, for it would be possible to do so and still claim the title appropriate: we really have presented a four-valued logic and argued that it is useful. But there is a fair bit more to do, some of it of theoretical interest, some of it practical. We begin with considerations closer to the practical.

§81.3.1. Epistemic states.

So far, in Part 1, we have been considering the situation in which the epistemic state of the computer could be represented by tables specifying for the various atomic formulas which of the four values in **4** each is to take. We called the mathematical equivalent of such a table a *set-up*; that is, a set-up s is a mapping from all atomic formulas into **4**: $s(p) \in \mathbf{4}$. Let S be the set of all set-ups, and recall that each $s \in S$ extends uniquely to map all formulas into **4**: $s(A) \in \mathbf{4}$.

Each set-up s represents (not what is true but) what the computer has been told. But can every epistemic state of the computer be represented by a set-up? If in fact, as in Part 1, only atomic sentences are affirmed or denied to the computer, of course; but not in general otherwise. For example, no single set-up can represent the state the computer should be in when it is told that either P, the Pirates, or O, the Orioles, won in 1971, but it isn't told which. Set-ups can, by judicious use of ***None***, represent some kinds of incomplete information, but not this kind. For any single set-up in which "either P or O" (with obvious meaning) is marked told True is a set-up in which either P or O is *also* marked told True and, hence, has too much information. Any such set-up would lead the computer to answer "Yes" either to the question, Did the Pirates win? or to the question, Did the Orioles win? And the computer should not be able to answer either of these questions, having been told only that either the Pirates or the Orioles won.

The solution to this problem is well known in the logical literature, going back to Carnap 1942 at least. It has been used in epistemic and doxastic logic by Hintikka 1962 and has also been worked out for computers by Isner 1972, 1975: one uses a *collection* of set-ups to represent a single epistemic state, the rough and partial idea being that the computer takes a formula as something it has been told if it comes out told True on *each* of the set-ups forming its current epistemic state. For example, when told that either the Pirates or the Orioles won, the computer would represent this information by building *two* set-ups, one in which the Pirates get ***T*** and the Orioles ***None***, and the other in which the Orioles get ***T*** and the Pirates ***None***. Later, when it is asked whether the Pirates won, it will say that it doesn't know, since the Pirates are not marked ***T*** in every state, and also not ***F*** in every state; and similarly if it is asked about the Orioles. But if it is asked "Did either the Pirates or the Orioles win?" then it will answer affirmatively, since that sentence is marked told True in both of the set-ups in its epistemic state.

Let us, therefore, at least for the duration of this Part, define an *epistemic state* as a nonempty collection of set-ups, a (nonempty) subset, that is, of S. (If we later omit "nonempty," please supply it, or identify the empty set with the unit set of the set-up that marks everything in sight with ***Both***.) We let ES be the set of all epistemic states, and use "E" as ranging over ES. Let E be an epistemic state. Then the "meaning" of E is that the computer has been told that the world is accurately (but perhaps incompletely) described by at least one of the set-ups in E. As from the beginning the possibility exists that such a description is inconsistent.

E represents the basis on which we want the computer to answer our questions. And let us now state more completely and more accurately how we want our questions answered, by defining the value of a sentence in an epistemic state; in symbols, E(A) for $E \in ES$ and A a formula. Note how the key idea of approximation is mobilized to give insight into what is going on: the value of a sentence in an epistemic state is to be determined by taking the *meet* of all its values in the separate states—the meet to be taken not in the logical lattice **L4** but in the approximation lattice **A4**. In notation:

$$\mathrm{E}(A) = \sqcap\{\mathrm{s}(A)\colon \mathrm{s}\in\mathrm{E}\}.$$

The idea of this definition is straightforward and intuitively appealing. In the first place, we noted that set-ups individually tend to give us more information than we've got about a formula, or in the language of approximation,

$$\mathrm{E}(A) \sqsubseteq \mathrm{s}(A) \qquad \text{for all } \mathrm{s}\in\mathrm{E}.$$

Now what we are saying is that E(*A*) should be defined so as to be *maximal* while retaining this relationship; i.e., E(*A*)—the value of *A* in E—should be the *greatest* lower bound of all the s(*A*) for s∈E.

EXAMPLE. Let E = {s, s′}, where

$$\left.\begin{array}{ll} \mathrm{s(P)} = \boldsymbol{T} & \mathrm{s'(P)} = \boldsymbol{None} \\ \mathrm{s(O)} = \boldsymbol{None} & \mathrm{s'(O)} = \boldsymbol{T} \\ \mathrm{s(B)} = \boldsymbol{T} & \mathrm{s'(B)} = \boldsymbol{F} \\ \mathrm{s(M)} = \boldsymbol{T} & \mathrm{s'(M)} = \boldsymbol{Both} \end{array}\right\} \text{ then: } \left\{\begin{array}{l} \mathrm{E(P)} = \boldsymbol{None} \\ \mathrm{E(O)} = \boldsymbol{None} \\ \mathrm{E(B)} = \boldsymbol{None} \\ \mathrm{E(M)} = \boldsymbol{T} \end{array}\right.$$

Further, though E(P) = E(O) = *None*, clearly E($P \vee$ O) = *T*.

Let us, as usual, relate this to marking told True and told False: it all amounts to saying that we should mark *A* told True in E if it is marked told True in all set-ups in E, and mark it told False if it is marked told False in all set-ups in E; recognizing, as always, that this recipe allows marking *A* with neither or both. On this account the similarities emerge to van Fraassen 1969a's supervaluations, to definitions of necessity and impossibility in modal logic (e.g., Kripke 1963), and to evaluation of epistemic operators in Hintikka 1962. But of course in all those cases set-ups are restricted to those which are consistent, nor is there any sense in which any of those logics are four-valued, or even three-valued. (Van Fraassen's formulas can take three values, but the third does not have a *logical* relation to the other two, nor is his semantics truth-functional.)

In now extending the account of question-answering, again treat only the simple question as to *A* in the context of an epistemic state E. It goes just as before: the computer answers Yes, No, Yes and No, or I don't know, according as the value E(*A*) of *A* in E is ***T***, ***F***, ***Both***, or ***None***. If, for example, the value of *A* in its current state is ***Both***, the computer answers "Yes and No." Of course, in this case the asker of the question will know that the answer is based on an inconsistency—and so will the computer. Indeed, this is how the computer would naturally *report* an inconsistency in an epistemic state; recall that the answer does not have the ontological force of "That's the way the world is," but rather has the epistemic force of "That's what I've been told (by people I trust to get it generally right)."

There are at least three situations in which the computer has to deal with formulas: when asked a question, as we have just discussed; when calculating or inferring, which we have discussed some and to which we shall return, and when a formula is input. It is this last which is now up for discussion, but further developments are going to be easier if some additional approximation lattices are introduced at this point.

§81.3.2. More approximation lattices. Note first that the family S of all set-ups constitutes a natural approximation lattice AS, where the order is pointwise:

$$\mathrm{s} \sqsubseteq \mathrm{s}' \text{ iff, for each atomic sentence } p,\ \mathrm{s}(p) \sqsubseteq \mathrm{s}'(p) \text{ (in } \mathbf{A4}\text{)}.$$

That is, one set-up approximates another if, for each atomic formula p, the information the first set-up gives about p approximates the information the other set-up gives about p. Our point is not only that AS is a complete lattice (we need that mathematically), but that it is *natural* to interpret its ordering as an approximation: if one increases the information on one of the atomic formulas, one increases the information in the set-up.

Since AS is infinite, for the first time in the course of these deliberations, the approximation-lattice ideas of limit and of continuity now come into their own. We won't dwell on this, but do point out one application. Let us say that a set-up is *finite* if it gives values other than ***None*** to only finitely many atomic formulas. Then every set-up s is the limit of a set of finite set-ups: $\mathrm{s} = \bigsqcup \mathrm{X}$ for some X a directed set of finite set-ups. This is important if the computer can directly represent only finite set-ups.

Moving up a level, we can also define a natural approximation-lattice ordering on the set ES of epistemic states. Naturally we want

$$\mathrm{E} \subseteq \mathrm{E}' \text{ implies } \mathrm{E}' \sqsubseteq \mathrm{E},$$

since the smaller epistemic state E gives more definite information; but the converse won't do (unless both E and E′ are closed upward; see below). The right definition, yielding the above as a special case, is as follows:

$$\mathrm{E} \sqsubseteq \mathrm{E}' \text{ iff every } \mathrm{s}' \in \mathrm{E}' \text{ is approximated by some } \mathrm{s} \in \mathrm{E}.$$

For example, let

S	S′	S″
$\mathrm{s}(p) = \boldsymbol{T}$	$\mathrm{s}'(p) = \boldsymbol{T}$	$\mathrm{s}''(p) = \boldsymbol{T}$
$\mathrm{s}(q) =$ ***None***	$\mathrm{s}'(q) = \boldsymbol{T}$	$\mathrm{s}''(q) =$ ***None***
$\mathrm{s}(r) =$ ***None***	$\mathrm{s}'(r) =$ ***None***	$\mathrm{s}''(r) = \boldsymbol{T}$

Then $E = \{s\}$ approximates $E' = \{s', s''\}$. Note that neither E nor E′ gives any information about q or r, but E′ tells us that $q \vee r$ is T.

It is not true that this ordering of ES yields a lattice; antisymmetry fails. There are two ways to make it a lattice, both of which we mention and neither of which we employ. We begin the first by defining an equivalence relation by

E *is equivalent to* E′ iff each approximates the other.

We then "divide through" by this relation: take equivalence classes. This easily turns out to be a complete lattice, and a natural approximation lattice (partly since the equivalence is natural).

The second uses the method of "representatives" instead of equivalence classes. Define a state E as *closed upward* if $s \sqsubseteq s'$ and $s \in E$ imply $s' \in E$. Where CES is the set of all nonempty closed-upward states, it constitutes a natural approximation lattice ACES under the ordering defined above. Indeed in this case it is obvious that the ordering we defined above does in fact agree with the superset relation, so that obviously we have a complete lattice. One might worry, however, that we have cut out some interesting states. Not so: define the *upward closure of* E by:

C(E) = the family of set-ups approximated by some set-up in E.

Clearly C(E) is both upward closed and equivalent to E; so if we liked we could use C(E) as the "representative" of E. (Note also that E and E′ are equivalent just in case $C(E) = C(E')$; everything fits.)

But we choose to stay with ES and its ordering even though it is not a lattice, for, although both the lattice of equivalence classes and the lattice ACES are mathematically convenient (indeed we constantly rely on the convenience of the latter), they depart from practicality: the computer cannot work with the elements of these lattices since these elements are grossly infinite.

Let us then define AES as ES supplied with the ordering above and also with a couple of lattice-like operations which (1) give results equivalent to those obtained by passing through ACES and (2) *preserve finiteness.* The most natural meet operation is obviously just union:

$$E \quad E' = E \cup E'$$

And the join:

$$E \sqcup E' = \{s \cup s' : s \in E,\ s' \in E'\}.$$

Also, analogously, for the general meet $\sqcap X$ and general join $\sqcup X$, where X is a subset of ES.

It is important that our valuation function $E(A)$ is not only monotonic in the argument E but, in an appropriate sense, continuous in AES; in spite of the fact that meets, used in the definition of E(A), are notoriously badly behaved in approximation lattices.

Certain elements of ES are of particular interest, namely those which characterize and are characterized by formulas. For each A, define the *truth set of A* and the *falsity set of A* as follows:

$$\mathrm{Tset}(A) = \{\mathrm{s}: \boldsymbol{T} \sqsubseteq \mathrm{s}(A)\}$$
$$\mathrm{Fset}(A) = \{\mathrm{s}: \boldsymbol{F} \sqsubseteq \mathrm{s}(A)\}$$

A is marked told True by all and only members of Tset(A), and told False by the members of Fset(A). Both of these sets are closed upwards, hence in CES. The next section in effect investigates some of their properties.

§81.3.3. Formulas as mappings: A new kind of meaning. Now we turn to a question of considerable interest, and a question on which our various approximation lattices can shed considerable light: How is the computer to interpret a truth-functional formula A as input? Clearly it is going to use A to modify its present epistemic state; and indeed it is not too much to say that defining how the computer uses the formula A to transform its present epistemic state into a new epistemic state is a way—and a *good* way—of giving A a meaning. Consequently we want to associate with formula A a transformation, a mapping from epistemic states into new epistemic states. Furthermore, we also want to know what the computer is to do when the formula A is *denied* to the computer; so actually we associate with a formula A *two* functions, one representing the transformation of epistemic state when A is affirmed, the other the transformation when A is denied. Let us call these two functions A^+ and A^-. How to define them?

Recall that A^+ is to map states into states: $A^+(\mathrm{E}) = \mathrm{E}'$. The key ideas in defining what we want E′ to be come from the approximation lattice. First, in our context we are assuming that the computer uses its input always to increase its information, or at least it never uses input to throw information away. (That would just be a different enterprise; it would be nice to know how to handle it in a theory, but we don't.) And we can say this accurately in the language of approximation: $\mathrm{E} \sqsubseteq A^+(\mathrm{E})$. Second, $A^+(\mathrm{E})$ should certainly say no less than the affirmation of A: $\mathrm{Tset(A)} \sqsubseteq A^+(\mathrm{E})$. Third and lastly, we clearly want $A^+(\mathrm{E})$ to be the *minimum mutilation* of E that renders A at least told True. "Minimum mutilation" is Quine's fine phrase, but in the approximation lattice we can give a sense to it that is *no longer merely metaphorical*: namely, we want the *least* of those epistemic states satisfying our first two conditions. That is, we should define

$$A^+(\mathrm{E}) = \mathrm{E} \sqcup \mathrm{Tset}(A);$$

for that is precisely the minimum mutilation of E that makes A at least told True. (Recall that, in any lattice, $\mathrm{x} \sqcup \mathrm{y}$ is the "least (minimum) upper bound.")

Having agreed on this as the definition of A^+, it is easy to see that $A^-(\mathrm{E})$ should be the minimum mutilation of E that makes A at least told false:

$$A^-(\mathrm{E}) = \mathrm{E}\sqcup\mathrm{Fset}(A).$$

The above definitions accurately represent the meaning of A as input, but they do involve a drawback: the Tsets and Fsets may be infinite, or at least large, and so do not represent something the computer can really work with. For this reason, and also for its intrinsic interest, we offer another explication of A^+ and A^-, this time inductive, but still very much involving the idea of minimum mutilation.

First, what is the computer to do to its present epistemic state, when an atomic formula, p, is affirmed? Recalling that p must be marked at least told True in the result, and that it will not be such unless it is such in each member of E, it is clear that what the computer must do is run through each set-up in E and add told True to p. This will make p ***T*** if it was ***None*** before, it will leave it alone if it was already either ***T*** or ***Both***, and will make it ***Both*** if it was ***F***. And this is obviously the *minimum* thing the computer can do. Defining $\mathrm{p}_{\boldsymbol{T}}$ as that set-up in which p has ***T*** and all other atoms have ***None***, we can say this technically as follows (note where minimum mutilation comes in):

$$p^+\mathrm{E} = \{\mathrm{s}\sqcup p_{\boldsymbol{T}}\colon \mathrm{s}\in\mathrm{E}\}$$

And, with $p_{\boldsymbol{F}}$ defined similarly,

$$p^-\mathrm{E} = \{\mathrm{s}\sqcup p_{\boldsymbol{F}}\colon \mathrm{s}\in\mathrm{E}\}.$$

The union is in the approximation lattice AS of all set-ups.

The recursive clauses, which represent a way of giving meaning to the connectives (different from—though of course related to—the usual "truth conditions" account), now come easily.

$$(A\&B)^+ = A^+\circ B^+$$

That is, to make $A\&B$ true by mimimum mutilation, first minimally mutilate to get B true, and then minimally mutilate the result to get A true as well. (The "$\circ$" is for composition of functions.) It had better turn out, and it does, that $(\mathrm{A}\&\mathrm{B})^+ = (\mathrm{B}\&\mathrm{A})^+$—i.e., that the order of minimal mutilation makes no difference. Next, obviously,

$$({\sim}A)^+ = A^-.$$

And

$$(A\vee B)^+ = \lambda\mathrm{E}(A^+(\mathrm{E})\sqcap B^+(\mathrm{E}))$$

That is, one makes the mimimum mutilations for A and B separately, and then one finds the best (maximum) among all the states that approximate both of these—which is just their set-theoretical union. For example, if E is a singleton $\{\mathrm{s}\}$ in which $p\vee q$ has ***None***, $(p\vee q)^+\mathrm{E}$ is obtained by "splitting" s into two new states, in one of which p has $\boldsymbol{T}$ while q and everything else stay the same, and in the second of which q has $\boldsymbol{T}$ while p and everything else remain fixed.

We give the clauses for A^- without comment:

$$
\begin{aligned}
(\sim A)^- &= A^+ \\
(A\&B)^- &= \lambda\mathrm{E}(A^-(\mathrm{E}) \sqcap B^-(\mathrm{E})) \\
(A \vee B)^- &= A^- \circ B^-
\end{aligned}
$$

We have given two separate accounts of the meaning of A as input (affirmed or denied), so we had better observe that they agree. A third account of some merit begins by defining A^+ and A^- as functions from *set-ups* s into states E:

$$
\begin{aligned}
A^+\mathrm{s} &= \{\mathrm{s} \sqcup \mathrm{s}' \colon \mathrm{s}' \in \mathrm{Tset}(A)\} \\
A^-\mathrm{s} &= \{\mathrm{s} \sqcup \mathrm{s}' \colon \mathrm{s}' \in \mathrm{Fset}(A)\}
\end{aligned}
$$

Then

$$
\begin{aligned}
A^+\mathrm{E} &= \cup\{A^+\mathrm{s} \colon \mathrm{s} \in \mathrm{E}\} \\
A^-\mathrm{E} &= \cup\{\mathrm{A}^-\mathrm{s} \colon \mathrm{s} \in \mathrm{E}\}
\end{aligned}
$$

And there are a number of other variations; e.g.,

$$(A\&B)^+\mathrm{E} = A^+\mathrm{E} \sqcup B^+\mathrm{E}$$

§81.3.4. More observations. What we have done is use the approximation lattices not only to spell out in reasonably concrete terms what the computer is to do when it receives a formula as affirmed or denied, but, further, we have given a new theoretical account of the meaning of formulas as certain sorts of mappings from epistemic states into epistemic states. It is clear that there remains work to be done here in finding the right abstract characterizations and general principles, unless it has already been done somewhere or other; but we make a few comments.

To set the stage, recall that Scott has observed that the family of all *continuous* functions from an approximation lattice into itself (or indeed another) naturally forms a new approximation lattice, and it is important that our A^+ and A^- functions are members in good standing. But the A^+ functions (we may drop reference now to the A^- functions, since $A^- = (\sim A)^+$) form but a limited subset of all these functions, and it would be desirable to characterize an appropriate subset, without, however, leaning too heavily on linguistic considerations. One feature they all have in common is that they are one and all (weakly) *ampliative*:

$$\mathrm{E} \sqsubseteq A^+\mathrm{E},$$

or, where $\boldsymbol{I}$ is the identity function on ES,

$$\boldsymbol{I} \sqsubseteq A^+.$$

And this feature is a hallmark of our entire treatment: the computer is never to throw away information, only to soak it up. It is easy to see that the family of all continuous functions "above" **I** themselves form an approximation lattice—the lattice of all ampliative and continuous functions—which is closed under such pleasant operations as composition. (**I** is the bottom of this lattice.)

Another feature of the A^+ is that they are *permanent*: once A^+ is done to a state E, it stays done, and does not have to be done again, no matter how much the computer later learns. In symbols:

$$A^+\mathrm{E} \sqsubseteq \mathrm{E}' \text{ implies } A^+\mathrm{E}' = \mathrm{E}'.$$

These three features taken together can very likely be taken as a proper intrinsic characterization of the "kind" of functions represented by our truth-functional formulas. For a function f is continuous, ampliative, and permanent just in case f can be characterized as improving the situation by some fixed amount. That is, just in case there is some fixed element E_0 such that $\mathrm{f(E)} = \mathrm{E} \sqcup \mathrm{E}_0$, for all E. And that sounds right.

The interested reader can verify that from these principles one can deduce what are perhaps the most amusing of the properties of the A^+ functions: composition is the same as join, hence is commutative and idempotent:

$$A^+ \circ B^+ = B^+ \circ A^+$$
$$A^+ \circ A^+ = A^+.$$

We do not like to leave this discussion on such an abstract note, and so we conclude with a more practical remark. What "permanence" in the above sense *means* for the computer is that it has a choice when it receives A as an input: it can, if it likes, "remember" the formula A in some convenient storage, or, if it prefers, it can "do" A to its epistemic state and then *forget* about it. Since the meaning of A is a permanent function, A will be permanently built into the computer's present *and future* epistemic states. In the next Part there will emerge important contrasts with this situation.

§81.3.5. Quantifiers again. Quantifiers introduce problems which *must* be worked through but which we do *not* work through. The chief difficulty comes from the fact that we must keep our set-ups and epistemic states finite for the sake of the computer, whereas a quantified statement contains infinitely much information if the domain is infinite. We are going to offer only some murky comments.

In the first place, we will stay with the substitutional interpretation of the quantifiers so as not to have to modify the definition of "set-up." So quantification is always with respect to a family of constants suitable for substitution: $\forall x A x$ is the generalized conjunction of all its instances, and $\exists x A x$ the generalized disjunction of its instances. So, given a substitutional range, the reader can supply the right definitions for $s(\forall xAx)$ and $s(\exists xAx)$. Second, we are going to suppose that the substitution range is infinite; otherwise there is no problem. Third, with considerable hesitation, we are going to attach the substitution range to the entire epistemic state E, rather than permit the various set-ups in E to come with different substitutional ranges.

The problem is not really how to answer questions about quantified formulas (though there may be difficulty in practice), but in how to treat them as input. Perhaps it is obvious what we want for the existential quantifier: given $\exists xAx$ as input, add a *new* constant c to the substitutional range, and then make the minimum mutilation that makes Ac True. But we are not yet clear how to justify this procedure in approximation terms.

The universal quantifier as input is where the real problem lies: it can lead from a finite state E (i.e., a finite collection of finite set-ups) to an infinite state E′. What is probably best is to apply the universal quantifier (mutilate minimally to make an instance true) only for a while; this will force the computer to *remember* the universally quantified formula so that it can be applied again later, if necessary. (What counts as "necessary" is: as much as is needed to answer the questions asked.) The various finite states obtained by repeatedly applying $\forall xAx$ in this way clearly have *as a limit* the minimum mutilation in which $\forall xAx$ is True.

Some of what is needed can be better appreciated from the point of view of Part 3, and we drop the matter for now.

§81.4. Part 3. Implicational inputs and rules.

In Part 1 we pretended that all information fed into the computer was atomic, so we could get along with set-ups. In Part 2 we generalized to allow information in the form of more complex truth-functional formulas, a generalization which required moving to epistemic states. Now we must recognize that it is *practically* important that sometimes we give information to the computer in the form of rules that allow it to modify its own representation of its epistemic state in directions we want. In other words, we want to be able to instruct the computer to make inferential moves that are not mere tautological entailments. For example, instead of physically handing the computer the whole list of Series winners and nonwinners for 1971, it is obviously cheaper to tell the computer: "the Pirates won; and further, if you've got a winner and a team not identical to it, that team must be a nonwinner" (i.e., $\forall x\forall y(Wx\&x\neq y\rightarrow\sim Wy)$). In the presence of an obviously needed table for identity and distinctness or else in the presence of a convention that different names denote different entities (not a bad convention for *practical* use in many a computer setting), one could then *infer* that "The Orioles won" is to be marked told False.

Your first thought might be that you could get the effect of "given A and B, infer C," or "if A and B, then C," by feeding the computer "$\sim A\vee\sim B\vee C$." But that won't work: the latter formula will tend to *split* the set-up you've got into three, in one of which A is marked told False, etc.; whereas what is wanted is (roughly) just to improve the single set-up you've got by adding told True to C provided that A and B are marked told True (and otherwise to leave things alone). (Connections of this idea with that of Belnap 1970 and 1973 need exploring.) It is (roughly) this idea we want to catch.

§81.4.1. Implicational inputs. Let us introduce "$A \rightarrow B$" as representing the implication of A to B; so what we have is notation in search of a meaning. But we have found in the previous section just the right way of giving meaning to an expression construed as an input: the computer is to improve its epistemic state in the minimum possible way so as to make the expression True. So let us look forward to treating $A \rightarrow B$ as signifying some mapping from states into states such that $A \rightarrow B$ is True in the resultant state.

Obviously if we are to pursue this line, we must know what it is for $A \rightarrow B$ to be True in a state. This is a delicate matter. One definition that suggests itself is making $A \rightarrow B$ True in a state E just in case $\mathrm{E}(A) \leq \mathrm{E}(B)$ (in the logical lattice **L4**); but although we don't have any knock-down arguments against the fruitfulness of this definition, we are pretty sure it is wrong. We think it will be more fruitful to define $A \rightarrow B$ closer to the following: modify *every* set-up you are considering to make $A \rightarrow B$ True in it. So let us first define what it is for $A \rightarrow B$ to be "True in a set-up"; and naturally for this we turn to the logical lattice **L4**; let us specify that $A \rightarrow B$ is True in s just in case $\mathrm{s}(A) \leq \mathrm{s}(B)$ (Note that we do *not* give $A \rightarrow B$ values in **4**; $A \rightarrow B$ is just True or False in s, never both or neither, since not merely "told.")

It might be tempting now to define $A \rightarrow B$ as True in state E if True in every set-up s in E, and False otherwise, but that would be wrong. The reason is that the Truth of $A \rightarrow B$ is *not* closed upward; $\mathrm{s} \sqsubseteq \mathrm{s}'$ and $A \rightarrow B$ True in s do not together guarantee that $A \rightarrow B$ is True in s′. But epistemic states are supposed to be equivalent to their upward closures. The next thing to try is to look just at the *minimal members* M(E) of each state E, i.e., those set-ups in E which are minimal with respect to the approximation ordering between set-ups. For, in any state E in which every set-up is approximated by some minimal set-up, nonminimal set-ups (those not in M(E)) can be thought of as redundant. In particular they do not contribute to the value of any formula and should not contribute to the value of implications. So it would be plausible to define $A \rightarrow B$ as True in a state if it is True in every minimal member. And indeed this will work if E is finite or if every s in E is finite; for then every descending sequence

$$\mathrm{s}_1 \sqsupseteq \ldots \mathrm{s}_i \sqsupseteq \ldots$$

in E is finite, and so in fact M(E) is equivalent to E. Of course for real applications on the computer this will always be so. But let us nevertheless give a definition that will work in the more general case: $A \rightarrow B$ is True in

E if, for every s$\in$E, there is some s$'\in$E such that s$'\sqsubseteq$ s, and $A\to B$ is True in s$'$.

We claim for this definition the merit of passing over equivalent states: given E and E$'$ equivalent, $A\to B$ will have the same truth value in each. The reason that if $A\to B$ is True in the closure of E then it is True in E as well (the hard part) is that there cannot be in the closure of E an infinitely descending chain of set-ups in which the truth value of $A\to B$ changes infinitely often. Sooner or later as you pass down the chain, the truth-value of $A\to B$ will have to stabilize as either True or False. And under the hypothesis that $A\to B$ is True in the closure of E, in each chain it will have to stabilize as True; which is enough to get it True in E. And the reason that there cannot be an infinitely descending chain of set-ups in which the truth value of $A\to B$ flickers is that any such flicker must be caused by a change in either s(A) or s(B); but since this function is monotonic in s, once having changed the value of A or B in the only permitted (downward) direction, one can never change it back up again. So at most the value of A can change twice, and similarly for B; which means that $A\to B$ can change at most four times.

One more note of profound caution: the notion of the Truth of $A\to B$ in E is dramatically different from the notion of A's being told True in E, in that the former is *not* monotonic in E, whereas the latter is: E $\sqsubseteq$ E$'$ guarantees that if A is at least $\boldsymbol{T}$ in E then it is so in E$'$, but does not guarantee that if $A\to B$ is True in E it is so in E$'$. (The falsity of $A\to B$ fares no better.) We shall see later how this influences the computer to manipulate $A\to B$ and A quite differently; now, however, we remark that this fact is not in conflict with Scott's thesis, since we have not got something that can be represented as a function from one approximation lattice into another. In particular, the usual characteristic function representing the set of E in which $A\to B$ is True will not work, since the two truth values True and False do not constitute an approximation lattice.

Now back to our enterprise of defining $A\to B$ in such a way as to make it a mapping from epistemic states E to states E$'$ in such a way as to represent minimum mutilation yielding Truth—in exact analogy with our results in the previous section.

Since we know what it is for $A\to B$ to be true in s, we know that it has a truth set: Tset $(A\to B)$ = {s: $A\to B$ true in s}. So one might try just defining

$$(A\to B)^{+}\mathrm{E} = \mathrm{E} \sqcup \mathrm{Tset}(A\to B)$$

as before. There may be something in the vicinity that works, but this doesn't; since one of the set-ups in which $A\to B$ is True is that in which every atomic formula has ***None***, this $(A\to B)^{+}$ is just the identity function (up to equivalence). It is also worth noting that Tset$(A\to B)$ is not closed upward and so not well-behaved. Nor will it do to try to close it upward—by the remark above, that would yield the family of *all* set-ups. In any event, we take a different—and, we think, intuitively plausible—path to defining $(A\rightarrow B)^+$ as a function that minimally mutilates E to make $A\rightarrow B$ true.

We propose first to define $A\rightarrow B$ on set-ups s, looking forward to the following extension to states:

$$(A\rightarrow B)^+\mathrm{E} = \bigcup\{(A\rightarrow B)^+\mathrm{s}\colon \mathrm{s}\in\mathrm{E}\}$$

So we are up to defining $(A\rightarrow B)^+$ on a set-up s—with the presumption doubtless that the value will be some state E′ (we may have to "split" s). The idea is, as always, that we want to increase the information in s as little as possible so as to make $A\rightarrow B$ true. If we keep firmly in mind that "increase of information" is no mere metaphor, but is relative to an approximation lattice, it turns out that we are guided as if by the hand of the Great Logician.

One case is easy. If $A\rightarrow B$ is *already* True in s, the minimum thing to do is to just leave s alone. Now in order to motivate the definition to come, consider all the ways that $p\rightarrow q$, for example, could be *False* in s. The possibilities, which refer to the *logical* lattice **L4**, are laid out under p and q below (ignore for now the right-hand column).

p	q	Essential to make $p\rightarrow q$ True
T	***None***	Raise q to ***T***
T	***Both***	Raise p to ***Both***
T	***F***	Raise p to ***Both*** and q to ***Both***
None	***F***	Raise p to ***F***
Both	***F***	Raise q to ***Both***

Now try the following. Keep one eye on the logical lattice **L4** (§81.2.2) and the other on the approximation lattice **A4** (§81.2.1) and use your third to verify the claims made in the right-hand column. For example, the first entry says, in effect, that if $p\rightarrow q$ is False because p is ***T*** and q is ***None*** then *it* does no good to raise p (in the approximation lattice **A4**), for the only place to which to raise it is to ***Both***; and (in the logical lattice **L4**) that still doesn't imply q (make $p\rightarrow q$ True). So q must be raised. (An important presupposition of these remarks is that we may speak only of "raising" in the approximation lattice **A4**, never of "lowering"; the computer is never to treat an input as *reducing* its information, never to treat it as a cause to "forget" something. And you will recall that this constraint is a local one, certainly not part of what we think is essential to the Complete Reasoner.)

Next note the following analysis of the table, where s is the current set-up and E′ is the new state. All the raisings of q occur when $\boldsymbol{T} \sqsubseteq \mathrm{s}(p)$, and all the raisings of p occur when $\boldsymbol{F} \sqsubseteq \mathrm{s}(q)$. Further, the raising of q consists in always making $\boldsymbol{T} \sqsubseteq \mathrm{E}'(q)$ and the raising of p consists in making $\boldsymbol{F} \sqsubseteq \mathrm{E}'(p)$. That is, as might have been expected, making $p \rightarrow q$ True consists in making q have at least $\boldsymbol{T}$ when p does and in making p have at least $\boldsymbol{F}$ when q does.

Let us divide the problem (and abandon the special case of atomic formulas). One thing we must do is make B have at least $\boldsymbol{T}$ when A does. Let us call the corresponding statement: $A \rightarrow_{\boldsymbol{T}} B$. We want to make B told True if A is, and in a minimal way. But we *already* know the minimal way of making B told True. So the following definition of $(A \rightarrow_{\boldsymbol{T}} B)^+$ is pretty well forced:

$$\begin{aligned} (A \rightarrow_{\boldsymbol{T}} B)^+\mathrm{s} &= B^+\{\mathrm{s}\} && \text{if } \mathrm{s} \in \mathrm{Tset}(A)\text{; i.e., if } \boldsymbol{T} \sqsubseteq \mathrm{s}(A), \\ &= \{\mathrm{s}\} && \text{if } \mathrm{s} \notin \mathrm{Tset}(A)\text{; i.e., if } \boldsymbol{T} \not\sqsubseteq \mathrm{s}(A). \end{aligned}$$

This account of $(A \rightarrow_{\boldsymbol{T}} B)^+$ matches very well the intuitions that led Ryle 1949 (see §6) to say that "if-then"s are inference tickets. For $(A \rightarrow_{\boldsymbol{T}} B)^+$ is exactly a license to the computer to infer the conclusion whenever it has got the premiss in hand. For example, if it finds that "The Pirates won" is marked $\boldsymbol{T}$, then "The Pirates won $\rightarrow_{\boldsymbol{T}}$ the Orioles didn't" will direct it to make the minimum mutilation that marks "the Orioles didn't" with at least $\boldsymbol{T}$. (Recall from the previous section that B^+ is the minimum mutilation making B at least $\boldsymbol{T}$.)

There is already much food for thought here, and a host of unanswered questions. We do note that Scott's thesis is not violated: $(A \rightarrow_{\boldsymbol{T}} B)^+$ is indeed a continuous function from the space of set-ups to that of states—and, with the previous extension, from states into states. That it is depends on the fact that Tsets are (1) always closed upward and (2) "open": if $\bigsqcup \mathrm{X} \in \mathrm{Tset}(A)$ for directed X, then $\mathrm{x} \in \mathrm{Tset}(A)$ for some $\mathrm{x} \in \mathrm{X}$. (The topological language fits the situation: it means that no point in X can be approached as the limit of a family of points lying entirely outside of X.) The point of this remark is to draw the consequence that we cannot sensibly use $A \rightarrow_{\boldsymbol{T}} B$ in the absence of these conditions; hence, since the Tset for $A \rightarrow_{\boldsymbol{T}} B$ is *not* closed upward, we cannot make sense of $(A \rightarrow_{\boldsymbol{T}} B) \rightarrow_{\boldsymbol{T}} C$. In contrast, all we need from B is the continuity of B^+; so $A \rightarrow_{\boldsymbol{T}} (B \rightarrow_{\boldsymbol{T}} C)$ is acceptable. (Note how the approximation idea and Scott's thesis guide us through the thicket.)

For its intrinsic interest, note that, in the lattice of all ampliative functions, we have

$$((A \rightarrow_{\boldsymbol{T}} B)^+ \circ A^+) \sqsupseteq B^+$$

but not

$$(A^+ \circ (A \rightarrow_{\boldsymbol{T}} B)^+) \sqsupseteq B^+.$$

Maybe this has something to do with some of the nonpermutative logics, and maybe not.

Turning back now to our principle task, the defining of $(A \rightarrow B)^+$, we have completed part of our task by defining $(A \rightarrow_T B)^+$, which makes B true if A is. The other part is by way of the function

$$\begin{aligned}(A \rightarrow_F B)^+ \mathrm{s} &= A^-\{\mathrm{s}\} &&\text{if } \mathrm{s} \in \mathrm{Fset}(B)\text{; i.e., if } \boldsymbol{F} \sqsubseteq \mathrm{s}(B),\\ &= \{\mathrm{s}\} &&\text{if } \{\mathrm{s}\} \notin \mathrm{Fset}(B)\text{, i.e., if } \boldsymbol{F} \not\sqsubseteq \mathrm{s}(B).\end{aligned}$$

This is the function that makes A told False, minimally, if B is.

Before pushing on to define $(A \rightarrow B)^+$, let us pause to note just a thing or two about $(A \rightarrow_T B)^+$. This family of functions has in common with the A^+ that each is ampliative:

$$\boldsymbol{I} \sqsubseteq (A \rightarrow_T B)\text{; i.e., } \mathrm{E} \sqsubseteq (A \rightarrow_T B)^+ \mathrm{E}.$$

In contrast, however, these new functions are *not* "permanent" in the sense defined at the end of Part 2. That means that, once the computer has "done" $(A \rightarrow_T B)^+$, it may have to do it again; this is a consequence of the fact that the truth set of $A \rightarrow B$ is not upward closed: adding new information can falsify $A \rightarrow B$. But there is one property in the vicinity that $(A \rightarrow_T B)^+$ shares with A^+: at least one doesn't have to do it twice in a row:

$$\mathrm{f} \circ \mathrm{f} = \mathrm{f}$$

for $\mathrm{f} = (A \rightarrow_T B)^+$.

Closely related to the permanence-impermanence distinction between the two sorts of ampliative functions is the way they behave under composition: all the truth-functional ampliative functions permute with each other ($A^+ \circ B^+ = B^+ \circ A^+$), but the $\rightarrow_T$ functions permute neither with each other nor with the truth functions. The clearest example of the latter is the inequality:

$$(p^+ \circ (p \rightarrow_T q)^+) \neq ((p \rightarrow_T q)^+ \circ p^+)$$

Applying the right-hand side to an s in which p and q each have ***None*** yields a state in which first p is made told True, and then, as a consequence of this, q is made told True, too. But applying the left-hand side to s does not fare so well: $p \rightarrow_T q$ does no work, since p is not at least $\boldsymbol{T}$ in s; so the outcome is only the marking of p as told True *without* changing q.

By noting that $(A \rightarrow_F B)^+ = ({\sim} B \rightarrow_T {\sim} A)^+$, we can be sure that this function has both the virtues and the shortcomings of $(A \rightarrow_T B)^+$—except that it has the additional shortcoming that not only is $(A \rightarrow_F B) \rightarrow_F C$ impossible, since the falsity set of $(A \rightarrow_F B)^-$ is not closed upward, but so is $A \rightarrow_F (B \rightarrow_F C)$, since $(B \rightarrow_F C)^-$ is not defined. (We can if we like have $A \rightarrow_T (B \rightarrow_F C)$.)

The shortcomings of the arrow functions make us see that we cannot define $(A \rightarrow B)^+$ as simply the composition of $(A \rightarrow_T B)^+$ and $(A \rightarrow_F B)^+$. For

$A\rightarrow B$ might not be True in the result. Intuitively, $(A\rightarrow_F B)^+$ might cause nothing to happen, since B is ***None*** in the set-up s in question; while $(A\rightarrow_T B)^+$ causes B to be marked not only told True (since A is) but told False *as well.* This can happen if B is a formula like $p\&{\sim}p$ which cannot be made told True without being made told False *as well.* Then, if A still has the value *T*, $A\rightarrow B$ will be false. So the composition of $(A\rightarrow_T B)^+$ with $(A\rightarrow_F B)^+$ (in either order) is *not* the minimum increase making $A\rightarrow B$ true in the result. As a special solution to this problem; one finds that $(A\rightarrow_F B)^+\circ(A\rightarrow_T B)^+\circ(A\rightarrow_F B)^+$ works admirably: first make A False if B is; then make B True if A is; then; once more, make A False if B is. Since $A\rightarrow B$ *is* true in the result, one need do nothing else; one has indeed found the minimum. In particular,

$$((A\rightarrow B)^+\circ(A\rightarrow B)^+)\mathrm{s} = (A\rightarrow B)^+\mathrm{s}$$
$$(A\rightarrow B)^+ = (A\rightarrow_T B)^+\circ(A\rightarrow_F B)^+\circ(A\rightarrow_T B)^+.$$

So we take this as a definition of what $A\rightarrow B$ *means* as a mapping of epistemic states into epistemic states.

We conclude this section with two remarks. First, we have offered no *logic* for rules $(A\rightarrow B)^+$; there is just much work to be done.

Second, $A\rightarrow B$ has been construed as a rule and has been given "input" meaning. It has been given no output meaning, and it is not intended that the computer answer questions about it. In particular, we have given no meaning to *denying* $A\rightarrow B$; $(A\rightarrow B)^-$ has not been given a sense. We are not sure whether this is a limitation to be overcome or just a consequence of our presenting $A\rightarrow B$ as a rule; for we do not know what it would mean to tell the computer *not* to use the rule $(A\rightarrow B)^+$. One might try to give sense to $(A\rightarrow B)^-$ by instructing the computer to make $\mathrm{E}(A)\not\subseteq\mathrm{E}(B)$; but this is an instruction that it is not always possible for the computer to carry out. Or the counterexample idea of §49 might work.

§81.4.2. Rules and information states. This last subsection is going to be altogether tentative, and altogether abstract, with just one concrete thought that needs remembering, which we learned from Isner: probably the best way to handle sophisticated information states in a computer is by a judicious *combination* of tables (like our epistemic states) and rules (like our $A\rightarrow B$ or a truth-functional formula that the computer prefers to remember, or a quantificational formula that it must remember). For this reason, as well as for the quite different reason that some rules may have to be used again (are not permanent, must be remembered), we can no longer be satisfied to represent what the computer knows by means of an epistemic state. Rather, this must be represented by a *pair* consisting of an epistemic state *and* a set of rules:

$$\langle \mathrm{R}, \mathrm{E}\rangle.$$

E is supposed to represent what the computer explicitly knows, and is subject to increase by application of the rules in the set R. For many purposes we should suppose that E is finite, but for some not.

Let us dub such a pair an *information state*, just so that we don't have to retract our previous definition of "epistemic state." But what is a rule? Of what is R a set? A good thing to mean by *rule*, or *ampliative* rule, in this context, might be: *any* continuous and ampliative mapping from epistemic states into epistemic states. As we mentioned above, the set of *all* continuous functions from an approximation lattice into itself has been studied by Scott; it forms itself a natural approximation lattice. It is, furthermore, easy to see that the ampliative continuous functions form a natural approximation lattice, and one which is an almost complete sublattice of the space of all the continuous functions: all meets and joins agree, except that the join of the empty set is the identity function I instead of the totally undefined function. Intuitively: the effect of an empty set of rules is to leave the epistemic state the way it was.

So much for the general concept of rule, of which the various functions A^+, A^-, $(A \rightarrow_T B)^+$, $(A \rightarrow_F B)^+$, and $(A \rightarrow B)^+$ are all special cases. We now have to say what a set R of rules *means*. Of course we want to express it as a mapping from epistemic states into epistemic states. Let us begin by saying that a rule ρ is *satisfied in* a state E if applying it to E does not increase information: $\rho(\mathrm{E}) = \mathrm{E}$, and by saying also that a set R of rules *is satisfied in* E if all its members are. Then what we want a set of rules to do is to make the minimum mutilation of E that will render all its members satisfied. Even if R is a unit set, simple application of its member might not work to satisfy it. And even if R is a finite set of rules, each of which is satisfied after its own application, the simple composition of R might not be adequate: All of this can be derived from considerations we adduced in defining $(A \rightarrow B)^+$. But there is a general construction which is bound to work.

Let R be a set of rules. Let $\mathrm{R}\circ$ be the closure of R under composition. This is a directed set; the composition $\mathrm{f}\circ\mathrm{g}$ of f and g will always provide an upper bound for both f and g if they are monotonic and ampliative. Now take the limit: $\bigsqcup \mathrm{R}\circ$.

Claim: take any E and any set R of rules. Then $\bigsqcup \mathrm{R}\circ(\mathrm{E})$ is the minimum mutilation of E in which all rules in the set R are satisfied. (Below we write R(E) for $\bigsqcup \mathrm{R}\circ(\mathrm{E})$.)

In this way we give meaning to the pair consisting of an epistemic state E and a set of rules R. There is that state R(E) consisting of "doing" the rules in all possible ways to E, and it is in regard to this state that we want our questions answered in the presence of E and R. Of course R(E) can be infinitely far off from E. This will certainly happen if the computer is dealing with infinitely many distinct objects and some rule involves universal quantification; so in practice, "I don't know" might have to mean either: "I haven't computed long enough" or "I have positive evidence that I haven't been told."

Because of the importance of computers that maintain both (1) sets of rules and (2) tables (epistemic states), the idea of information states $\langle R, E\rangle$ should be studied in detail. We close this section with just a few idle definitions in the area which might or might not turn out to be fruitful.

When are two states equivalent? There seem to be at least two ideas: $\langle R_1, E_1\rangle$ is *currently equivalent* to $\langle R_2, E_2\rangle$ just in case $R_1(E_1) = R_2(E_2)$; which is to say, they give the same answers to the same questions. And they are *strongly equivalent* if adding the same information to each always produces currently equivalent results: $\langle R_1, E_1 \sqcup E\rangle$ is currently equivalent to $\langle R_2, E_2 \sqcup E\rangle$, for all E. Such information states would answer alike not only all present questions, but also all future questions asked after the addition of the same information to each.

We defined a rule ρ as *satisfied* in an epistemic state E if $\rho(E) = E$. We could similarly define a rule as satisfied in an information state $\langle R, E\rangle$ in one of two ways: *currently satisfied* if satisfied in E, and *ultimately satisfied* if satisfied in R(E). A third notion brings in just the set R: perhaps saying that a rule ρ is *in force* in R might be defined by: $\rho \sqsubseteq R$; i.e., ρ approximates R. But this is *not* a "relevant" idea of being in force; e.g., for each A the rule $(A \rightarrow A)^+$ is in force in every R.

PROBLEM. What is a relevant idea?

§81.4.3. Closure.

Lest it have been lost, let us restate the principal aim of this section: to propose the *usefulness* of the scheme of tautological entailments as a guide to inference in a certain setting, namely, that of a reasoning, question-answering computer threatened with contradictory information. No reader of this book can possibly suppose that Larger Applications have not occurred to us; e.g., application of some of the ideas to a logic of imperatives, or to doxastic logic, or to the development of The One True Logic. But because of our fundamental conviction that logic is occasionally practical, we did not want these possibilities to loom so large as to shut out the light required for dispassionate consideration of our far more modest proposal.

§82. Rescher's hypothetical reasoning: An amended amendment.

Rescher 1964—henceforth HR—proposes a way of reasoning from a set of hypotheses that may include some of our beliefs and also hypotheses contradicting those beliefs. The aim of this section is to point out what we take to be a fault in Rescher's proposal and to suggest a modification of it, using a nonclassical logic, which avoids that fault. We neither attack nor defend the broader aspects of Rescher's proposal, but assume that it is at least *prima facie* worthwhile and therefore worthy of amendment; consequently, we try to tinker as little as possible. In particular, the use of a nonclassical logic which we propose does *not* replace any use by HR of classical logic—in those places where Rescher is classical, we shall be classical, too. The amendment introduces a nonclassical logic at a point where HR uses no logic at all.

It turns out that we have changed our mind since Belnap 1979, from which this section is drawn, as to *which* nonclassical logic is appropriate, thus amending our own amendment. For the sake of clarity we will call attention to this change of mind below.

§82.1. HR-consequence. We begin with a description of Rescher's proposal. Suppose we have a set of hypotheses P constituted by (a) some of our beliefs, and (b) an additional hypothesis that is inconsistent with those beliefs. We may still want to say something about the consequences of P—such is the topic of getting clear on counterfactual conditionals as addressed by HR.

The first of three elements of Rescher's proposal is *modal categorization* of all sentences in our language. A *modal family* M is a list $M(1), \ldots, M(n)$ of nonempty sets of sentences, called *modal categories*, (1) each of which is a proper subset of its successors, (2) each of which contains the classical logical consequences of each of its members (but is not necessarily closed under conjunction), and (3) the last of which contains all sentences. This definition is slightly at variance with HR, p. 46, but not (we think) in any way that makes a difference. If each member of a family is also closed under conjunction, we will speak of a *conjunction-closed modal family*; and we note that all modal categories of such, except $M(n)$, are consistent (on pain of violation of *proper* subsethood—see HR, p. 47).

It is part of the proposal of HR that reasoning from a set of hypotheses P is to be carried out in the context of some modal family M. In application to the belief-contravening hypothesis case, we let $M(1)$ be the hypothesis H together with all its consequences, and then sort our beliefs into the remaining categories $M(2), \ldots, M(n)$ according to how determined we are to hold on to them, where a lower index indicates a higher degree of epistemic (or doxastic) adhesion—the beliefs in the lower-numbered categories are those with which we intend to stick, if we can. This sorting is perhaps the critical notion of HR, and a good deal is said there about the principles on which it might be based. But the amendment we have in mind does not pertain thereto, and accordingly we shall say no more about it.

The second element of Rescher's proposal begins to tell us how to put the hypotheses P together with a modal family M in order to tease out the consequences of P. This is done through the instrumentality of "preferred maximally mutually compatible (PMMC)" subsets of P, relative to M. And these may be defined inductively, by defining PMMC(i) for each i ($1 \leq i \leq n$), assuming that the work has already been done for $i' < i$. Choose a member X

of PMMC($i-1$), or let X be the empty set if $i=1$. If all the members of $P \cap M(i)$ can be consistently (classical sense) added to X, add them, and put the result in PMMC(i). Otherwise, form *each* result of adding to X as many members as possible of $P \cap M(i)$ without getting (classical) inconsistency, and put *each* such result in PMMC(i). All PMMC(i) having been defined, PMMC ("the PMMC subsets of P") is defined as PMMC(n).

If one wants a more set-theoretical definition, it could go like this. PMMC(i) (for $1 \leq i \leq n$) is the set of all sets of sentences S such that there is a set U such that $U \in$ PMMC($i-1$) (or U is the empty set if $i=1$) and (a) U is a subset of S, (b) S is a subset of $P \cap M(i)$, (c) S is classically consistent, and (d) no proper superset S' of S satisfies (a)–(c).

The third and last element of the proposal of HR is to define the consequences of P relative to M as those sentences which are (classical) consequences of *every* member of PMMC. For this notion, when M is understood, we use the notation

$$P \rightarrow A$$

of HR, which we can read as "A is an HR-consequence of P" (relative to M). We note explicitly that the use of the arrow here is intended to remind the reader of HR, not of relevant implication or entailment; see Example 2 below in order to reinforce this point.

It is also convenient to use

$$P \not\rightarrow A$$

for the failure of HR-consequence. Evidently $P \not\rightarrow A$ holds just in case there is some member of some PMMC(i) which contains or (classically) implies $\sim A$.

It is useful to have a transparent notation representing how the hypotheses of a set P fall into modal categories of a fixed family M; to avoid endless subscripts, we introduce it by way of example.

$$(A, B, C \,/\, D, E \,/\, F, G)$$

represents that the sentences to the left of a given slash fall into a narrower ("more fundamental", "more important"—HR, p. 47) modal category than any sentence to the right of the slash, but that sentences unseparated by slashes are themselves modally indistinguishable. (This is related to but distinct from the notation of HR, p. 50.) As a special case we write, for example.

$$(A, B, C, D)$$

(no slashes) to indicate that all members of P are modally indistinguishable. And

$$(\sim A \,/\, B, C \,/\, A)$$

might indicate that we are considering a case in which we believe that A as a "fact," and are wondering what would have happend if instead $\sim A$, in the context of "laws" B and C.

EXAMPLE 1. $(Cvb\ /\ Fb \supset \sim Ib,\ Fv \supset \sim Iv,\ Cvb \supset [(Fb \equiv Fv)\&(Ib \equiv Iv)]\ /\ Fb,\ Iv,\ \sim Cvb) \rightarrow (Fb\&Fv) \vee (Ib\&Iv))$.

This is about Bizet (b) and Verdi (v) of whom HR gives a slightly different account on pp. 67–68: under the hypothesis Cvb that they are compatriots, together with strongly held beliefs about disjointness of the French and Italians and about what necessary conditions for being compatriots are, together with more weakly held beliefs about the nationality of Bizet and Verdi, and that they are not compatriots, we can HR-conclude that they are either both French or both Italian. (We can also conclude that they are either both non-French or both non-Italian; but this is less interesting, since it does not use statements in the weakest modal category.)

The remaining examples are kept wholly unrealistic in order to make certain points in the simplest possible way.

EXAMPLE 2. $(p,\ \sim p \vee q) \rightarrow q$. If P is consistent, its *HR-consequences* (as we shall say) are just its classical consequences.

We can see from Example 2 that HR-consequence is not being treated by us as a competitor to tautological entailment; the interest of the program seems to us to derive entirely from the apparatus of modal categorization and its effect on the PMMCs in the presence of inconsistencies.

EXAMPLE 3. $(p\ /\ \sim p,\ q) \rightarrow p\&q$. HR, p. 53, notes of a similar example that "q is an 'innocent bystander,' not involved in the contradiction at all," and that the modal categorization is irrelevant to getting q (but of course not p). That seems right, and we shall make much of it.

§82.2. Objections. We have an objection to the concept of HR-consequence as described in the preceding section: it is entirely too sensitive to the way in which conjunction figures in the description of our beliefs. This complaint must not be taken too far: some segregation of our premisses is essential for Rescher's program to get under way at all—certainly the belief-contravening hypothesis must be separated out, and certainly the categorization of our beliefs requires segregation—not everything must be inextricable.

But, within categories, Rescher's method gives wildly different accounts depending on just how many ampersands are replaced by commas, or vice versa. It depends too much on how our doxastic subtheory of a certain category is itself separated into sentential bits. The trouble is seen bare in

Example 4. $(p\ /\ {\sim}p\&q) \overset{\sim}{\rightarrow} q$; that is, HR does *not* get the "innocent bystander" q of Example 3 if, in describing the relevant beliefs, one uses an ampersand instead of a comma. That seems to us wrong. Furthermore, consider

Example 5. Let $P = (p, {\sim}p\&q)$, where modal categorization yields $({\sim}p\ /\ p, {\sim}p\&q)$. Here, because ${\sim}p$ is bound up with q in P, its narrower modal categorization cannot on Rescher's account come into play. So P has *no* HR-consequences other than tautologies. But a sensible account should let P yield ${\sim}p$ because of its membership in a more ferocious category—and of course q because of its not participating in the contradiction at all.

So sometimes HR doesn't get consequences that we think it should. But sometimes it gets too many. Consider the following pair.

Example 6. $(p\ /\ q, {\sim}q\&{\sim}p) \rightarrow q$, since one can add q but not ${\sim}q\&{\sim}p$ consistently to p.

Example 7. $(p\ /\ q, {\sim}q, {\sim}p) \overset{\sim}{\rightarrow} q$, since one can add ${\sim}q$ consistently to p, so that at least one PMMC omits having q as a (classical) consequence.

It seems to us that Example 6 gets q only "deviously," because its negation ${\sim}q$ "happens" to be tied to ${\sim}p$. Example 7 seems to us right.

Here we were looking mostly at examples in which A, B, and $A\&B$ were all modally indistinguishable. We do *not* mean to imply that we can always settle the consequence question for $A\&B$ as a hypothesis in a certain context by looking at the question for A and B separately in that same context; for one or both of A and B might be in a narrower category than $A\&B$. But *if* $A\&B$, A, and B are modally indistinguishable, it seems a hard saying that the consequence question for $A\&B$ should be different from that for A and B separately.

Since different ways of articulating our beliefs (of a single modal category) give different results under Rescher's proposal and since we do not want this, evidently we have to have some views about which articulations we most want to reflect.

Policy: try to reflect *maximum* articulation. We note that this is a policy and not a whim. For the opposite policy—the agglutinative policy—gives entirely too few interesting results in central cases. Consider the very central case in which some finite P is inconsistent. Then, if we represent P by a single sentence, the conjunction of its members, evidently we will have no HR-consequences beyond tautologies. In contrast, if we maximally articulate P, we may be able to isolate the effect of its contradiction, adding the consistent bits and obtaining something entertaining. Or, which seems just as important, we may be able to block a consequence by freeing for use some conjunct of a conjunction which is itself not consistently available, as in Example 6–7.

§82.3. Candidate amendments. So much for complaints. Our aim is to modify HR minimally so as to avoid them. Our strategy is to amend the definition of HR-consequence at only one place. We are going to keep the first element, the apparatus of modal categorization, untouched. We shall also retain the third element, the account of consequence in terms of PMMC: *A* is to be a consequence of *P*, relative to *M*, just in case it is a (classical) consequence of every PMMC.

Further, we are going to keep the outline of the second element, the definition of PMMC. We change it at only one place. Rescher considers the addition, at the *i*th stage, of only formulas in *M*(*i*); good. But he also allows only the addition of formulas that are actually in *P*. This is what we suggest changing. We suggest allowing also the addition of formulas in a larger set, *P**, which can be thought of as the *articulation* of *P*, the freeing of its contents from such notational bondage as they might have in *P*. All this is to be done before the application of the device of modal categorization to get PMMC.

In what follows we shall experiment with various possible articulations *P**. In all cases, please help lessen the pains of repetition by picturing the definition of PMMC in §82.2 as containing "*P**" wherever "*P*" occurs. (Hence, the Rescher proposal can be described in these new terms by simply identifying *P** with *P*.)

§82.3.1. The first thing one might try is to define *P** as the closure of *P* under classical consequence, but this is ridiculous; for typically *P* is inconsistent, so *P** would contain every sentence. It follows that the (amended) HR-consequences of *P* would be determined entirely by the modal family *M* and be wholly independent of *P* itself! In short, we would be giving up all of Rescher's gains. So much for classical consequence.

§82.3.2. The second thing one might try is to define *P** as the closure of *P* under *relevant* consequence, in the sense of the concept of "tautological entailment" of §15 or its generalization to quantifiers as in §40. Please notice that it won't do to count on some kind of relevant idea of entailment to do all the work. For it is quite essential, we should say, that in Rescherian consideration of belief-contravening hypotheses we give *classical consistency* its proper role, not letting in any inconsistent consequences. But at the level at which we are working, it is not unfair to say that relevant entailment just doesn't care about contradictions at all: $(p, \sim p, q)$ relevantly implies $p\&\sim p$ as well as q.

So the idea is to use a judicious combination of relevance notions and classical notions. First use relevant implication to articulate our hypotheses

P; i.e., define P^* as the collection of all relevant consequences of *P*. Then use modal categorization and plain old classical logic to tease out its (amended) HR-consequences. Since contradictions do not relevantly imply everything, we can at least be sure that this proposal does not have the same defect as that of the first thing we tried.

The proposal gets some examples right. We ignore its virtues, however, because in other cases it gives results that deviate not only from HR-consequence, but from what we think is correct. Consider

EXAMPLE 8. $(p\ /\ {\sim}p, q)$ does not on this proposal yield *q*, although as indicated in our remark on Example 3, we agree with Rescher that this *P* should give the "innocent bystander" *q*. The reason it does not is that the implication from *A* to $A \vee B$ is relevantly OK, and so P^* will contain ${\sim}\mathrm{p} \vee {\sim}q$. Since ${\sim}p \vee {\sim}q$ must be in every modal category containing ${\sim}p$, it certainly does not have a weaker modal standing than *q*. So in its turn it will form with *p* the basis of a member of PMMC—which, since it is consistent and has ${\sim}q$ as a classical consequence, cannot have *q* as a consequence.

§82.3.3. For a while, after discovering this, we fooled around with some related proposals which paid attention to the fact that ${\sim}p \vee {\sim}q$ "threatens" contradiction when put with *p* in a way that *q* does not—sense can be made out of this by looking at the four-valued representation of the set $(p, {\sim}p \vee {\sim}q)$ according to the pattern of §81. But although there may be something in the vicinity, as conjectured in Belnap 1977b, p. 50, we do not now know what it is. Instead we think that the trouble lies deeper, and that in fact it is to be found in too free use of the principle of "disjunction introduction," as Fitch 1952 labels the inference from *A* (or *B*) to $A \vee B$.

It is not that we have started thinking that the consequence from *A* to $A \vee B$ is somehow doubtful. But we are not speaking of a matter of consequence; instead, we are searching for principles for articulating sets of hypotheses, and we already know that such principles may be far weaker than consequence.

In any event, consideration of Example 8 makes it plausible to suggest that we replace relevance logic, in its role of defining the set P^* that articulates *P*, by some logic that in a natural way bars disjunction introduction. And there is such a logic: the logic of "analytic implication" of Parry 1933. (See §29.6.)

The idea behind Parry's system is that *A* shall not analytically imply *B* unless every variable occurring in *B* "already" occurs in *A*—so that, in this sense, *B* does not "enlarge the content" of *A*. Of course the inference from *A* to $A \vee B$ fails this test.

But it turns out that, although we may be on the right track, Parry's own system is not enough help. For Parry wishes to maximize the implicates of

A relative to the above idea of analytic implication and, hence, allows the inference from $\sim p$ and q to $\sim p \vee \sim q$—note that all the variables of the conclusion do indeed lie among those already in the premisses. And since this inference is allowed, if we define P^* as the closure of P under Parry's analytic implication, we won't get q from $(p \,/\, \sim p, q)$, since q will be missing from among the consequences of every consistent extension of the set $(p, \sim p \vee \sim q)$, one of which, at least, will be in PMMC—exactly as for Example 8.

The upshot is that, for our purposes, analytic implication is No Good.

§82.3.4 So relevance logic and analytic implication are too strong to give satisfying results in defining P^*. The weakest solution to the problems so far discussed is just to let P^* be the closure of P under "conjunction elimination" (§23.1), the inference from $A\&B$ to A (or B). But this is too weak. At the very least we must allow dissolution of conjunctions inside of disjunctions, as in the following example, which merely adds r as a hypothesis and then uniformly disjoins $\sim r$ to the elements of Example 4.

EXAMPLE 9. $(r, \sim r \vee p \,/\, \sim r \vee (\sim p \& q))$ does not yield q either as an HR-consequence or when P^* is defined as the closure of P under conjunction elimination. But it should; just as in Example 4, q is an "innocent bystander," which becomes apparent if we put $\sim r \vee q$ in P^* because $\sim r \vee (\sim p \& q)$ is.

Further, any of our other examples can be modified in a parallel routine way to make the same point: if we buy into the principle of dissolution of conjunctions at all, we need it as well for conjunctions lying under disjunctions.

Evidently there are other ways in which conjunctions can be hidden. If we think of our notation restricted to conjunction, disjunction, and negation, conjunctions can lie under double negations as well or be concealed as denied disjunctions. And the disjunctions under which conjunctions might lie might themselves be hidden or concealed, so that we should be adding further principles of articulation; but we postpone this for a paragraph.

What about "conjunction introduction" (§23.1), the principle that gets $A\&B$ from A and B? Should P^* be closed under conjunction introduction? It does not matter in a direct way, since at any stage of the formulation of PMMC at which $A\&B$ could be added, A and B (which must be in any modal category containing $A\&B$ and which must together be consistent with any set with which $A\&B$ is consistent) could be added instead; evidently the classical consequences of a set with $A\&B$ are exactly the same as the set with A and B instead of $A\&B$. But, on the one hand, it does keep our thinking straight to have P^* closed under conjunction introduction, since it reinforces the doctrine that it is irrelevant whether our hypotheses are articulated with conjunctions or with commas; and, on the other, it allows us to state further principles of articulation, needed for hidden conjunctions and the like, in a somewhat briefer manner than would otherwise be possible.

The next (and penultimate) candidate suggestion is that, in addition to the principles of conjunction elimination and introduction, we should use as our standard of articulation just the equivalence principles sanctioned by a new logic, one that is stricter than either relevance logic or Parry's analytic implication: the logic of *analytic containment* of Angell 1976, 1989. We describe it by reference to the following *analytic equivalence* principles:

1 $A\&B \rightleftarrows B\&A$
2 $(A\&B)\&C \rightleftarrows A\&(B\&C)$
3 $A\vee(B\&C) \rightleftarrows (A\vee B)\&(A\vee C)$
4 $\sim\sim A \rightleftarrows A$
5 $\sim(A\vee B) \rightleftarrows \sim A\&\sim B$
6 $(A\&A) \rightleftarrows A$

In addition, we suppose that these analytic equivalences are closed under *transitivity*, *symmetry*, and *replacement* (if A and A' are equivalent, so are $\ldots A \ldots$ and $\ldots A' \ldots$). Observe that it is easy to add the following duals by taking a detour through negation:

7 $A\vee B \rightleftarrows B\vee A$
8 $(A\vee B)\vee C \rightleftarrows A\vee(B\vee C)$
9 $A\&(B\vee C) \rightleftarrows (A\&B)\vee(A\&C)$
10 $\sim(A\&B) \rightleftarrows \sim A\vee\sim B$
11 $(A\vee A) \rightleftarrows A$

In the present context, these are to be used to generate closure conditions on P^* in the following straightforward way: if $(\ldots A \ldots)$ is in P^*, then so is $(\ldots A' \ldots)$ if A is equivalent to A' by any of the above principles; and of course we are still supposing that P^* is closed under conjunction elimination and introduction.

Let us say just a few words about Angell's system. Angell sharply distinguishes the concept of containment from deducibility and sets out only to formalize the former: A is said to *analytically contain* B if A is analytically equivalent to $A\&B$. Angell accepts the Parry intuitions for containment: A does not contain $A\vee B$. But he goes further, suggesting that it is not enough, as it is for Parry, to have B's variables occur in A. It must furthermore be the case that variables occurring in B positively also occur in A positively and that those occurring in B negatively also occur in A negatively. This immediately rules out the Parry-acceptable (and relevance-acceptable) inference from $\sim p$ and q to $\sim p\vee\sim q$, since q occurs negatively in the consequence but not in the hypotheses. In this way the problem of Example 8 is avoided. Positively put: if P^* is defined as suggested, then $(p\ /\ \sim p, q) \rightarrow q$, just as in Example 3. Indeed, using the sharp normal form theorem of Angell 1977, 1978, we can be sure that P^* contains no formula with a negative occurrence of q, so that q must be consistently addable to every member of each PMMC(i), hence in every member of PMMC.

One equivalence deducible from the above is

$$12 \qquad A\&(B\vee C) \rightleftarrows A\&(B\vee C)\&(A\vee C),$$

by means of which we are led to:

EXAMPLE 10. $(p\ /\ {\sim}p, q, r\vee{\sim}q) \not\rightrightarrows q$ when P^* is defined as suggested via analytic containment. (Compare Examples 3 and 8.) Reason: ${\sim}p$ conspires with $r\vee{\sim}q$ to put ${\sim}p\vee{\sim}q$ in P^*, via the above equivalence, and the rest of the reasoning is as in Example 8. This is in definite contrast to HR-consequence, which continues to get q even when $r\vee{\sim}q$ is added, as above, to the hypotheses of Example 3. So if a case is to be made against this suggestion, it could be based on this example.

There is a subtle question here, on which we have shifted views. Belnap 1979 was inclined to think that adding the hypothesis $r\vee{\sim}q$, in which q has a negative occurrence, is enough to render q no longer a bystander of shining innocence; and accordingly Belnap 1979 remained with Angell's analytic containment as the standard of articulation of our beliefs, in spite of Example 10. But, in the meantime, reflection on this example and, in particular, meditation on the curious way in which Angell's principles act so as to bar the production of q, have led us to suggest that P^* should be defined in terms of an even stricter standard than that provided by analytic containment in the sense of Angell.

§82.4. Conjunctive containment. In order to come up with a truly stable suggestion, let us reconsider the matter almost from the beginning. What is wanted is a definition of P^*, where P^* is supposed to be the "articulation" of P (see the beginning of §82.3). "Articulation" immediately suggests "conjunction elimination," except that, as we noted, conjunctions can be buried or even concealed as denied disjunctions, etc. But this immediately suggests an absolutely straightforward account of P^*, an account without detours. First follow the idea of §22.1.1 by defining a *positive* or *consequent* part of a formula (written in &, ∨, and ~) as one lying under an even number of negations, and a *negative* or *antecedent* part as one lying under an odd number of negations. Then define P^* as the smallest superset of P that contains both $\ldots B \ldots$ and $\ldots C \ldots$ whenever it contains either $\ldots (B\&C) \ldots$ with the conjunction as a positive part, or $\ldots (B\vee C) \ldots$ with the disjunction as a negative part. If quantifiers are present, one wants also to instantiate in all possible ways each $\forall xB$ as a positive part and each $\exists xB$ as a negative part. Let us call this the *strictest conjunctive closure* P^* of P. Strictest conjunctive closure clearly has the following properties, where we use "A^*" for $\{A\}^*$.

CONJUNCTIVE CLOSURE FACTS. $*$ is a closure operation: $\varnothing^* = \varnothing$; $P \subseteq P^*$; $P \subseteq Q$ implies $P^* \subseteq Q^*$; $P^{**} = P^*$. Furthermore, $*$ distributes over union: $(P \cup Q)^* = (P^* \cup Q^*)$; accordingly, $A \in P^*$ just in case $A^* \subseteq P^*$, and $A^* \subseteq (P^* \cup Q^*)$ just in case $A^* \subseteq P^*$ or $A^* \subseteq Q^*$.

Our view is that P^* so defined is just right: neither too small, like P itself, nor too big, like the preceding proposed P^*s. In the first place, note that this P^* will get Example 10 right, like HR, instead of wrong, like the Angell-induced P^*; for, clearly, for this example, since P contains *no* conjunctions as positive parts or disjunctions as negative parts, P^* will just be P itself. Agreement with HR is thereby assured: we are bound to obtain the "innocent bystander" q from the P of Example 10, as desired. In the second place, this P^* will get Example 9 right as well; for the suggestion is *precisely* a generalization of "putting $\sim r \vee q$ in P^* because $\sim r \vee (\sim p \& q)$ is," which is all that our analysis of Example 9 required. But, beyond these two examples and others, we think that the idea of strictest conjunction closure is so closely bound up with our intuitions about what P^* should be that we predict that no one will find examples that would lead us to change our minds yet once again. It particularly gives us confidence that the idea extends in such a uniquely determined way to the quantifiers.

Working back this time from the closure principle to a relation, we may say that a *set P conjunctively contains* a formula A *in the strictest sense* if A belongs to P^*. And we may say that a *formula A* conjunctively contains a formula B in the strictest sense just in case $\{A\}$ conjunctively contains B in the strictest sense. Conjunctive containment in the strictest sense is clearly reflexive and transitive—and doesn't very often hold; $A\&(B\&C)$, for example, does not conjunctively contain $A\&B$—in the strictest sense.

There are a number of containment relations that are less strict than conjunctive containment in the strictest sense or, equivalently, there are closure operations less minimal than strictest conjunctive closure—which would nevertheless give us the same results. For example, we could without making any difference whatsoever, close P also under commutativity of conjunction, or disjunction; or under double negation. Let us be clear what we mean by "the same results." In §82.1 we defined "the consequences of P relative to a modal family M" in terms of PMMC, the preferred maximal mutually compatible subsets of P^* (the substitution of P^* for P was made in §82.3). Accordingly, in the present context, we will say that another closure operation ! *equivalently extends* * provided that, for all P and modal families M, $P!$ delivers exactly the same consequences as P^* on this definition, and provided furthermore that $P!^* = P!$.

We leave as an open question the existence and determination of a *strongest* closure operation equivalently extending *. In the meantime, the following seems a fine candidate for a strong closure operation equivalently extending *: let $A \in P!$ just in case every member of A^* is classically equivalent to a conjunction of members of P^*. We offer ! as our current candidate for *conjunctive closure* (but not in the strictest sense). In other words, we are enlarging P^* in two ways. First, we are adding all conjunctions of its members; we mentioned below Example 9 that this addition cannot create a nonequivalence, even though it is certainly not in the spirit of pure conjunctive *dissolution*. Second, we are adding formulas classically equivalent to those we already have (*that* could not possibly create a nonequivalence), but only provided that the addition does not lead by closure under * outside of those we already have (up to classical equivalence). For example, $\{p\}!$ will contain $p \vee p$; but it will not contain $p \vee (q \& p)$; for, although the latter is classically equivalent to the member p of $\{p\}^*$, there are members such as $p \vee q$ of $(p \vee (q \& p))^*$ which are not classically equivalent to any conjunction of members of $\{p\}^* = \{p\}$.

Accordingly, our current candidate for "conjunctive containment" (but not in the strictest sense) is this: P *conjunctively contains* A just in case $A \in P!$. And, if it turns out that ! is in fact the strongest closure operation equivalently extending *, then we propose to drop the qualifier "current" and offer this as our final candidate for "conjunctive containment," and ! for "conjunctive closure."

Which of the analytic containments 1–11 of Angell survive as conjunctive containments? Obviously all entries 1–6 of Angell's original list are all right, in both directions; and so are 7, 8, and 10. Also, 9 is all right from right to left; but of course its failure from left to right is (a) obvious, and (b) precisely what is needed to obtain the right result on Example 10, as can plainly be seen from our discussion thereof. 11 is acceptable from left to right. Somewhat surprisingly, however, 11 fails from right to left: A does not contain $A \vee A$. The counterexample does not come when A is some variable p, but instead when A is a conjunction, say $p \& q$; for $(p \& q \vee p \& q)^*$ has a member, say $p \vee q$, that is not classically equivalent to any member of $(p \& q)^* = \{p \& q, p, q\}$.

One might suppose we are on the track here of a proof that ! is not the *strongest* closure operation equivalently extending *; one might suppose that one would have an even stronger such operation, say †, by adding as a new closure condition that $A \vee B$ should be in $P^\dagger$ whenever both A and B are in $P^\dagger$ (this would guarantee the "containment" of $A \vee A$ in A). But in fact the so-defined closure operation † would *not* be equivalent to $*$; the following discriminates between them.

EXAMPLE 11. Let P be $\{p, q, \sim p, \sim q\}$, and let the modal family M in question put $p \vee q$ (and accordingly its classical consequences) in the most ferocious category M_1, with every other formula being in the weaker category M_2; so the picture of $P \cup \{p \vee q\}$ is $(p \vee q \,/\, p, q, \sim p, \sim q)$. If one uses $*$ or ! or indeed the HR-principle itself then P does *not* yield $p \vee q$ (that is, $P \not\rightarrow p \vee q$); for nothing in M_1 will be in the closure of P used to make entries into PMMC, and so the entire matter will be settled by M_2—which certainly does not favor $p \vee q$ over $\sim p \& \sim q$. On the other hand, if one used $P^\dagger$ or the Angell closure (which contains it) then, because $p \vee q$ would be in the closure $P^\dagger$ of P used in the definition of PMMC, $p \vee q$ would in fact be in every member of PMMC and one would therefore have $P \rightarrow p \vee q$.

This example shows that in fact † is not equivalent to $*$ or to !. So much for the facts. We may yet wonder whether we *ought* to add a principle putting $A \vee B$ in the closure of P (a principle that is used in computing the consequences of P according to the HR plan) whenever that closure contains both A and B. In considering this matter, we have not been able to find an example more decisive than Example 11; and we certainly do not think that intuitions on that example run very deep. But if we keep in mind that we are looking for principles of containment rather than principles of inference and that we know from previous examples that we definitely do not want to say either that A contains $A \vee B$ or that B contains $A \vee B$ or even that $\{A, \sim B\}$ (or $\{\sim A, B\}$) contains $A \vee B$, then it seems not so very difficult to deny that A and B together also fail to contain $A \vee B$—and, accordingly, that A fails to contain $A \vee A$.

As a final note, we remark on two inelegant features of our concept of conjunctive containment. In the first place, it is not closed under uniform substitution: p conjunctively contains $p \vee p$, but the statement fails if $p \& q$ is substituted uniformly for p. In the second place, mutual conjunctive containment does not survive as a replacement principle: with reference to the list of Angell principles displayed between Examples 9 and 10, numbers 1–6 are verified, but numbers 9 and 11, which arise therefrom by replacement (and transitivity), are not. Both of these inelegancies seem to us to be essential concomitants of the conceptual analysis on which the enterprise is founded.

§83. Relevance logic in computer science (by Stuart C. Shapiro). Artificial Intelligence (AI) is the branch of computer science that uses computational methods to study the kinds of processing that make up human intelligence. One means of pursuing this study is by building computer models (i.e., writing computer programs) that perform intellectual tasks, but recently more and more AI researchers have become concerned with the logical foundations of such processes. It is not surprising, then, that a group of AI researchers have been attracted to relevance logic as an appropriate foundation for human and computer reasoning systems.

We can categorize the uses of relevance logic that have been suggested in the AI literature in two groups: those which have made use of, or modified, **R**'s proof theory to design AI reasoning systems; those which have stressed the four-valued semantics of **R**.

§83.1. Use of the proof theory. One of the first suggestions that **R** would be useful for Artificial Intelligence reasoning systems was by Shapiro and Wand 1976. Their first point is that, "In a question-answering system, an implication has imperative as well as declarative content: an implication ought to be a useful inference rule" (Shapiro and Wand 1976 p. 8; see also Hewitt 1972). In this view, an implication, such as $A \rightarrow B$, is also treated as a rule that says, "If you want to know the truth of B, check the truth of A." If A is irrelevant to B (the worst case being that A is a contradiction), this is not a reasonable rule.

Shapiro and Wand modify the notation of $\mathbf{FR}_{\rightarrow\&}$ (§27.2) to eliminate the subproof structure. They suggest that the knowledge base (KB) of a reasoning system be considered to contain "assertions of the form $\langle A, \varphi, \alpha\rangle$, where A is some formula, $\varphi \in \{0, 1\}$, and α is a set." (The angle brackets were not in the original.) The rules of inference they present are:

hyp. $\langle A, 0, \{k\}\rangle$ may be added to the KB as long as $\{k\}$ is a singleton set such that no assertion of the form $\langle B, 0, \{k\}\rangle$ is already in KB.

add. $\langle A, 0, \{k\}\rangle$ may be removed from KB and replaced by $\langle A\&B, 0, \{k\}\rangle$.

→E. If $\langle A, \varphi, \alpha\rangle \in$ KB and $\langle A\rightarrow B, \xi, \beta\rangle \in$ KB, then $\langle B, 1, \alpha\cup\beta\rangle$ may be added to KB.

→I. If $\langle A, 0, \{k\}\rangle \in$ KB and $\langle B, \varphi, \beta\rangle \in$ KB and $k \in \beta$, then $\langle A\rightarrow B, 1, \beta-\{k\}\rangle$ may be added to KB.

&E. If $\langle A\&B, \varphi, \alpha\rangle \in$ KB, then $\langle A, 1, \alpha\rangle$ may be added to KB and $\langle B, 1, \alpha\rangle$ may be added to KB.

&I. If $\langle A, \varphi, \alpha\rangle \in$ KB and $\langle B, \xi, \alpha\rangle \in$ KB, then $\langle A\&B, 1, \alpha\rangle$ may be added to KB.

Using the later terminology of Martins and Shapiro (see §83.1.1 below), we may refer to φ as the *origin tag* and α as the *origin set* of the assertion. All assertions whose origin tags are 0 are hypotheses entered into the KB by some user. All assertions of the form $\langle A, 1, \alpha\rangle$ are derived assertions which have been derived under the set of assumptions $\{\langle B, 0, \{k\}\rangle \mid k \in \alpha\}$.

Shapiro and Wand discuss the use of their system for using hypothetical reasoning to derive new rules:

> Consider a universe of discourse, α, and the new, hypothetical world produced by assuming $\langle P, 0, \{p\}\rangle$. If, in this hypothetical world, we can derive $\langle Q, 1, \alpha\cup\{p\}\rangle$, we can then derive the new deduction rule $\langle P\rightarrow Q, 1, \alpha\rangle$ in the original universe by use of $\rightarrow$I. This is a productive rule in the sense that if we later learn that $\langle P, \varphi, \beta\rangle$ is true, we can derive $\langle Q, 1, \alpha\cup\beta\rangle$. . . . The rules of $\mathbf{FR}_{\rightarrow\&}$ are precisely the right ones to ensure that any derived [rules] are in fact relevant to the hypothetical situation. (pp. 16–17)

Shapiro and Wand also use origin sets to define the notion of a *context*: "A context is a set γ and is said to contain the set of assertions $\{\langle A, \varphi, \alpha\rangle \mid \alpha \subseteq \gamma\}$" (p. 15). They point out, in the light of suggestions made by Shapiro 1971 (pp. 107–109), that the rules of $\mathbf{FR}_{\rightarrow\&}$ can be used to keep contradictory contexts separate and that origin sets can be used to discover and remove the source of contradictions if any arise during reasoning. These ideas were subsequently key ideas in Belief Revision systems and Assumption-based Truth Maintenance systems (see below).

§83.1.1. SWM. The work of Shapiro and Wand 1976 was continued by Martins and Shapiro, whose work is described in a series of papers (Martins 1983; Martins and Shapiro 1981, 1983, 1984, 1986a, 1986b, 1986c, 198+, 1988; see also Martins 1987). The logic developed by Martins and Shapiro, called SWM, operates on *supported wffs*, which are expanded versions of the assertion triples of Shapiro and Wand, and which we shall here refer to as *assertions*. An SWM assertion $\mathfrak{A}$ is a quadruple, $\langle A, \tau, \alpha, \rho\rangle$, where A is a wff called the wff of $\mathfrak{A}$, τ is a member of the set {hyp, der, ext}, and is called the *origin tag* (OT) of $\mathfrak{A}$, α is a set of wffs and is called the *origin set* (OS) of $\mathfrak{A}$, and ρ is a set of sets of wffs and is called the *restriction set* (RS) of $\mathfrak{A}$. If $\mathfrak{A}$ is the assertion $\langle A, \tau, \alpha, \rho\rangle$, the functions wff, ot, os, and rs are defined so that $\mathrm{wff}(\mathfrak{A}) = A$, $\mathrm{ot}(\mathfrak{A}) = \tau$, $\mathrm{os}(\mathfrak{A}) = \alpha$, and $\mathrm{rs}(\mathfrak{A}) = \rho$. (The notation in most of Martins and Shapiro's papers differs slightly from that given here.)

A set of hypotheses, α, is known to be inconsistent as soon as an assertion is derived whose wff is a contradiction and whose os is α, or as soon as two assertions, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, are derived for which $\mathrm{wff}(\mathfrak{A}_1) = {\sim}\mathrm{wff}(\mathfrak{A}_2)$ and $\alpha = \mathrm{os}(\mathfrak{A}_1)\cup\mathrm{os}(\mathfrak{A}_2)$. The rules of inference of SWM guarantee that for every derived assertion $\mathfrak{A}$, $\mathrm{os}(\mathfrak{A})$ contains every hypothesis wff that was used in $\mathfrak{A}$'s derivation and only those hypothesis wffs and that $\mathrm{rs}(\mathfrak{A})$ contains every set of hypothesis wffs known to be inconsistent with $\mathrm{os}(\mathfrak{A})$. The rules of inference do not allow the derivation of any assertion $\mathfrak{A}$ for which $\mathrm{os}(\mathfrak{A})$ would be a set of hypothesis wffs already known to be inconsistent.

§83.1.1.1. Rules of inference of SWM. To make the rules of inference of SWM easier to state, several functions are defined.

First, to prevent any use of a context already known to be inconsistent, the rules require all parent assertions to be *combinable*, as defined by:

$$\text{Combine}\,(\mathfrak{A}_1, \mathfrak{A}_2) = \forall r \in \text{rs}(\mathfrak{A}_1)\text{: } r \not\subseteq \text{os}(\mathfrak{A}_2) \text{ and } \forall r \in \text{rs}(\mathfrak{A}_2)\text{: } r \not\subseteq \text{os}(\mathfrak{A}_1)$$

The OT "hyp" tags assertions that are hypotheses; "der" tags assertions that are normal derived assertions; "ext" tags derived assertions whose later use is restricted. To prevent irrelevancies from arising, the rule of And Introduction must be restricted to parent assertions with the same OS. However, if $\mathfrak{A}_1 = \langle A, t_1, o_1, r_1\rangle$ and $\mathfrak{A}_2 = \langle B, t_2, o_2, r_2\rangle$ are two assertions, it seems intuitively unobjectionable for a reasoner to assert $\mathfrak{A}_3 = \langle A\&B, t_3, o_1 \cup o_2, r_3\rangle$. There is, in fact, nothing wrong with this as long as certain rules are prevented from acting on $\mathfrak{A}_3$ or any of its descendants. For this reason, $\mathfrak{A}_3$ and all its descendants are given an OT of "ext". The function Λ correctly computes OTs:

$$\Lambda(a, b) = \begin{cases} \text{ext} & \text{if } a = \text{ext} \quad \text{or} \quad b = \text{ext} \\ \text{der} & \text{otherwise} \end{cases}$$

The final four functions are used in the computation of RSs to ensure that no two sets in an RS overlap and that all are disjoint with the OS.

$$\begin{aligned} &\psi(R, O) = \{\alpha \mid (\alpha \in R \text{ and } \alpha \cap O = \emptyset) \\ &\qquad \text{or } (\exists \beta \in R)[\beta \cap O \neq \emptyset \text{ and } \alpha = \beta - O]\} \\ &\sigma(R) = \{\alpha \in R \mid {\sim}(\exists\beta)(\beta \neq \alpha \text{ and } \beta \in R \text{ and } \beta \subset \alpha)\} \\ &\mu(\{r_1, \ldots, r_m\}, \{o_1, \ldots, o_n\}) = \sigma(\psi(r_1 \cup \cdots \cup r_m, o_1 \cup \cdots \cup o_n)) \\ &\int(O) = \mu(\{r \mid \exists H \in O\text{: } r = \text{rs}(H)\}, \{o \mid \exists H \in O\text{: } o = \text{os}(H)\}) \end{aligned}$$

Given these functions the rules of inference of **SWM** are:

Hypothesis (hyp): For any wff A and sets of wffs $R_1, \ldots, R_n$ $(n \geq 0)$, such that $\forall r \in \{R_1, \ldots, R_n\}\text{: } r \cap \{A\} = \emptyset$ and $\forall r,s \in \{R_1, \ldots, R_n\}\text{: } r \not\subset s$, we may add the assertion $\langle A, \text{hyp}, \{A\}, \{R_1, \ldots, R_n\}\rangle$ to the knowledge base, provided that A has not already been introduced as a hypothesis.

Negation Introduction $({\sim}\text{I})$

$$\begin{array}{rl} \text{From} & \langle A, t_1, o, r\rangle, \\ & \langle {\sim}A, t_2, o, r\rangle, \\ \text{and} & \{H_1, \ldots, H_n\} \subset o, \\ \text{infer} & \langle {\sim}(H_1 \& \ldots \& H_n), \Lambda(t_1, t_2), o - \{H_1, \ldots, H_n\}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \int(o - \{H_1, \ldots, H_n\})\rangle. \end{array}$$

$$\begin{array}{rl} \text{From} & \mathfrak{A}_1 = \langle A, t_1, o_1, r_1\rangle, \\ & \mathfrak{A}_2 = \langle {\sim}A, t_2, o_2, r_2\rangle, \\ & o_1 \neq o_2, \\ & \text{Combine}(\mathfrak{A}_1, \mathfrak{A}_2), \\ \text{and} & \{H_1, \ldots, H_n\} \subset (o_1 \cup o_2), \end{array}$$

infer $\langle {\sim}(H_1 \& \cdots \& H_n), \text{ext}, (o_1 \cup o_2) - \{H_1, \ldots, H_n\},$
$\int((o_1 \cup o_2) - \{H_1, \ldots, H_n\})\rangle$.

This rule may be applied before URS (see below).

Negation Elimination (${\sim}$E): From $\langle {\sim}{\sim}A, t, o, r\rangle$, infer $\langle A, \Lambda(t, t), o, r\rangle$.

And Introduction (&I)
From $\langle A, t_1, o, r\rangle$
and $\langle B, t_2, o, r\rangle$,
infer $\langle A \& B, \Lambda(t_1, t_2), o, r\rangle$.
From $\mathfrak{A}_1 = \langle A, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle B, t_2, o_2, r_2\rangle$,
$o_1 \neq o_2$,
and Combine $(\mathfrak{A}_1, \mathfrak{A}_2)$,
infer $\langle A \& B, \text{ext}, o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.

And Elimination (&E)
From $\langle A \& B, t, o, r\rangle$,
and $t \neq \text{ext}$,
infer either $\langle A, \text{der}, o, r\rangle$
or $\langle B, \text{der}, o, r\rangle$ or both.

Or Introduction (*truth-functional*) ($\vee$I)
From $\langle A, t, o, r\rangle$,
infer either $\langle A \vee B, \Lambda(t, t), o, r\rangle$
or $\langle B \vee A, \Lambda(t, t), o, r\rangle$, for any wff B.

Or Introduction (*intensional*) ($\oplus$I)
From $\langle {\sim}A \rightarrow B, t_1, o, r\rangle$
and $\langle {\sim}B \rightarrow A, t_2, o, r\rangle$,
infer $\langle A \oplus B, \Lambda(t_1, t_2), o, r\rangle$.

Or Elimination ($\oplus$E)
a. From $\mathfrak{A}_1 = \langle A \oplus B, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle {\sim}A, t_2, o_2, r_2\rangle$,
and Combine$(\mathfrak{A}_1, \mathfrak{A}_2)$,
infer $\langle B, \Lambda(t_1, t_2), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.
From $\mathfrak{A}_1 = \langle A \oplus B, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle {\sim}B, t_2, o_2, r_2\rangle$,
and Combine$(\mathfrak{A}_1, \mathfrak{A}_2)$,
infer $\langle A, \Lambda(t_1, t_2), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.
b. From $\mathfrak{A}_1 = \langle A \oplus B, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle A \rightarrow C, t_2, o_2, r_2\rangle$,
$\mathfrak{A}_3 = \langle B \rightarrow C, t_3, o_2, r_2\rangle$,

and Combine($\mathfrak{A}_1$, $\mathfrak{A}_2$),
infer $\langle C, \Lambda(t_1, \Lambda(t_2, t_3)), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.

Implication Introduction ($\rightarrow$I)
From $\langle B, \text{der}, o, r\rangle$
and any hypothesis $H \in o$,
infer $\langle H \rightarrow B, \text{der}, o - \{H\}, \int(o - \{H\})\rangle$.

Modus Podens—Implication Elimination, part 1 (MP)
From $\mathfrak{A}_1 = \langle A, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle A \rightarrow B, t_2, o_2, r_2\rangle$,
and Combine($\mathfrak{A}_1$, $\mathfrak{A}_2$),
infer $\langle B, \Lambda(t_1, t_2), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.

Modus Tollens—Implication Elimination, part 2 (MT)
From $\mathfrak{A}_1 = \langle A \rightarrow B, t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle \sim B, t_2, o_2, r_2\rangle$,
and Combine($\mathfrak{A}_1$, $\mathfrak{A}_2$),
infer $\langle \sim A, \Lambda(t_1, t_2), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.

Updating of Restriction Sets (URS): From $\langle A, t_1, o_1, r_1\rangle$, and $\langle \sim A, t_2, o_2, r_2\rangle$, we *must* replace each hypothesis $\langle H, \text{hyp}, \{H\}, R\rangle$ such that $H \in (o_1 \cup o_2)$ by $\langle H, \text{hyp}, \{H\}, \sigma(R \cup ((o_1 \cup o_2) - H))\rangle$. Furthermore, we *must* also replace every assertion $\langle F, t, o, r\rangle$ (t = der or t = ext) such that $o \cap (o_1 \cup o_2) \neq \emptyset$ by $\langle F, t, o, \sigma(r \cup \{(o_1 \cup o_2) - o\})\rangle$. However, the rule $\sim$I may be applied before the restriction sets are updated.

$\forall$ *Introduction* ($\forall$I): From $\langle B(t), \text{der}, o \cup \{A(t)\}, r\rangle$, in which $A(t)$ is a hypothesis that uses a term (t) never used in the system prior to A's introduction, and t is not in o or r, infer $\langle \forall(x)[A(x) \rightarrow B(x)], \text{der}, o, \int(o)\rangle$. (According to this rule of inference, the universal quantifier can be introduced only in the context of an implication. This is not a drawback, as it may seem at first, since the role of the antecedent of the implication ($A(x)$) is to define the type of objects that are being quantified.)

$\forall$ *Elimination—Universal Instantiation* ($\forall$E)
From $\mathfrak{A}_1 = \langle \forall(x)[A(x) \rightarrow B(x)], t_1, o_1, r_1\rangle$,
$\mathfrak{A}_2 = \langle A(c), t_2, o_2, r_2\rangle$,
and Combine($\mathfrak{A}_1$, $\mathfrak{A}_2$),
where c is any individual symbol,
infer $\langle A(c) \rightarrow B(c), \Lambda(t_1, t_2), o_1 \cup o_2, \mu(\{r_1, r_2\}, \{o_1, o_2\})\rangle$.

$\exists$ *Introduction* ($\exists$I)
From $\langle A(c), t, o, r\rangle$, where c is an individual constant,
infer $\langle \exists(x)[A(x)], \Lambda(t, t), o, r\rangle$.

$\exists$ *Elimination* ($\exists$E)
From $\langle \exists(x)[A(x)], t, o, r\rangle$

infer $\langle A(c), \Lambda(t, t), o, r\rangle$
where c is any individual constant that was never used before.

The rules of $\sim$I (part 1), &I (part 1), and $\oplus$I are applicable only to assertions that have the same OS and the same RS. This condition is not as constraining as it may seem at first glance, since Martins and Shapiro prove that, if two assertions have the same OS, they also have the same RS. In fact, this justifies a different view of the data base of assertions. One may think of the KB as containing a set of wffs. For every wff A and every assertion $\mathfrak{A}$ in which $A = \text{wff}(\mathfrak{A})$, A is a wff of type $\text{ot}(\mathfrak{A})$ in the context $\text{os}(\mathfrak{A})$ and in every context γ such that $\text{os}(\mathfrak{A}) \subseteq \gamma$. Two contexts α and β are known to be inconsistent if, in the previous way of thinking, there is an assertion $\mathfrak{A}$ such that $\alpha = \text{os}(\mathfrak{A})$ and $\beta \in \text{rs}(\mathfrak{A})$ or $\beta = \text{os}(\mathfrak{A})$ and $\alpha \in \text{rs}(\mathfrak{A})$. The rules of inference of SWM apply with the obvious modifications. However, Martins and Shapiro show that if one restricts the reasoner to consideration of wffs in a single context, not known to be inconsistent, the Combine test need never be made, and, if a new contradiction is uncovered within the context, the removal of any wff in the OS of the contradictory assertion will restore the context to the status of not being known to be inconsistent. This is the logical basis for assumption-based truth maintenance, or belief revision (Martins 1987; Martins and Shapiro 1981, 1984, 1986a, 198+, 1988).

§83.1.1.2. Example. The main advantages of SWM are that the OSs show precisely the hypotheses required to derive each assertion, so that, when a contradiction is found, no innocent hypothesis will be blamed, and, once a set of hypotheses is found to be contradictory, reasoning will no longer occur in the context formed by that set of hypotheses. In actual computer reasoning systems based on SWM, the user may explicitly decide to reason in a context known to be inconsistent.

As an example of SWM, we show the derivation that the existence of the Russell set is self-inconsistent.

1 $\langle\exists(s)[\text{Set}(s)\&\forall(x)[\text{Set}(x)\rightarrow((x \in s\rightarrow \sim(x \in x))\&(\sim(x \in x)\rightarrow x \in s))]]$,
hyp, $\{1\}, \{\ \}\rangle$ hyp
2 $\langle\text{Set}(R)\&\forall(x)[\text{Set}(x)\rightarrow((x \in R\rightarrow \sim(x \in x))\&(\sim(x \in x)\rightarrow x \in R))]$,
der, $\{1\}, \{\ \}\rangle$ $\exists$E 2
3 $\langle\text{Set}(R)$, der, $\{1\}, \{\ \}\rangle$ &E 2
4 $\langle\forall(x)[\text{Set}(x)\rightarrow((x \in R\rightarrow \sim(x \in x))\&(\sim(x \in x)\rightarrow x \in R))]$,
der, $\{1\}, \{\ \}\rangle$ &E 2
5 $\langle((R \in R\rightarrow \sim(R \in R))\&(\sim(R \in R)\rightarrow R \in R))$, der, $\{1\}, \{\ \}\rangle$ $\forall$E 4,3
6 $\langle(R \in R)\rightarrow \sim(R \in R)$, der, $\{1\}, \{\ \}\rangle$ &E 5
7 $\langle R \in R$, hyp, $\{7\}, \{\ \}\rangle$ hyp
8 $\langle \sim(R \in R)$, der, $\{1, 7\}, \{\ \}\rangle$ MP 7, 6
9 $\langle \sim(R \in R)$, ext, $\{1\}, \{\{7\}\}\rangle$ $\sim$I 7, 8

URS is now required by the presence of 7 and 8. Every assertion with an OS of $\{1\}$ now has $\{7\}$ added to its RS, and every assertion with an OS of $\{7\}$ now has $\{1\}$ added to its RS. The two hypotheses are now:

1′ $\langle\exists(s)[\mathrm{Set}(s)\&\forall(x)[\mathrm{Set}(x)\to((x\in s\to{\sim}(x\in x))\&({\sim}(x\in x)\to x\in s))]],$
hyp, $\{1\}, \{\{7\}\}\rangle$ hyp; URS 7, 8
7′ $\langle R\in R,$ hyp, $\{7\}, \{\{1\}\}\rangle$ hyp; URS 7, 8

Other revised assertions will be shown when and only when they are about to be used.

9′ $\langle{\sim}(R\in R),$ ext, $\{1\}, \{\{7\}\}\rangle$ ${\sim}$I 7, 8; URS 7, 8
5′ $\langle((R\in R\to{\sim}(R\in R))\&({\sim}(R\in R)\to R\in R)),$
der, $\{1\}, \{\{7\}\}\rangle$ $\forall$E 4, 3; URS 7, 8
10 $\langle{\sim}(R\in R)\to R\in R,$ der, $\{1\}, \{\{7\}\}\rangle$ &E 5′
11 $\langle R\in R,$ ext, $\{1\}, \{\{7\}\}\rangle$ MP 9′, 10

URS is now required by the presence of 11 and 9′. In this case, $o_1 = o_2 = o_1\cup o_2 = \{1\}$; so hypothesis 1 becomes:

1″ $\langle\exists(s)[\mathrm{Set}(s)\&\forall(x)[\mathrm{Set}(x)\to$
$((x\in s\to{\sim}(x\in x))\&({\sim}(x\in x)\to x\in s))]],$ hyp, $\{1\}, \{\{\ \}\}\rangle$
hyp; URS 7, 8, URS 11, 9′

The existence of the empty set in the RS of 1″ means that 1″ is self-inconsistent and not combinable with any other assertion. Within the context of the hypothesis $\{1\}$ we may reason about the Russell set, but that hypothesis may not be combined with any other; so the contradiction has been isolated.

§83.1.2. Implementations. Martins and Shapiro implemented a computer reasoning system, SNeBR (Martins 1983; Martins and Shapiro 1983, 1984, 1986c, 1988), based on a version of **SWM** for the nonstandard propositional connectives of SNePS, the Semantic Network Processing System (Shapiro 1979, Shapiro and Rapaport 1987).

Ohlbach and Wrightson 1984 used the Markgraf Karl Refutation Procedure (Raph 198+), a resolution-based theorem prover, to show that

$$(A\to(B\to B))\to(A\to(A\to(B\to B)))$$

follows from the axioms of $\mathbf{T}_{\to}$ (see §8.13).

Thistlewaite and McRobbie have implemented KRIPKE, an **R**-based automatic theorem prover (see Malkin 1987 and Thistlewaite, McRobbie and Meyer 1988).

Brachman, Gilbert, and Levesque 1985 mention their intention to implement an inference mechanism based on a relevance logic as part of the KRYPTON knowledge representation/reasoning system.

§83.2. Use of the four-valued semantics of R. Belnap 1975, 1977 was the first to suggest that the four-valued semantics of **R** make it a useful model for computer reasoning systems. A revised version of these papers appears as §81 of this volume; so the discussion will not be repeated here, beyond noting the meaning, in a computer reasoning context, of the four values. Most data-base management systems assume what in Artificial Intelligence has been called the Closed World assumption (Reiter 1978). This is that the data base contains all relevant true information, so whatever information is not in the data base is false. The Closed World assumption is unreasonable for any reasoning system that might learn new facts. For such a system, false assertions as well as true assertions may be explicitly stored in the data base. An assertion that is not stored in the data base as either true or false must only be assumed to be unknown. True, false, and unknown are three of the four truth values. The fourth, "both," is used if more than one informant put information into the data base and one informant said that an assertion was true while another said that it was false. Perhaps a single informant at one time said that the assertion was true, and at another time said that it was false. Perhaps the actual situation changed, so that an assertion that was true at one time later became false, or maybe a simple error was made in entering information, and this led to a contradiction. Of course, an assertion's having a truth value of "both" indicates some problem to be resolved in the data base, unless it is true in one context and false in another. However, until the problem is resolved, the use of **R** can prevent the contradiction from polluting the data base with every possible conclusion (derivable from a contradiction in standard logics).

The Closed World assumption is also unreasonable for a data-base management system or reasoning system that, for reasons of speed, must produce information before it can develop all the implications of its stored data. Such a system might not find some information, not because it was not in or implied by its data base, but because it was not given enough time (or other resources). Call the information retrievable by such a system within its resource limits its *explicit beliefs* and all the information it could retrieve, given an arbitrary amount of resources, *its implicit beliefs*. Semantics for relevance logics appropriate for the set of explicit beliefs of such systems have been discussed by Levesque 1984, 1984a; Fagin and Halpern 1985, 1987; Frisch 1985, 1986; and Lakemeyer 1986 (see also Levesque 1986).

Lakemeyer 1987 extends the model of Levesque 1984a to one that an agent can use to hold metabeliefs (beliefs about its own beliefs) and reason about them efficiently.

Mitchell and O'Donnell 1986 (see also O'Donnell 1985) are particularly interested in the use of **R** for data base systems that may have errors in the data. They present two versions of realizability semantics for relevance logic, show soundness for the first, and soundness and completeness over a nonstandard set of models for the second.

Patel-Schneider 1985, 1985a presents a decidable variant of relevance logic including quantifiers as an appropriate logic for reasoning systems.

Allowing "unknown" as a truth value invites one to consider inferences based on lack of knowledge; e.g., if P is unknown, conclude Q. The Closed World assumption then amounts to $\forall P$ (if P is unknown then $\sim P$), but less overriding rules are useful for the sort of default reasoning people seem to engage in. (The favorite example in Artificial Intelligence is: if x is a bird and it is not known that x doesn't fly then x does fly.) If a previously unknown datum, used for one of these lack-of-knowledge inferences, is later learned to be false, the earlier conclusion may no longer be justified. This phenomenon, of once valid conclusions becoming invalid through the gaining of knowledge, has been termed "nonmonotonicity," and several nonmonotonic logics have been proposed as the foundation of such reasoning (see Perlis 1987). Sandewall 1985, 1985a discusses a functional approach to nonmonotonic logic with the four-valued semantics of **R**.

A particular kind of data base used in Artificial Intelligence is the *inheritance net* (see Touretzky 1987). Thomason, Horty, and Touretzky 1986 discuss inheritance nets in which nodes represent either individuals or kinds and in which there are two kinds of links. The link $p \rightarrow q$ means that p is a q (or all ps without exception are qs), and the link $p \not\rightarrow q$ means that p is not a q (or ps are not qs, again without exception). They give a proof theory and a model theory for inference in these nets, show the soundness and completeness of the proof theory relative to the model theory, and show that the four-valued semantics of **R** is an appropriate interpretation of this logic. This work has been extended in Horty, Thomason, and Touretzky 1987 and in Horty and Thomason 1988.

M. Fitting 1988 and 198+ investigates a generalization of the four values (topological bilattices) to represent a space of incomplete or contradictory evidences for and against.

Not all connections between relevance logic and computer science come by way of AI. A recent and potentially fruitful application of relevance logic comes by way of Girard's 1987 completely independent invention of what he calls "linear logic," but which Avron 1987 shows (omitting the novel connectives Girard labels "exponentials") is **R** minus contraction and distribution. Girard proposes linear logic as a logical foundation for the study of parallelism (much as intuitionistic logic has been proposed as the logical foundation of the traditional sequential programs). The notation and terminology that Girard uses are richly inventive, and Avron 1987 provides a useful Rosetta Stone for translating some of Girard's language into the usual relevance-logic framework. It should be remarked that although the dropping of contraction seems well motivated in the context of parallelism, it is perhaps an open question whether the dropping of distribution is similarly principled. Girard provides both semantical and proof-theoretical characterizations of his system of linear logic, and there would seem to be much room for a two-way flow of results and notions between work on linear logic and work on the more familiar (at least to the authors if not the readers of this volume) relevance logics.

Other relevant papers are Rezus 198+, 198+a, Belnap 1990, Bollen 1985, 1987, and Blair and Subrahmanian 1987.

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THIS BIBLIOGRAPHY is meant to serve three purposes: (1) It gives a complete listing of all citations in the main body of both volumes of this book. (2) It gives as complete a listing as possible of all items that treat not only the main systems of this book but also related relevant, paraconsistent, and conditional systems. (3) It attempts to cover a wide range of philosophical and/or technical items that discuss the various candidates, both classical and nonclassical, for the correct logics of entailment and implication.

If I were to attempt to list by name those who aided in the completion of the bibliography by sharing both their works and their knowledge, I would end up mentioning a significant fraction of the authors listed here. I owe them gratitude not only for their help but also for the opportunity to learn of, to read, and to enjoy some of the most important philosophical work of the century. Most of all I thank Susan, who made the enterprise worthwhile.

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INDEX OF SUBJECTS

Special symbols are listed in a separate table after this index; see also under the heading "notation for." Greek letters are alphabetized after roman letters. Curly braces, { }, are used for simultaneous expression of alternate readings. A phrase such as "**E** et al." refers to neighbors of **E** (in the sense of Chapter V) such as **T** and **R**; and a phrase such as "$\mathbf{E}^{\forall\exists x}$ et al." refers to the various neighbors $\mathbf{T}^{\forall\exists x}$, $R^{\forall\exists x}$, etc.

System names are listed in the usual alphabetical order, so that "E" and "FE", for example, do not occur together. We assume an automatic cross-reference among entries having the same boldface portion. In this connection, we remind the reader of the conventions on system nomenclature explained in the Preface to Volume I: (1) boldface indicates a principal system (of propositional logic); (2) subscripts indicate fragments (e.g. $\mathbf{E}_+$ is the positive fragment of **E**); (3) superscripts indicate extensions (by adding the superscripted connectives); and (4) lightface prefixes indicate different formulations of the same underlying system. All the lightface prefixes occurring in Volume II are listed under "logic, formulations of." Also prominent in this volume is the use of "$-\mathbf{W}$", or sometimes just "**W**", to indicate a variant system without contraction; e.g. $\mathbf{T}-\mathbf{W}=\mathbf{TW}=\mathbf{T}$ without the contraction axiom.

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SPECIAL SYMBOLS

ONLY SYMBOLS with more than local employment are listed. Also consult the Index of subjects under "notation for."

GRAMMATICAL SYMBOLS

→	if-then; implication; entailment	xxiii
⇆	co-entailment	xxiii
¯	overbar for negation	xxiii
~	negation	xxiii
¬	intuitionist negation, Boolean negation	52
&	conjunction (sometimes omitted)	xxiii
∨	disjunction	xxiii
&	generalized conjunction	338
⋁	generalized disjunction	338
∘	co-tenability, fusion, binary structure operator (sometimes omitted)	xxiii
□	necessity	xxiii
◇	possibility	263
⥽	strict "implication"	52
⊃	material or intuitionist "implication"	4
≡	material "equivalence"	5
⊢	consecution	281
∀	universal quantifier	5
∃	existential quantifier	5
=	identity	4
:	such that	168

PROOF-THEORETICAL SYMBOLS

⊢	provability; deducibility	17
$\vdash_T$	provability in T	29
⊣	unprovability	378
⇆	mutual provable entailment	204

Set-theoretic Symbols

$\in$	membership	35
$\varnothing$	empty set	206
$\cup$	join, union	63
$\cap$	meet, intersection	95
$\times$	Cartesian product	95
$\bigcup$	generalized join,	531
$\bigcap$	generalized meet	531
$\bigtimes$	generalized product	95
$\subseteq, \supseteq$	subset, superset	88
$\subset, \supset$	proper subset, proper superset	88

Semantic Symbols

$\Vdash$	consequence	144
$\vDash$	truth	158
$\Dashv$	nontruth	224
$\lvert A\rvert, \lVert A\rVert$	interpretation of A	185
$\uparrow$	up arrow	239
$\downarrow$	down arrow	239
$\approx$	equivalence relation	269
$\lesssim, \gtrsim$	ordering relations	250
$\triangleright, \triangleleft$	ordering relations	378
$\gtrsim$	ordering relation	405

Algebraic Symbols

$\leq, <, \geq, >$	ordering relations	xxvi
$\wedge$	lattice greatest lower bound; meet (sometimes omitted)	89
$\vee$	lattice least upper bound; join	89
$\bigwedge$	generalized meet	88
$\bigvee$	generalized join	88
$\circ$	operation in algebra—often intensional conjunction, fusion	369
$\otimes$	special multiplication, sometimes fusion	361
$[\ ]$	equivalence class	
$*$	used to mark designated values of a matrix	141
$\bot$	bottom element of a structure	359
$\top$	top element of a structure	359
$\sqsubseteq, \sqsupseteq$	square ordering relations	509

$\sqcup$	square join	509
$\sqcap$	square meet	509
$\bigsqcup$	generalized square join,	509
$\bigsqcap$	generalized square meet	509

MISCELLANEOUS SYMBOLS

$\Rightarrow$	use-language conditional	210
$\Leftrightarrow$	use-language biconditional	210
$\Leftarrow$	use-language reverse conditional	211
$\neq$, $\notin$, $\nvdash$, $\nvDash$, $\npreceq$, $\nsucceq$, $\nsubseteq$	negated signs	
$\langle\ \rangle$	angle brackets, usually ordered *n*-tuples	34